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ON THE STRICTLY PARALLEL PLANES OF THE SPACES WITH LINEAR CONNECTION

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ABSTRACT. In this paper are proved two theorems. The first theorem generalizes one Wong's result. Namely, it is proved that if M_n is a simply-connected differentiable manifold with linear connection which admits a parallel r-plane and if the corresponding double tensor vanishes identically, then that r-plane is strictly parallel. The second theorem gives a necessary and sufficient conditions for a paracompact differentiable manifold to admit a linear connection such that given r linearly independent vector fields $X_{(1)},\dots,X_{(r)}$ and s linearly independent 1-forms $w^{(1)},\dots,w^{(s)}$ are parallel. These conditions are given by (3). Three corollaries are given at the end.

Suppose that M_n is an n-dimensional differentiable manifold with linear connection. An r-plane Σ in M_n (i.e. a field of r-planes in M_n) is said to be parallel if, for any two points A and B a vector in $\Sigma(A)$ is displaced into a vector in $\Sigma(B)$ by the parallel transport along any curve from A to B. An r-plane Σ in M_n is said to be strictly parallel if the r-plane Σ has a basis of r parallel differentiable vector fields.

A necessary and sufficient condition for an r-plane to be parallel is that one, and therefore everyone, of its bases $\{\lambda^i_{(\alpha)}\}$ ($\alpha \in \{1,\dots,r\}$) satisfies the recurrent relations of the form

$$\lambda^i_{(\alpha);k} = A^\beta_{\alpha k}\lambda^i_{(\beta)} \tag{1}$$

where $A^\beta_{\alpha k}$ ($\alpha,\beta \in \{1,\dots,r\}$) are components of a covariant vector for fixed α and β ([1],[2]). The components

$$A^\beta_{\alpha kl} = \partial A^\beta_{\alpha k}/\partial x^l - \partial A^\beta_{\alpha l}/\partial x^k + A^\delta_{\alpha k}A^\beta_{\delta l} - A^\delta_{\alpha l}A^\beta_{\delta k} \tag{2}$$

$(\alpha, \beta, \delta \in \{1, \ldots, r\},\ k, l \in \{1, \ldots, n\})$ transform as a tensor skew-symmetric in k and l, if α and β are fixed. Furthermore, for fixed k and l, $A^{\beta}_{\alpha kl}$ has tensor character with respect to α and β under changes of basis ([2]). So $A^{\beta}_{\alpha kl}$ is called a double tensor.

Wong ([2], Theorem 6.1) has proved locally that a parallel plane in M_n is strictly parallel if and only if the double tensor associated with it vanishes identically. Now we shall give a similar theorem in the global case.

Theorem 1. *Assume that M_n is simply-connected differentiable manifold with linear connection and assume that M_n admits a parallel r-plane Σ. If the double tensor associated with Σ vanishes identically, then Σ is a strictly parallel plane.*

Proof. Suppose that M_n is a connected topological space. We choose a point $x \in M_n$ and r linearly independent vectors $X_{(1)}, \ldots, X_{(r)} \in \Sigma(x)$. For an arbitrary point $y \in M_n$ there exists a path $z(t)$ which connects x and y. The vectors $X_{(1)}, \ldots, X_{(r)}$ can be parallel transported along the path $z(t)$ to the point y. We will prove that the transported vectors at the point y do not depend on the choice of the path $z(t)$. It is sufficient to prove that if $z(t)$ is a closed path, i.e. $y = x$, then each of the vectors $X_{(1)}, \ldots, X_{(r)}$ transports at the same vector.

For an arbitrary point w there exists a coordinate neighbourhood U_w and r parallel linearly independent vector fields on U_w which form a basis for Σ reduced on U_w. It follows from the above Wong's theorem. Now we have an open cover $\mathcal{U}$ for M_n. From the Lemma of factorization ([3]) an arbitrary closed path which is null-homotopic in M_n is equivalent to a composition of finite number of $\mathcal{U}$-lassos. Since M_n is simply-connected manifold, the closed path $z(t)$ is null-homotopic and the Lemma of factorization can be applied. So it is sufficient to prove our statement that $X_{(1)}, \ldots, X_{(r)}$ are independent of the parallel transport in case the path $z(t)$ is a $\mathcal{U}$-lasso. Now it is sufficient to prove that each vector $X \in \Sigma(w)$ transports at the same vector, if the path $z(t)$ is contained in a set $U_w \in \mathcal{U}$. It is satisfied because if $Y_{(1)}, \ldots, Y_{(r)}$ are parallel vector fields on U_w, then

$$c_1 Y_{(1)} + \ldots + c_r Y_{(r)}$$

is a parallel vector field where $c_1, \ldots, c_r$ are constants.

We have constructed r vector fields on M_n. It is obvious that they are parallel, differentiable and they are linearly independent vector fields, i.e. Σ is a strictly parallel plane. If the manifold M_n is not connected, then the above statement can be applied to each component of the connection. □

Theorem 2. *Assume that M_n is a paracompact differentiable manifold and assume that $X_{(1)}, \ldots, X_{(r)}$ are linearly independent vector fields on M_n and $w^{(1)}, \ldots, w^{(s)}$ are linearly independent 1-forms on M_n. Then there exists a linear connection on M_n*

such that $X_{(p)}$ and $w^{(q)}(p \in \{1,\dots,r\},\ q \in \{1,\dots,s\})$ are parallel vector fields and 1-forms if and only if for each $p \in \{1,\dots,r\}$ and for each $q \in \{1,\dots,s\}$

$$w^{(q)}(X_{(p)}) = C_p^q = \text{const.} \tag{3}$$

Proof. Let us suppose that $X_{(p)}$ has components $\lambda^i_{(p)}$ and $w^{(q)}$ has components $\mu_j^{(q)}$ with respect to a local coordinate system. Then (3) is equivalent to

$$\lambda^i_{(p)}\mu_i^{(q)} = C_p^q = \text{const.} \tag{4}$$

If there exists a linear connection such that $X_{(p)}$ and $w^{(q)}$ are parallel ($p \in \{1,\dots,r\}$, $q \in \{1,\dots,s\}$), then it is obvious that (4) is satisfied.

