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ASYMPTOTIC ALMOST PERIODIC DISTRIBUTIONS, STRUCTURAL THEOREM

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Dedicated to Academician Ljubomir Iliev
on the Occasion of his Eightieth Birthday

ABSTRACT. For distributions $T \in D'(\mathbf{R}^k \times \mathbf{R}^m)$ the asymptotic almost periodicity on $\mathbf{R}^m$ is defined with their asymptotic behaviour. Equivalent assertions are proved that a distribution is asymptotic almost periodic with the given asymptotic behaviour.

Introduction. The theory of asymptotic almost periodic numerical functions has been well elaborated with different applications, especially in the theory of differential equations. In [2] one can find such results collected and systematized. Already by Schwartz [6] the definition and some properties of almost periodic distributions have been placed. The asymptotic almost periodicity in spaces of vector valued continuous functions was elaborated in [5]. Cioranescu in [1] introduced the asymptotic almost periodicity of distributions. Her results were in the one-dimensional case and they were applied to a differential equation.

Our definition of the asymptotic almost periodic distributions is in the multidimensional case and it gives simultaniously the measure of the asymptotic behaviour. It is suitable to discuss solutions of partial differential equations in the space of Schwartz's distributions (see [7]), where the situation with the almost periodicity is much more complicated than in case of differential equations. For example, solutions of $\Delta_2 U(x,y) = 0$ are: $u = 1$, periodic in x and y; $u(x,y) = \sin x \cos x/(\cos^2 x + \mathrm{sh}^2 y)$, periodic in x, but not in y; $u(x,y) = (x^2 - y^2)/(x^2+y^2)^2$, which is almost periodic neither in x nor in y.

In the definition of the almost periodic distributions we follow the idea and the results of Schwartz [6] and Cioranescu's [1] ideas in case of asymptotic almost periodic distributions.

1. Notations and definitions. By K we denote a compact set in $\mathbf{R}^k$; $K \subset\subset \mathbf{R}^k$ means: for every compact set K belonging to $\mathbf{R}^k$.

For $h = (h_1, \dots, h_m) \in \mathbf{R}^m$, $h \to \infty$ means that $h_i \to \infty$, $i = 1, \dots, m$. If $j = (j_1, \dots, j_{k+m}) \in \mathbf{N}_0^{k+m}$, then $|j| = j_1 + \cdots + j_{k+m}$. $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

A function $u : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$, or a distribution U defined on $\mathbf{R}^k \times \mathbf{R}^m$ we also denote by $u(\cdot,\cdot)$ or by $U(\cdot,\cdot)$ respectively.

The notations and definitions for spaces and subspaces of distributions are as in [6]. For the space of distributions $\boldsymbol{D}'(\mathbf{R}^k \times \mathbf{R}^m)$ we write in short $\boldsymbol{D}'_{k+m}$. The restriction of the distribution $T \in \boldsymbol{D}'_{k+m}$ on $\mathbf{R}^k \times \mathbf{R}^m_+$ we denote by T^+.

A filter of distributions $\{T_a, a \in A\} \subset \boldsymbol{D}'_{k+m}$ is said to be **uniformly convergent in $\boldsymbol{D}'_{k+m}$** if $\langle T_a(\cdot + x, \cdot + y), \phi(\cdot,\cdot)\rangle$ converges uniformly in $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m_+$ for every $\phi \in \boldsymbol{D}_{k+m}$.

A sequence $\{\delta_n\} \in \boldsymbol{D}_{k+m}$ is called a δ-sequence if $\delta_n \geq 0$; $\int \delta_n \, dx = 1$, $n \in \mathbf{N}$ and $\operatorname{supp} \delta_n \subset [-a_n, a_n]$, $a_n \to 0$, when $n \to \infty$. If $\phi \in \boldsymbol{D}_{k+m}$, then $\delta_n * \phi \to \phi$ in $\boldsymbol{D}_{k+m}$, when $n \to \infty$.

Throughout this paper we denote by c a positive and measurable function defined on $\mathbf{R}^m$ and such that $c(h) \to 0$, when $h \to \infty$.

Definition A.[3] *We say that distribution $T \in \boldsymbol{D}'_{k+m}$ has the S-asymptotics related to c on $\mathbf{R}^m$ if $\{T(\cdot,\cdot+h)/c(h);\ h \in \mathbf{R}^m_+\}$ converges in $\boldsymbol{D}'_{k+m}$ to U, when $h \to \infty$.*

We write in short $T \overset{s}{\sim} c(h) \cdot U, h \in \mathbf{R}^m_+$.

