Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae
mathematicae publicationes
19 (1993) 267-272

ON K-CONTACT MANIFOLDS

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ABSTRACT. A type of K-contact manifold with characteristic vector field ξ belonging to the k-nullity distribution satisfying the condition $R(\xi, Y).C = 0$ is investigated, where R is the curvature transformation and C is the conformal curvature tensor.

1. Introduction. In this paper we consider a K-contact manifold M^{2m+1} with characteristic vector field ξ belonging to the k-nullity distribution. In a recent paper [1] M. C. Chaki and M.Tarafdar proved that if in a Sasakian manifold M^n $(n > 3)$ the relation $R(X,Y).C = 0$ holds, where $R(X,Y)$ is considered as a derivation of tensor algebra at each point of the manifold for tangent vectors and C is the conformal curvature tensor, then the manifold is locally isometric with a unit sphere $S^n(1)$. In this paper we have generalised the rezult of Chaki and Tarafdar by taking the weaker hypothesis $R(\xi, Y).C = 0$ instead of $R(X,Y).C = 0$ in a K-contact manifold.

2. K-contact manifolds. A $(2m+1)$-dimensional C^∞ manifold M^{2m+1} is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$. For a given contact form η it is well known that there exists a unique vector field ξ (called the characteristic vector field) on M such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A Riemannian metric g is said to be an associated metric if there exists a tensor field Φ of type (1,1) such that $d\eta(X,Y) = g(X, \Phi Y), \eta(X) = g(X, \xi)$ and $\Phi^2 = -I + \eta \otimes \xi$. The structure (Φ, ξ, η, g) on M^{2m+1} is called a contact metric structure and M^{2m+1} is called a contact metric manifold [2].

Given a contact metric structure (Φ, ξ, η, g) we define a tensor field h by $h = \frac{1}{2}(\mathcal{L}_\xi \Phi)$ where $\mathcal{L}$ denotes the Lie differentiation. h is a symmetric operator which anticommutes with Φ and hence if λ is an eigenvalue of h with eigenvector x, then $-\lambda$ is also an eigenvalue of the eigenvector Φx. Clearly $h\xi = 0$ and it is easy to see that ξ is a killing vector field with respect to g if $h = 0$.

A contact metric manifold for which ξ is a killing vector field is called a K-contact manifold [2], [3]. A K-contact Riemannian manifold is called Sasakian [2] if

$$(\nabla_X\Phi)(Y) = g(X,Y)\xi - \eta(Y)X \tag{2.1}$$

hold, where the operator of covariant differentiation with respect to g is denoted by ∇.

The k-nullity distribution [4] of a Riemannian manifold for a real number k is a ditribution

$$\begin{aligned} &N(k) : x \to N_x(k) = \\ &= \{z \in T_xM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y), X,Y \in T_xM\}. \end{aligned} \tag{2.2}$$

Thus if ξ belongs to the k-nullity distribution, then we get

$$R(X,Y)\xi = k(g(Y,\xi)X - g(X,\xi)Y) = k[\eta(Y)X - \eta(X)Y] \tag{2.3}$$

From (2.2) it is clear that if $k = 1$, then the manifold becomes a Sasakian one.

A Sasakian manifold is K-contact but the converse is not true in general. However a 3-dimensional K-contact manifold is Sasakian.

3. Preliminaries. In a K-contact Rimannian manifold the following relations hold: [3], [5]

$$\Phi(\xi) = 0, \tag{3.1}$$

$$\eta(\xi) = 1, \tag{3.2}$$

$$\Phi^2 x = -x + \eta(X)\xi, \tag{3.3}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{3.4}$$

$$g(X,\xi) = \eta(X), \tag{3.5}$$

$$\nabla_X\xi = -\Phi X, \tag{3.6}$$

$$s(X,\xi) = (n-1)\eta(X), \tag{3.7}$$

$$g(R(\xi,X)Y,\xi) = g(X,Y) - \eta(X)\eta(Y), \tag{3.8}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi, \tag{3.9}$$

$$(\nabla_X\Phi)(Y) = R(\xi,X)Y. \tag{3.10}$$

The above formulas will be used in the next section.