Inversely, let us suppose that the matrix $\left[\lambda^i_{(p)}\mu_i^{(q)}\right]$ has constant components and its rank is q. It is well-known from the matrix theory that there exist invertible matrices $[c^p_\alpha]_{r\times r}$ and $\left[d^\beta_q\right]_{s\times s}$ with constant components such that

$$\left[c^p_\alpha \lambda^i_{(p)} d^\beta_q \mu_i^{(q)}\right] = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\lambda^i_{(p)}\mu_i^{(q)}$ do not depend on the coordinate system, then it follows that $[c^p_\alpha]$ and $\left[d^\beta_q\right]$ do not depend on the coordinate system either.

We can define new vector fields $\lambda'^i_{(\alpha)}$ and 1-forms $\mu_i'^{(\alpha)}$ by $\lambda'^i_{(\gamma)} = c^p_\gamma \lambda^i_{(p)}$ for $\gamma \in \{1,\dots,r\}$, and $\mu_j'^{(\alpha)} = d^\alpha_\beta \mu_j^{(\beta)}$ for $\alpha \in \{1,\dots,q\}$ and $\mu_j'^{(\beta)} = d_\gamma^{\beta-(n-s)}\mu_j^{(\gamma)}$ for $\beta \in \{n-s+q+1,\dots,n\}$. Since the vector fields are linearly independent and also for the 1-forms, it can be verified that $r \le n-s+q$. The new vector fields and 1-forms satisfy

$$w'^{(\alpha)}(X'_{\beta)}) = \delta^\alpha_\beta \tag{5}$$

($\beta \in \{1,\dots,r\}$, $\alpha \in \{1,\dots,q,n-s+q+1,\dots,n\}$).

Since M_n is a paracompact manifold, then there exists a locally finite open covering of coordinate neighbourhoods $\{U_i\}$ such that the sets $\bar{U}_i$ are compact. Then ([3]) there exist functions $\{f_i\}$ such that

1) $0 \le f_i \le 1$,

2) $\operatorname{supp}(f_i) \subseteq U_i$,

3) $\sum_i f_i(x) = 1$.

In an arbitrary coordinate neighbourhood U_i can be chosen $n-r$ vector fields $X'_{(\gamma)}$ $(\gamma \in \{r+1, \dots, n\})$ such that

$$w'^{(\alpha)}(X'_{(k)}) = \delta_k^\alpha \tag{6}$$

where $k \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, q, n-s+q+1, \dots, n\}$, and $X'_{(1)}, \dots, X'_{(n)}$ are linearly independent vector fields. Then $X'_{(1)}, \dots, X'_{(n)}$ generate a flat linear connection $\Gamma_{(i)}$ on U_i ([4]) such that $X'_{(1)}, \dots, X'_{(n)}$ are parallel vector fields on U_i. Since $X'_{(1)}, \dots, X'_{(n)}$ are linearly independent, it follows from (6) that $w'^{(1)}, \dots, w'^{(q)}, w'^{(n-s+q+1)}, \dots, w'^{(n)}$ are parallel 1-forms. Let us define a connection Γ on M_n by

$$\Gamma^j_{kl} = \sum_i f_i \Gamma^j_{kl(i)}.$$

Then using that $\nabla_Y K = \sum_i f_i (\nabla_Y)_i K$ for an arbitrary tensor field K, where ∇_Y and $(\nabla_Y)_i$ are covariant derivatives with respect to the connections Γ and $\Gamma_{(i)}$ respectively, we obtain that $X'_{(1)}, \dots, X'_{(r)}, w'^{(1)}, \dots, w'^{(q)}, w'^{(n-s+q+1)}, \dots, w'^{(n)}$ are parallel vector fields and 1-forms with respect to the connection Γ. So $X_{(1)}, \dots, X_{(r)}$ and $w^{(1)}, \dots, w^{(s)}$ are parallel vector fields and 1-forms with respect to the connection Γ. □

In particular if $s = 0$ or $r = 0$ we can obtain the following two corollaries:

Corollary 1. *Assume that M_n is a paracompact differentiable manifold and assume that $X_{(1)}, \dots, X_{(r)}$ are linearly independent vector fields on M_n. There exists a linear connection on M_n such that $X_{(1)}, \dots, X_{(r)}$ are parallel vector fields.*

Corollary 2. *Assume that M_n is a paracompact differentiable manifold and assume that $w^{(1)}, \dots, w^{(s)}$ are linearly independent 1-forms on M_n. There exists a linear connection on M_n such that $w^{(1)}, \dots, w^{(s)}$ are parallel 1-forms.*

Suppose that $n + 1 = 2^{4\alpha+\beta}(2s+1)$ where α and s are non-negative integer numbers and $\beta \in \{0,1,2,3\}$. Adams ([5]) has proved that the maximal number of linearly independent vector fields on S^n is equal to $k(n) = 2^\beta + 8\alpha - 1$. Using this result and Corollary 1 we obtain

Corollary 3. *S^n admits a linear connection and strictly parallel $k(n)$-plane, but it does not admit parallel $k(n)+1$-plane for each linear connection on S^n.*

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