Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae mathematicae publicationes
19 (1993) 204-214

Mathematical Structures - Informatics - Mathematical Modelling
Papers dedicated to Academician L. Iliev on the Occasion of his Eightieth Birthday

A SIMPLE PROOF OF THE BIEBERBACH CONJECTURE

PAVEL G. TODOROV

Dedicated to Academician Ljubomir Iliev on the Occasion of his Eightieth Birthday

ABSTRACT. We give a simple proof of the Bieberbach conjecture which throws a bridge over de Branges' proof [15]-[16] and the Weinstein proof [23] of this conjecture.

1. Introduction. Let S denote the class of all functions

$$f(z)=\sum_{n=1}^{\infty}a_nz^n,\quad a_1=1; \tag{1}$$

analytic and univalent in the unit disc $|z|<1$. Bieberbach [1] conjectured that the inequalities

$$|a_n|\le n,\quad n=2,3,\dots, \tag{2}$$

hold, where the equalities hold only for the Koebe function

$$\mathcal{K}(z)=\frac{z}{(1-z)^2}=\sum_{n=1}^{\infty}nz^n\in S \tag{3}$$

and its rotations $e^{-i\alpha}\mathcal{K}(ze^{i\alpha})\in S$ where α is real.

Bieberbach [1], Löwner [2], Garabedian and Schiffer [3], Pederson and Schiffer [4], and Pederson [5] proved the Bieberbach conjecture (2) for the functions (1) for $n=2,3,4,5,6$, respectively. (Further, Charzyńsky and Schiffer [6], Garabedian, Ross and Schiffer [7] and Ozawa [8] and Gong Sheng [9] gave other proofs for the cases $n=4$, and $n=6$, respectively.)

Let S^2 denote the subclass of odd functions

$$f_2(z)=\sqrt{f(z^2)}=\sum_{n=1}^{\infty}b_{2n-1}z^{2n-1},\quad f(z)\in S,\quad b_1=1, \tag{4}$$

of the class S. From (1) and (4) it follows that

$$a_n = \sum_{k=1}^{n} b_{2k-1} b_{2(n+1-k)-1}, \quad n = 1, 2, \ldots, \quad b_1 = 1. \tag{5}$$

From (5) and the Cauchy inequality it is clear that

$$|a_n| \le \sum_{k=1}^{n} |b_{2k-1}|^2, \quad n = 2, 3, \ldots, \quad b_1 = 1. \tag{6}$$

Because of (6), Robertson [10] conjectured that over the class S^2 the inequalities

$$\sum_{k=1}^{n} |b_{2k-1}|^2 \le n, \quad n = 2, 3, \ldots, \quad b_1 = 1, \tag{7}$$

hold, where the equalities hold only for such functions (4) which correspond to the Koebe function (3) and its rotations. Thus the Robertson conjecture (7) over the class S^2 implies the Bieberbach conjecture (2) over the class S. Robertson [10] and Friedland [11] proved (7) for $n = 2, 3$ and $n = 4$, respectively, and hence, the Bieberbach conjecture (2) for $n = 2, 3, 4$.

The logarithmic coefficients of the function $f(z) \in S$ are generated by the Taylor expansion

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n z^n, \quad |z| < 1, \quad \log 1 = 0. \tag{8}$$

For the coefficients in (4) and (8), Lebedev and Milin [12] found the inequalities

$$\sum_{k=1}^{n+1} |b_{2k-1}|^2 \le (n+1) \exp\left(\frac{1}{4(n+1)} \sum_{k=1}^{n} (k|c_k|^2 - \frac{4}{k})(n - k + 1) \right) \tag{9}$$

for $n = 1, 2, \ldots$. Because of (9), Milin [13] conjectured that over the class S the inequalities

$$\sum_{k=1}^{n} (k|c_k|^2 - \frac{4}{k})(n - k + 1) \le 0, \quad n = 1, 2, \ldots, \tag{10}$$

hold, where the equalities hold only for the Koebe function (3) and its rotations. Thus from (9) and (6) it follows that the Milin conjecture (10) implies the Robertson conjecture (7) and the Bieberbach conjecture (2). Grinshpan [14] proved (10) for $n = 1, 2, 3$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for $n = 2, 3, 4$, respectively.