Remarks. a) If $\{T(\cdot,\cdot + h)/c(h);\ h \in \mathbf{R}^m_+\}$ converges in $\boldsymbol{D}'_{k+m}$, then $\{T(\cdot + x, \cdot + y + h)/c(h);\ h \in \mathbf{R}^m_+\}$ converges in $\boldsymbol{D}'_{k+m}$ uniformaly in $x \in K_1 \subset\subset \mathbf{R}^k$ and $y \in K_2 \subset\subset \mathbf{R}^m$, when $h \to \infty$. This follows from the relation

$$\langle T(\cdot + x, \cdot + y + h)/c(h), \phi(\cdot,\cdot)\rangle = \langle T(\cdot,\cdot + h)/c(h), \phi(\cdot = x, \cdot - y)\rangle$$

and the fact that the set $\{\phi(\cdot - x, \cdot - y);\ x \in K_1,\ y \in K_2\}$ is bounded in $\boldsymbol{D}_{k+m}$.

b) If T_1 and T_2 belong to $\boldsymbol{D}'_{k+m}$, $T_1 = T_2$ on $\mathbf{R}^k \times \mathbf{R}^m_+$ and T_1 has the S-asymptotics related to c on $\mathbf{R}^m$, then T_2 has the S-asymptotics related to c on $\mathbf{R}^m$, as well, with the same limit. This follows from

$$\langle (T_1(\cdot,\cdot + h) - T_2(\cdot,\cdot + h)), \phi(\cdot,\cdot)\rangle = 0$$

for every $\phi \in \boldsymbol{D}_{k+m}$ and enough large h ([4], p. 100).

We denote by

$(\boldsymbol{CB})_{k+m}$ the space of continuous and bounded functions $f(x,y)$, $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m$, with the norm $\|f\| = \sup_{x \in \mathbf{R}^k, y \in \mathbf{R}^m} |f(x,y)|$.

$(\boldsymbol{CB})_{k+m,0} = \{w \in (\boldsymbol{CB})(\mathbf{R}^k \times \mathbf{R}^m_+); \lim_{y \in \mathbf{R}^m_+, y \to \infty} w(x,y) = 0, \text{ uniformly in } x \in \mathbf{R}^k\}$.

$(CB)_{ap,m} = \{v \in (CB)_{k+m}; \{v(\cdot, \cdot+h); h \in \mathbf{R}^m\}$ is relatively compact in $(CB)_{k+m}\}$. $(CB)_{ap,m}$ is the space of **almost periodic functions in** y.

$(CB)_{aap,m} = \{u \in (CB)(\mathbf{R}^k \times \mathbf{R}^m_+), u = v + w$ on $\mathbf{R}^k \times \mathbf{R}^m_+$, where $v \in (CB)_{ap,m}$ and $w \in (CB)_{k+m,0}\}$. $(CB)_{aap,m}$ is the space of **asymptotic almost periodic functions in** y.

$B'_{k+m} = D_L(\mathbf{R}^k \times \mathbf{R}^m)'$.

$B'_{k+m,0} = \{W \in B'_{k+m}; W(\cdot, \cdot + h) \to 0$ uniformly in D_{k+m}, when $h \in \mathbf{R}^m_+, h \to \infty\}$.

We recall that $W^+(\cdot, \cdot + h) \to 0$, $h \in \mathbf{R}^m_+$, $h \to \infty$ in $B'(\mathbf{R}^k \times \mathbf{R}^m_+)$ if and only if $\langle W^+(\cdot + x, \cdot + y + h), \phi(\cdot, \cdot)\rangle \to 0$, when $h \to \mathbf{R}^m_+$, $h \to \infty$, for every $\phi \in D(\mathbf{R}^k \times \mathbf{R}^m_+)$ uniformly in $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m_+$ (see [6], p.61). If $W \in B'_{k+m,0}$, then $W^+(\cdot, \cdot + h) \to 0$ in $B'(\mathbf{R}^k \times \mathbf{R}^m_+)$ when $h \in \mathbf{R}^m_+$, $h \to \infty$. Conversely, if for $W \in B'_{k+m}$, $W^+(\cdot, \cdot + h) \to 0$ in $B'(\mathbf{R}^k \times \mathbf{R}^m_+)$, then $W \in B'_{k+m,0}$.

2. Almost periodic and asymptotic almost periodic distributions.

Definition 1. *A distribution $T \in B'_{k+m}$ is said to be almost periodic on $\mathbf{R}^m$ if the set $\{T(\cdot, \cdot + h); h \in \mathbf{R}^m\}$ is relatively compact in B'_{k+m}.*

The set of almost periodic distribututions on $\mathbf{R}^m$ we denote by $B'_{ap,m}$.

Let us recall some properties of almost periodic distributions ([6], [7]):

1. If $T \in B'_{ap,m}$, then $D^j T \in B'_{ap,m}$, as well. ($j = (j_1, \ldots, j_{k+m})$, $j \in \mathbf{N}_0^{k+m}$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$).

2. $T \in B'_{ap,m}$ if and only if

$$T = \sum_{|j| \le i} D^j g_j, \; g_j \in (CB)_{ap,m}, \; |j| \le i.$$

3. $T \in B'_{ap,m}$ if and only if $(T * \phi) \in (CB)_{ap,m}$ for every $\phi \in D_{k+m}$.

4. $T \in B'_{ap,m}$ if and only if for every sequence $\{c_n\} \subset \mathbf{R}^m$ there exists a subsequence $\{b_n\}$ such that $\{T(\cdot, \cdot + b_n)\}$ converges in B'_{k+m}, when $n \to \infty$, that means that $\{T(\cdot + x, \cdot + y + b_n)\}$ converges in D'_{k+m}, when $n \to \infty$, uniformly in $x \in \mathbf{R}^k$ and $y \in \mathbf{R}^m$.

5. If $T \in B'_{ap,m}$, $T \ne 0$ and if $\lim_{n\to\infty} T(\cdot, \cdot + b_n) = S$ in B'_{k+m} for a sequence $\{b_n\} \subset \mathbf{R}^m$, then $S \ne 0$ and $\lim_{n\to\infty} S(\cdot, \cdot - b_n) = T$ in B'_{k+m}, as well.

Definition 2. *A distribution $T \in B'_{k+m}$ is said to be asymptotic almost periodic on $\mathbf{R}^m$, in short $T \in B'_{aap,m}$, if $T = P + Q$, where $P \in B'_{ap,m}$ and $Q \in B'_{k+m,0}$.*

Proposition. *If $T \in B'_{aap,m}$ and $T = P + Q$, where $P \in B'ap, m$ and $Q \in B'_{k+m,0}$ this decomposition of T is unique.*

Proof. Suppose on the contrary that $T = P + Q = R + S$, $R \in \boldsymbol{B}'_{ap,m}$ and $S \in \boldsymbol{B}'_{k+m,0}$. Let $\{c_n\}$ be a sequence belonging to $\mathbf{R}^m_+$ and such that $c_n \to \infty$, $n \to \infty$. By property 4 there exists a subsequence $\{b_n\}$ such that $P(\cdot, \cdot + b_n) \to P_1$ and $R(\cdot, \cdot + b_n) \to R_1$ in $\boldsymbol{B}'_{k+m}$ consequently uniformly convergent in $\boldsymbol{D}'_{k+m}$ and $S(\cdot, \cdot + b_n) \to 0$ uniformly in $\boldsymbol{D}'_{k+m}$, when $n \to \infty$.

Now from $\lim_{n\to\infty}(P+Q)(\cdot, \cdot + b_n) = \lim_{n\to\infty}(R+S)(\cdot, \cdot + b_n)$ uniformly in $\boldsymbol{D}'_{k+m}$ it follows that $P_1 = R_1$. By property 5 we have $P = R$, therefore $Q = S$. □

Theorem. *For $T \in \boldsymbol{B}'_{k+m}$ and $T \neq 0$ the following assertions are equivalent:*

1. *$T \in \boldsymbol{B}'_{aap,m}$, $T = P + Q$, where $P \in \boldsymbol{B}'_{ap,m}$ and $Q \in \boldsymbol{B}'_{k+m,0}$, $Q \overset{s}{\sim} c(h).U$, $h \in \mathbf{R}^m_+$.*

2. *There exist $p \in \mathbf{N}_0$ and $F_j \in (\boldsymbol{CB})_{aap,m}$, $f_j = v_j + w_j$, where $v_j \in (\boldsymbol{CB})_{ap,m}$, $w_j \in (\boldsymbol{CB})_{k+m,0}$ and $\{w_j(x,h)/c(h); h \in \mathbf{R}^m_+\}$, $|j| \leq p$, converges when $h \to \infty$ uniformly in $x \in K \subset\subset \mathbf{R}^k$, such that*

$$T = \sum_{|j|\leq p} D^j f_j, \quad \text{on } \mathbf{R}^k \times \mathbf{R}^m_+.$$