De Branges [15]–[16] proved the Milin conjecture (10) for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all

$n = 2, 3, \ldots$. Fitzgerald and Pommerenke [17]–[18], Fitzgerald [19] and Korevaar [20] simplified the proof of de Branges. The authors [15]–[20] use the Löwner differential equation [2]

$$\frac{\partial f(z,t)}{\partial t} = z\varphi(z,t)\frac{\partial f(z,t)}{\partial z} \tag{11}$$

for the family of analytic and univalent functions

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad |z| < 1, \quad 0 \le t < +\infty, \tag{12}$$

where $f(z,0) = f(z)$, and the function $\varphi(z,t)$ is analytic in $|z| < 1$ with

$$\mathrm{Re}\varphi(z,t) > 0, \quad |z| < 1, \quad \varphi(0,t) = 1, \quad 0 \le t < +\infty. \tag{13}$$

If the logarithmic coefficients for the functions (12) are generated by the Taylor expansion

$$\log \frac{f(z,t)}{e^t z} = \sum_{n=1}^{\infty} c_n(t) z^n, \quad |z| < 1, \quad 0 \le t < +\infty, \quad \log 1 = 0, \tag{14}$$

where

$$c_1(t) = e^{-t} a_2(t), \quad 0 \le t < +\infty, \quad a_2(0) = a_2, \tag{15}$$

then with the help of the Löwner equation (11), de Branges proved the general inequality

$$\sum_{k=1}^{n} \left(k|c_k(t)|^2 - \frac{4}{k} \right) \sigma_k(t) \le 0, \quad 0 \le t < +\infty, \tag{16}$$

for any positive integer $n \ge 1$, where $\sigma_k(t)$ are the de Branges functions

$$\sigma_k(t) = k \sum_{\nu=0}^{n-k} (-1)^{\nu} \frac{(2k+\nu+1)_{\nu}(2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!} e^{-\nu t - kt} \tag{17}$$

for $0 \le t < +\infty$ and $k = 1, 2, \ldots, n$, where $(a)_{\nu}$ for an arbitrary number a denotes

$$(a)_{\nu} = a(a+1)\ldots(a+\nu-1), \quad \nu = 1, 2, \ldots; \quad (a)_0 = 1. \tag{18}$$

The functions(17) are the unique solution of the de Branges system of differential equations

$$\begin{gathered} \sigma_k(t) - \sigma_{k+1}(t) = -\frac{\sigma'_k(t)}{k} - \frac{\sigma'_{k+1}(t)}{k+1}, \\ 0 \le t < +\infty, \quad k = 1, 2, \ldots, n, \quad \sigma_{n+1}(t) = 0, \end{gathered} \tag{19}$$

with initial conditions

$$\sigma_k(0) = n - k + 1, \quad k = 1, 2, \ldots, n. \tag{20}$$

De Branges proved that the sign of the derivative with respect to t of the left–hand side of (16) is determined by the signs of the derivatives of the functions (17), which have the representation

$$-\frac{\sigma_k'(t)}{k} = e^{-kt}\binom{n+k+1}{n-k}\,{}_3F_2\left(\begin{matrix}-n+k,\ k+\frac{1}{2},\ n+k+2\\ k+\frac{3}{2},\ 2k+1\end{matrix}\,\middle|\, e^{-t}\right) \tag{21}$$

for $0 \le t < +\infty$ and $k = 1, 2, \ldots, n$, where ${}_3F_2$ for arbitrary numbers $\alpha_1, \alpha_2, \alpha_3$ and $\beta_{1,2} \neq 0, -1, -2, \ldots$ denotes the Clausen hypergeometric function (for references, see [22]),

$$ {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2\end{matrix}\,\middle|\, x\right) = \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu(\alpha_2)_\nu(\alpha_3)_\nu}{(\beta_1)_\nu(\beta_2)_\nu}\frac{x^\nu}{\nu!}, \quad |x| < 1, \tag{22}$$

keeping in mind (18). Indeed, for $1 \le n \le 29$, de Branges verified by a computer that all functions ${}_3F_2$ in (21) are positive for $0 < t < +\infty$ (for references, see [15]–[16] and [19]–[20]). For every positive integer $n \ge 1$, de Branges established that the positivity of the functions ${}_3F_2$ in (21) follows from the Askey and Gasper inequality [21]

$$\begin{gathered} {}_3F_2\left(\begin{matrix}-n+k,\ k+\frac{1}{2},\ n+k+2\\ k+\frac{3}{2},\ 2k+1\end{matrix}\,\middle|\, e^{-t}\right) > 0,\\ 0 < t < +\infty, \quad n \ge 1, \quad k = 1, 2, \ldots, n; \end{gathered} \tag{23}$$

the proof of which was simplified by Kazarinoff [22]. Therefore, the left–hand side of (16) is an increasing function with respect to t, and hence, the inequality (16) holds since $\sigma_k(+\infty) = 0$ for $k = 1, 2, \ldots, n$ according to (17). In particular, for $t = 0$, from (16) and(20) it follows that the inequalities (10) hold where

$$c_k = c_k(0), \quad k = 1, 2, \ldots, \tag{24}$$

are the coefficients in (8) for the function $f(z) = f(z, 0)$.

Weinstein [23] proved the Milin conjecture (10) for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all $n = 2, 3, \ldots$, in another way. Indeed, for $|z| < 1$, Weinstein found the identity

$$\sum_{n=1}^{\infty} z^{n+1} \sum_{k=1}^{n} \left(\frac{4}{k} - k|c_k|^2\right)(n - k + 1) = \int_0^{\infty} \frac{e^t w}{1 - w^2}\left(\sum_{k=1}^{\infty} A_k(t) w^k\right) dt, \tag{25}$$

where c_k are the coefficients in (8) (or in (24)), $A_k(t)$ are the functions

$$A_k(t) = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\varphi(re^{i\theta},t). \tag{26}$$
$$\left|2\left(1+\sum_{\nu=1}^{k}\nu c_\nu(t)r^\nu e^{i\nu\theta}\right) - kc_k(t)r^k e^{ik\theta}\right|^2 d\theta \geq 0$$

for $0 \lesssim t < +\infty$, $k = 1, 2, \ldots$, $0 < r < 1$, where φ is the function in (11) and (13) and $c_\nu(t)$ are the coefficients in (14), and $w = w(z,t)$ is the function determined by the equation

$$\frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2}, \quad |z| < 1, \quad 0 \leq t < +\infty. \tag{27}$$

With the help of the theory of Legendre polynomials and the associated Legendre functions, Weinstein showed that in the Taylor expansion

$$\frac{e^t w^{k+1}}{1-w^2} = \sum_{n=0}^{\infty} \Lambda_k^n(t) z^{n+1}, \quad k = 1, 2, \ldots, \tag{28}$$

for $|z| < 1$ and $0 \leq t < +\infty$, all coefficients $\Lambda_k^n(t) \geq 0$, which, according to (25)–(26), proves the Milin conjecture (10) for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all $n = 2, 3, \ldots$.

We shall note that the function w, determined by (27), can be found in Pick [24], Schiffer and Tammi [25], Jakubowski, Zielińska and Zyskowska [26] and de Branges [16]. In Goodman [27], w is known as the bounded Koebe function. Indeed, if $\mathcal{K}^{-1}(z)$ is the inverse function of the Koebe function (3), then the function w has the explicit form

$$w \doteq w(z,t) = \mathcal{K}^{-1}\left(\frac{e^{-t}z}{(1-z)^2}\right) = \tag{29}$$
$$= \frac{2z + e^t(1-z)^2 - (1-z)\sqrt{e^t[4z + e^t(1-z)^2]}}{2z}, \sqrt{e^{2t}} = e^t,$$

and it maps the disc $|z| < 1$ onto the disc $|w| < 1$ except for a slit along the negative real axis from -1 to $\mathcal{K}^{-1}(-e^{-t}/4) = 1 - 2e^t + 2\sqrt{e^t(e^t-1)}$, where

$$|w(z,t)| \leq |z|, \quad |z| < 1, \quad 0 \leq t < +\infty, \tag{30}$$

where, for $0 < t < +\infty$, the equality holds only for $z = 0$.