3. *For every $\phi \in \boldsymbol{D}_{k+m}$, $T*\phi = v+w \in (\boldsymbol{CB})_{aap,m}$ and $\{w(x,h)/c(h); h \in \mathbf{R}^m_+\}$ converges uniformly in $x \in K \subset \mathbf{R}^k$, when $h \to \infty$.*

4. *For a δ-sequence $\{\delta_n\}$, $(T*\delta_n) \in (\boldsymbol{CB})_{aap,m}$, $n \in \mathbf{N}$ and for every $\phi \in \boldsymbol{D}_{k+m}$ there exists a function $v \in (\boldsymbol{CB})_{ap,m}$ such that $((T*\phi)(x,h) - v(x,h))/c(h)$ converges uniformly in $x \in K \subset\subset \mathbf{R}^k$, when $h \to \infty$, $h \in \mathbf{R}^m_+$.*

Proof. 1⇒2. Suppose that $T = P + Q$, $P \in \boldsymbol{B}'_{ap,m}$ and $Q \in \boldsymbol{B}'_{k+m,0}$. By property 2 we have $P = \sum_{|j|\leq i} D^j v_j$, $v_j \in (\boldsymbol{CB})_{ap,m}$. Also, by Remarques in [6], p. 58, $Q^+ = \sum_{|j|\leq q} D^j w_j$, where $w_j \in (\boldsymbol{CB})_{k+m,0}$ and by Proposition and Remarks b) we have $w_j(x,h)/c(h)$ converges when $h \in \mathbf{R}^m_+$, $h \to \infty$ uniformly in $x \in K \subset\subset \mathbf{R}^k$, $|j| \leq i$.

Since the zero belongs to $(\boldsymbol{CB})_{ap,m}$ and to $(\boldsymbol{CB})_{k+m,0}$, as well, we have proved that 1⇒2.

2⇒1. Suppose that T has the form given in assertion 2. Then

$$T = \sum_{|j|\leq p} D^j v_j + \sum_{|j|\leq p} D^j w_j \quad \text{on} \quad \mathbf{R}^k \times \mathbf{R}^m_+.$$

By property 2, $\sum_{|j|\leq p} D^j v_j \equiv p \in \boldsymbol{B}'_{ap,m}$. It remains the second sum. We will prove that $D^j w_j \in \boldsymbol{B}'_{k+m,0}$, $|j| \leq p$. Since $T \in \boldsymbol{B}'_{k+m}$ we have $Q = T - P \in \boldsymbol{B}'_{k+m}$ and we have just seen that $P \in \boldsymbol{B}'_{ap,m}$. Now, we have to prove that $Q \in \boldsymbol{B}'_{k+m,0}$ using the fact that $Q^+ = \sum_{|j|\le p} D^j w_j$. For $\phi \in D(\mathbf{R}^k \times \mathbf{R}^m_+)$ and $|j| \le p$

$$\langle D^j w_j(\cdot + x, \cdot + y + h), \phi(\cdot,\cdot)\rangle = (-1)^{|j|} \int_{\mathbf{R}^{k+m}} w_j(a+x, b+y+h)\phi(a,b)\, da\, db \to 0$$

when $h \to \infty$ uniformly in $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m_+$.

Thus we proved that $Q^+(\cdot, \cdot + h) \to 0$ in $B'(\mathbf{R}^k \times \mathbf{R}^m_+)$ hense $Q \in B'_{k+m,0}$.

In the same way we can prove that $D^j w_j(\cdot, \cdot + h)/c(h)$ converges in D'_{k+m} when $h \in \mathbf{R}^m_+$, $h \to \infty$. Now, 2⇒1 follows from Remarks b).

1⇒3. If $T = P + Q$, where $P \in B'_{ap,m}$ and $Q \in B'_{k+m,0}$, then we have to prove that $T * \phi = P * \phi + Q * \phi \in (CB)_{aap,m}$ for every $\phi \in D_{k+m}$. By property 3, $P * \phi = v \in (CB)_{ap,m}$. For the second addend $Q * \phi = w$ we have that it belongs to $(CB)_{k+m}$.

From relation

$$w(x,y) = (Q * \phi)(x,y) = \langle Q(\cdot + x, \cdot + y)(\check{\phi}(\cdot,\cdot)\rangle, \quad \check{\phi}(x,y) = \phi(-x,-y),$$

it follows the asked properties of w.