The explicit form of the coefficients $\Lambda_k^n(t)$ in (28) is unknown. Now we shall give a method with the help of which we discover that the Taylor expansion (28) generates the derivatives (21) of the de Branges functions (17). This fact and the Askey and Gasper inequality (23), applied to the Weinstein identity (25)–(26), prove immediately the Milin conjecture (10) for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all $n = 2, 3, \ldots$. Thus we show that in the final stage the de Branges proof [15]–[16] and the Weinstein proof [23] of the Milin conjecture (10) for all $n = 1, 2, \ldots$ are one and the same.

2. An explicit form of the Taylor expansion of the function (28) in the powers of z. Our discovery is realized by the following

Theorem 1. *If $w = w(z,t), |z| < 1,\ 0 \le t < +\infty$, is the bounded Koebe function, determined by the equation (27) (or (29)), then for $|z| < 1$ and $0 \le t < +\infty$, the Taylor expansion (28) has the explicit form*

$$\frac{e^t w^{k+1}}{1-w^2} = -\sum_{n=k}^{\infty} \frac{\sigma'_{nk}(t)}{k} z^{n+1}, \quad k = 1, 2, \ldots, \tag{31}$$

with the full notations $\sigma_{nk}(t) \equiv \sigma_k(t),\ n \ge k,\ k = 1, 2, \ldots,\ 0 \le t < +\infty$, of the de Branges functions (17) and $\sigma'_{nk}(t) \equiv \sigma'_k(t)$, $n \ge k$, $k = 1, 2, \ldots$, $0 \le t < +\infty$, of their derivatives (21), respectively.

Proof. Let z be an arbitrary fixed point in the disc $|z| < 1$ and r be a fixed real number such that $|z| < r < 1$. Then, on the basis of inequality (30) for the function w determined by equation (27), by the Cauchy theorem of residues, we shall have the integral representation formula

$$\frac{w^{k+1}}{1-w^2} = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta^{k+1}}{1-\zeta^2} \frac{\dfrac{e^t(1+\zeta)}{(1-\zeta)^3}}{\dfrac{e^t\zeta}{(1-\zeta)^2} - \dfrac{z}{(1-z)^2}} d\zeta \tag{32}$$

for $0 \le t < +\infty$ and $k = 1, 2, \ldots$, where the integration along the circle $|\zeta| = r$ is performed in positive direction. From (32) we obtain the Taylor expansion as follows

$$\begin{aligned}
\frac{w^{k+1}}{1-w^2} &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta^k}{(1-\zeta)^2} \frac{d\zeta}{1 - \dfrac{(1-\zeta)^2}{e^t\zeta}\dfrac{z}{(1-z)^2}} = \\
&= \sum_{\nu=k+1}^{\infty} \frac{z^\nu}{(1-z)^{2\nu}} \frac{e^{-\nu t}}{2\pi i} \int_{|\zeta|=r} \frac{(1-\zeta)^{2\nu-2}}{\zeta^{\nu-k}} d\zeta = \\
&= \sum_{\nu=k+1}^{\infty} (-1)^{\nu-k-1} \binom{2\nu-2}{\nu-k-1} e^{-\nu t} \sum_{n=\nu}^{\infty} \binom{n+\nu-1}{n-\nu} z^n = \\
&= e^{-(k+1)t} \sum_{n=k}^{\infty} z^{n+1} \sum_{\nu=0}^{n-k} \lambda_\nu(n,k) e^{-\nu t}
\end{aligned} \tag{33}$$

for $|z| < 1$, $0 \le t < +\infty$, $k = 1, 2, \ldots$, where

$$\lambda_\nu(n,k) = (-1)^\nu \binom{2k+2\nu}{\nu}\binom{n+k+\nu+1}{n-k-\nu} \tag{34}$$

for $0 \le \nu \le n-k$, $n \ge k$, $k = 1, 2, \ldots$. From (34) it follows that

$$\frac{\lambda_\nu(n,k)}{\lambda_{\nu-1}(n,k)} = \frac{(-n+k+\nu-1)(k-\frac{1}{2}+\nu)(n+k+\nu+1)}{(k+\frac{1}{2}+\nu)(2k+\nu)\nu} \tag{35}$$

for $1 \le \nu \le n-k$, $n \ge k+1$, $k = 1, 2, \ldots$ with

$$\lambda_0(n,k) = \binom{n+k+1}{n-k}, \quad n \ge k, \quad k = 1, 2, \ldots. \tag{36}$$

From (35) and (36) we deduce that

$$\lambda_\nu(n,k) = \binom{n+k+1}{n-k}\frac{(-n+k)_\nu(k+\frac{1}{2})_\nu(n+k+2)_\nu}{(k+\frac{3}{2})_\nu(2k+1)_\nu\,\nu!} \tag{37}$$

for $0 \le \nu \le n-k$, $n \ge k$, $k = 1, 2, \ldots$, having in mind the notation (18). With the help of (37), having in mind (22), the expansion (33) takes the form

$$\frac{e^t w^{k+1}}{1-w^2} = \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1}, \quad |z| < 1, \quad 0 \le t < +\infty, \quad k = 1, 2, \ldots, \tag{38}$$

where

$$\Lambda_k^n(t) = e^{-kt}\binom{n+k+1}{n-k}\,{}_3F_2\left(\begin{matrix} -n+k,\ k+\frac{1}{2},\ n+k+2 \\ k+\frac{3}{2},\ 2k+1 \end{matrix}\,\middle|\, e^{-t}\right) \tag{39}$$

for $0 \le t < +\infty$, $n \ge k$, $k = 1, 2, \ldots$. (Compare with (28), where the coefficients $\Lambda_k^n(t)$ are not determined explicitly.) Now, according to the Askey and Gasper inequality (23), from (39) it follows that the inequalities

$$\Lambda_k^n(t) > 0, \quad 0 < t < +\infty, \quad n \ge k, \quad k = 1, 2, \ldots, \tag{40}$$

hold. Therefore, from (40), (38) and the Weinstein identity (25)–(26) we obtain immediately that the Milin conjecture (10) is true for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) are true for all $n = 2, 3, \ldots$. The comparison of our formula (39) and the de Branges formula (21) with the full notation $\sigma'_{nk}(t) \equiv \sigma'_k(t)$ for $n \ge k$, $k = 1, 2, \ldots$, $0 \le t < +\infty$, yields the identity

$$\Lambda_k^n(t) = -\frac{\sigma'_{nk}(t)}{k}, \quad 0 \le t < +\infty, \quad n \ge k, \quad k = 1, 2, \ldots, \tag{41}$$

which throws a bridge over the de Branges proof [15]–[16] and the Weinstein proof [23] of the Milin conjecture (10) for all $n = 1, 2, \ldots$. Finally, from (41) and (38) we obtain (31).

This completes the proof of Theorem 1.

In particular, for $t = 0$, from (27) it follows that $w = w(z, 0) = z$, and (31) is reduced to

$$\frac{z^{k+1}}{1-z^2} = -\sum_{n=0}^{\infty} \frac{\sigma'_{n+k,k}(0)}{k} z^{n+k+1}, \quad |z| < 1, \quad k = 1, 2, \ldots. \tag{42}$$

On the other hand, we have the expansion

$$\frac{z^{k+1}}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+k+1}, \quad |z| < 1, \quad k = 1, 2, \ldots. \tag{43}$$

The comparison of (42) and (43) yields immediately the relations

$$\sigma'_{nk}(0) = -k, \quad n \geq k, \quad k \geq 1, \quad n - k = 2m, \quad m = 0, 1, 2, \ldots, \tag{44}$$

and

$$\sigma'_{nk}(0) = 0, \quad n \geq k, \quad k \geq 1, \quad n - k = 2m + 1, \quad m = 0, 1, 2, \ldots, \tag{45}$$

which Fitzgerald and Pommerenke [17]–[18] obtained with the help of the Askey and Gasper special Jacobi polynomials theory [21]. According to (44)–(45) and (21), the left–hand side of the Askey and Gasper inequality (23), for $t = 0$, is equal to $\binom{n+k+1}{n-k}^{-1}$ if $n - k$ is even, and to 0 if $n - k$ is odd. From (44)–(45) and (19) it follows that

$$\sigma_{nk}(0) - \sigma_{n,k+1}(0) = 1, \quad n \geq k, \quad k \geq 1, \quad \sigma_{n,n+1}(0) = 0, \tag{46}$$

where we have used the full notations of the de Branges functions and their derivatives defined in Theorem 1. By an indication on k, the relation (46) leads us to the relation

$$\sigma_{nk}(0) = n - k + 1, \quad n \geq k, \quad k = 1, 2, \ldots. \tag{47}$$

The relation (47) expresses the initial conditions (20) for the de Branges functions (17) in full notations.