3⇒2. Suppose that $T * \phi \in (CB)_{aap,m}$, $\phi \in D_{k+m}$. By Theorem 9.3 in [2], $T * \phi$ belongs to $(CB)_{aap,m}$ if and only if for every sequence $\{c_n\} \subset \mathbf{R}^m_+$, $c_n \to \infty$, when $n \to \infty$, there exists a subsequence $\{b_n\}$ such that $(T * \phi)(x, y + b_n)$ converges uniformly in $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m_+$, when $n \to \infty$. Since for $\phi \in D(\mathbf{R}^k \times \mathbf{R}^m_+)$

$$\begin{aligned} \lim_{n\to\infty} (T * \check{\phi})(x, y+b) &= \lim_{n\to\infty} \langle T(\cdot + x, \cdot + y + b_n), \phi(\cdot,\cdot)\rangle \\ &= \lim_{n\to\infty} \langle T^+(\cdot + x, \cdot + y + b_n), \phi(\cdot,\cdot), \end{aligned}$$

$T^+(\cdot, \cdot + b_n)$ converges in $B'(\mathbf{R}^k \times \mathbf{R}^m_+)$. By the cited Remarques in [6], p. 58, there exists $q \in \mathbf{N}$ such that

$$T = \sum_{|j|\le q} D^j f_j \quad \text{on} \quad \mathbf{R}^k \times \mathbf{R}^m_+, \quad f_j \in (CB)_{k+m}, \quad |j| \le q,$$

where f_j, $|j| \le q$, have the property: for every $\{c_n\} \subset \mathbf{R}^m_+$, $c_n \to \infty$, when $n \to \infty$, there exists $\{b_n\} \subset \{c_n\}$ such that $f_j(x, y + b_n)$ converges when $n \to \infty$, uniformly in $x \in \mathbf{R}^k$ and $y \in \mathbf{R}^m_+$, $|j| \le q$. By the mentioned theorem in [2], it follows that $f_j \in (CB)_{app,m}$, $|j| \le q$. Therefore $f_j = v_j + w_j$, where $v_j \in (CB)_{ap,m}$, $w_j \in (CB)_{k+m,0}$, $|j| \le q$, and w_j has the asked asymptotic behaviour. Consequently 3⇒2.

3⇒4 is trivial because of $\delta_n \in D_{k+m}$, $n \in \mathbf{N}$.

4⇒3. First we have to prove that the set $A = \{\delta_i(\cdot + x, \cdot + y);\ i \in \mathbf{N},\ x \in \mathbf{R}^k,\ y \in \mathbf{R}^m\}$ is dense in D_{k+m}. Suppose that $U \in D'_{k+m}$ and

$$\langle U(\cdot,\cdot), \delta_i(\cdot + x, \cdot + y)\rangle = (U * \check{\delta}_i)(x,y) = 0$$

for every $i \in \mathbf{N}$, $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m$. Then for any $\phi \in \boldsymbol{D}_{k+m}$, $\langle (U * \check{\delta}_i), \phi\rangle = 0$, $i \in \mathbf{N}$, and

$$\langle U, \phi\rangle = \lim_{i\to\infty} \langle U, \delta_i * \phi\rangle = \lim_{i\to\infty} \langle U * \check{\delta}_i, \phi\rangle = 0.$$

It follows that $U = 0$ and that A is dense in $\boldsymbol{D}_{k+m}$. □

We know that $T * \phi \in (\boldsymbol{CB})_{k+m}$ for every $\phi \in \boldsymbol{D}_{k+m}$. If the assertion 4 is true, then for every $\{c_n\} \subset \mathbf{R}^m_+$, $c_n \to \infty$, when $n \to \infty$, there exists a subsequence $\{b_n\}$ such that $(T * \delta_i)(x, y + b_n)$ converges uniformly in $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m_+$, when $n \to \infty$, $i \in \mathbf{N}$. This is also true for the sequence in $n \in \mathbf{N}$, $(T(\cdot,\cdot) * \delta_I(\cdot + a, \cdot + b))(x, y + b_n) = (T * \delta_i)(x + a, y + b + b_n)$ for every $i \in \mathbf{N}$ and for some fixed $a \in \mathbf{R}^k$, $b \in \mathbf{R}^m$. By the Banach-Steinhaus theorem it follows that $(T * \phi)(x, y + b_n)$ converges uniformly in $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m_+$, when $n \to \infty$, for every $\phi \in \boldsymbol{D}_{k+m}$. Consequently, $(T * \phi) \in (\boldsymbol{CB})_{aap,m}$.

We know that the decomposition of $(T * \phi) = v + w$ is unique as an element of $(\boldsymbol{CB})_{aap,m}$. Therefore w has the asked property.

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