Theorem 2. *The Milin functional on the left–hand side of* (10) *has the following integral representation*

$$\sum_{k=1}^{n} \left(k|c_k|^2 - \frac{4}{k} \right) (n - k + 1) = \int_0^{+\infty} dt \sum_{k=1}^{n} A_k(t) \frac{\sigma'_{nk}(t)}{k} \tag{48}$$

for $n = 1, 2, \ldots$, where c_k are the coefficients in (8) (or in (24)), $A_k(t)$ are the Weinstein functions (26) and $\sigma'_{nk}(t)$ are the derivatives (21) of de Branges functions (17) in full notations defined in Theorem 1.

Proof. We shall find the Taylor expansion of the right–hand side of the Weinstein identity (25) in the power of z with the help of our Taylor expansion (31). Thus we obtain the Taylor expansion

(49)
$$\sum_{k=1}^{\infty} A_k(t) \frac{e^t w^{k+1}}{1-w^2} = -\sum_{k=1}^{\infty} A_k(t) \sum_{n=k}^{\infty} \frac{\sigma'_{nk}(t)}{k} z^{n+1} =$$
$$= -\sum_{n=1}^{\infty} z^{n+1} \sum_{k=1}^{n} A_k(t) \frac{\sigma'_{nk}(t)}{k}, \quad |z| < 1, \quad 0 \le t < +\infty.$$

Now with the help of (49) we obtain the Taylor expansion

$$\int_0^{\infty} \frac{e^t w}{1-w^2} \left(\sum_{k=1}^{\infty} A_k(t) w^k \right) dt = -\sum_{n=1}^{\infty} z^{n+1} \int_0^{\infty} dt \sum_{k=1}^{\infty} A_k(t) \frac{\sigma'_{nk}(t)}{k} \tag{50}$$

for $|z| < 1$. Finally, the comparison of the right–hand side of (50) and the left–hand side of (25) yields (48).

This completes the proof of Theorem 2.

With the help of our identity (48), the de Branges formula (21), the Askey and Gasper inequality (23) and the Weinstein functions (26), the Milin conjecture (10) becomes evidently true for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) become evidently true for all $n = 2, 3, \ldots$.

The case of equality in (10) follows directly from (48) because the integrand must be identically zero, i.e. $A_k(t) \equiv 0$, $0 \le t < +\infty$, $1 \le k \le n$ $(n \ge 1)$. From (26) and (24) it follows that

$$kc_k e^{ik\theta} = -2 - 2\sum_{\nu=0}^{k-1} \nu c_\nu e^{i\nu\theta}, \quad 1 \le k \le n, \quad n \ge 1, \quad c_0 = 0, \tag{51}$$

where θ is real. For $k \ge 2$, the summation in (51) is taken over $1 \le \nu \le k-1$. In particular, for $n = 1$ from (51) it follows that $a_2 = -2e^{-i\theta}$, having in mind (24) and (15). Bieberbach [1] proved that this is possible only for the Koebe function (3) and its rotations. The general case leads us to the same result. Indeed, for any positive integer $n \ge 1$, we obtain, by an induction on k, that the solution of the system (51) is given by the logarithmic coefficients $c_k = (-1)^k 2e^{-ik\theta}/k$, $1 \le k \le n$, of the Koebe function $f(z) = z/(1 + ze^{-i\theta})^2$. Hence, the cases of equality in the Robertson conjecture (7) and the Bieberbach conjecture (2) for any positive integer $n \ge 2$ are also attained only by the Koebe function (3) and its rotations.

Theorem 1 and Theorem 2 have been reported at the March 1993 Meeting 879 (-30-10) of the American Mathematical Society in Knoxville, Tennessee.

REFERENCES

[1] L. BIEBERBACH, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.* **138** (1916), 940-955.

[2] K. LÖWNER, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, *Math. Ann.* **89** (1923), 103-121.

[3] P. R. GARABEDIAN and M. SCHIFFER, A proof of the Bieberbach conjecture for the fourth coefficient, *J. Rational Mech. Anal.* **4** (1955), 427-465.

[4] R. N. PEDERSON and M. SCHIFFER, A proof of the Bieberbach conjecture for the fifth coefficient, *Arch. Rational Mech. Anal.* **45** (1972), 161-193.

[5] R. N. PEDERSON, A proof of the Bieberbach conjecture for the sixth coefficient, *Arch. Rational Mech. Anal.* **31** (1968), 331-351.

[6] Z. CHARZYŃSKI and M. SCHIFFER, A new proof of the Bieberbach conjecture for the fourth coefficient, *Arch. Rational Mech. Anal.* **5** (1960), 187-193.

[7] P. R. GARABEDIAN, G. G. ROSS and M. SHIFFER, On the Bieberbach conjecture for even n, *J. Math. Mech.* **14** (1965), 975-989.

[8] M. OZAWA, An elementary proof of the Bieberbach conjecture for sixth coefficient, *Kodai Math. Semin. Rep.* **21** (1969), 129-132.

[9] G. SHENG, A simple proof of the Bieberbach conjecture for sixth coefficient, *Scientia Sinica* **23** (1980) No. 1, 1-15.

[10] M. S. ROBERTSON, A remark on the odd schlicht functions, *Bull. Amer. Math. Soc.* **42** (1936), 366-370.

[11] S. FRIEDLAND, On a conjecture of Robertson, *Arch. Rational Mech. Anal.* **37** (1970), 255-261.

[12] I. M. MILIN, On the coefficients of univalent functions, *Dokl. Akad. Nauk USSR* **176** (1967), 1015-1018 (in Russian); English transl. *Soviet Math. Dokl.* **8** (1967), 1255-1258.

[13] I. M. MILIN, Univalent functions and orthonormal systems, Izd. "Nauka", Moscow, 1971 (in Russian); English transl. *Transl. Math. Monographs*, **49** (1977) Amer. Math. Soc., Providence, R. I.

[14] A. Ž. GRINSHPAN, Logarithmic coefficients of functions of Class S, *Sibirsk. Mat. Ž.* **13** (1972), 1145-1157 (in Russian); English transl. *Siberian Math. J.*, **13** (1972), 793-801.

[15] L. DE BRANGES, A proof of the Bieberbach conjecture, Preprint E-5-84, Leningrad Branch of the V. A. Steklov Mathematical Institute, 1984, 1-21.

[16] L. DE BRANGES, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), 137-152.

[17] C. H. FITZGERALD and CH. POMMERENKE, The de Branges theorem of univalent functions, *Trans. Amer. Math. Soc.* **290** (1985), 683-690.

[18] C. H. FITZGERALD and CH. POMMERENKE, The de Branges Theorem of univalent functions, *Serdica, Bulgaricae mathematicae publicationes* **13** (1987) No. 1, 21-25 (in Russian).

[19] C. H. FITZGERALD, The Bieberbach conjecture: Retrospective, Notices, *Amer. Math. Soc.* **32** (1985), 2-6.

[20] J. KOREVAAR, Ludwig Bieberbach's conjecture and its proof by Louis de Branges, *Amer. Math. Monthly* **93** (1986), 505-514.

[21] R. ASKEY and G. GASPER, Positive Jacoby polynomial sums, II, *Amer. J. Math.* **98** (1976), 709-737.

[22] N. D. KAZARINOFF, Special functions and the Bieberbach conjecture, *Amer. Math. Monthly* **95** (1988), 689-696.

[23] L. WEINSTEIN, The Bieberbach conjecture, *International Mathematics Research Notices* **5** (1991) 61-64.

[24] G. PICK, Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiet, *Sitzungsber. Akad. Wiss. Wien, Abt. IIa*, **126** (1917), 247-263.

[25] M. SHIFFER and O. TAMMI, The fourth coefficient of a bounded real univalent function, *Ann. Acad. Sci. Fenn., Ser. A, I. Math.* No. **354** (1965), 1-32.

[26] Z. J. JAKUBOWSKI, A. ZIELIŃSKA and K. ZYSKOWSKA, Sharp estimation of even coefficients of bounded symmetric univalent functions, *Ann. Polon. Math.* **40** (1983), 193-206.

[27] A. W. GOODMAN, Univalent Functions, Vols. I, II, Mariner Publishing Company, Tampa, Florida, 1983.

Pavel Georgiev Todorov
20 Maritsa Avenue
4002 Plovdiv
BULGARIA