4. K-contact manifold with the characteristic vector field ξ belonging to the k-nullity distribution.

If ξ belongs to the k-nullity distribution, then

$$R(X,Y)\xi = k[g(Y,\xi)X - g(X,\xi)Y] = k[\eta(Y)X - \eta(X)Y]. \tag{4.1}$$

Putting $X = \xi$ in (4.1) by using (3.2) we get

$$R(\xi,Y)\xi = k[\eta(Y)\xi - Y]. \tag{4.2}$$

If possible, let us suppose that $k = 0$. Then form (4.2), (3.9) and (3.3) we get

$$\Phi^2 X = 0, \tag{4.3}$$

which is a contardiction. Thus we can state the following Theorem:

Theorem 1. *In a K-contact manifold the real number k for the k-nullity distribution cannot be zero.*

5. K-contact manifold with $R(\xi,X).C = 0$.

We have for the conformal curvature tensor C

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y]$$

$$+\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \tag{5.1}$$

where Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S [6], i.e.,

$$g(QX,Y) = S(X,Y). \tag{5.2}$$

Hence

$$\begin{aligned} \eta(C(X,Y)Z) &= g(C(X,Y)Z,\xi) = \eta(R(X,Y)Z) \\ &+[\frac{r}{(n-1)(n-2)} - \frac{n-1}{n-2}][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &-\frac{1}{n-2}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]. \end{aligned} \tag{5.3}$$

Putting $Z = \xi$ in (5.3) we get

$$\eta(C(X,Y)\xi = 0. \tag{5.4}$$

Again putting $X = \xi$ in (5.3) we get

$$\begin{aligned} \eta(C(\xi,Y)Z) &= (\frac{r}{n-1} - 1)\frac{1}{n-2}[g(Y,Z) - \eta(Y)\eta(Z)] \\ &- \frac{1}{n-2}[S(Y,Z) - (n-1)\eta(Y)\eta(Z)]. \end{aligned} \tag{5.5}$$

Now

$$\begin{aligned} (R(\xi,X).C)(U,V)W &= R(\xi,X)C(U,V)W - C(R(\xi,X)U,V)W \\ &- C(U,R(\xi,X)V)W - C(U,V)R(\xi,X)W. \end{aligned} \tag{5.6}$$

In virtue of $R(\xi,X).C = 0$ we have from (5.6)

$$R(\xi,X)C(U,V)W - C(R(\xi,X)U,V)W - C(U,R(\xi,X)V)W - C(U,V)R(\xi,X)W = 0$$

or,

$$\begin{aligned} & g(R(\xi,X)C(U,V)W,\xi) - g(C(R(\xi,X)U,V)W,\xi) \\ & -g(C(U,R(\xi,X)V)W,\xi) - g(C(U,V)R(\xi,X)W,\xi) = 0. \end{aligned} \tag{5.7}$$

Taking into account the fact that the characteristic vector field ξ belongs to the k-nullity distribution we obtain from (5.7) that

$$\begin{aligned} & k[C(U,V,W,X) - \eta(X)\eta(C(U,V)W) + \eta(U)\eta(C(X,V)W) \\ & +\eta(V)\eta(C(U,X)W) + \eta(W)\eta(C(U,V)X) - g(X,U)\eta(C(\xi,V)W) \\ & -g(X,V)\eta(C(U,\xi)W) - g(X,W)\eta(C(U,V)\xi)] = 0, \end{aligned} \tag{5.8}$$

where $C(U,V,W,X) = g(C(U,V)W,X)$ and $R(U,V,W,X) = g(R(U,V)W,X)$.

Putting $X = U$ in (5.8) we get

$$\begin{aligned} & k[C(U,V,W,U) + \eta(V)\eta(C(U,U)W) + \eta(W)\eta(C(U,V)U) \\ & -g(U,U)\eta(C(\xi,V)W) - g(U,V)\eta(C(U,\xi)W + g(U,W)\eta(C(U,V),\xi)] = 0. \end{aligned} \tag{5.9}$$

Using (5.4) we get from (5.9)

$$\begin{aligned} & k[C(U,V,W,U) + \eta(V)\eta(C(U,U)W) + \eta(W)\eta(C(U,V)U) \\ & -g(U,U)\eta(C(\xi,V)W) - g(U,V)\eta(C(U,\xi)W] = 0. \end{aligned} \tag{5.10}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at each point. Then the sum for $1 \leq i \leq n$ of the relation (5.10) for $U = e_i$ gives

$$k(n-1)\eta(C(\xi, V)W) = 0 \tag{5.11}$$

Since $n \neq 1$ either $k = 0$ or $\eta(C(\xi, V)W) = 0$.
But in virtue of Theorem 1 we must have

$$\eta(C(\xi, V)W) = 0. \tag{5.12}$$

Thereofore using (5.4), (5.12) and Theorem 1 we get from (5.8) that

$$\begin{aligned} &C(U, V, W, X) - \eta(X)\eta(C(U, U)W) + \eta(U)\eta(C(X, V)W) \\ &+\eta(V)\eta(C(U, X)W) + \eta(W)\eta(C(U, V)X) = 0. \end{aligned} \tag{5.13}$$

From (5.12) and (5.5) we get

$$S(V, W) = (\frac{r}{n-1} - 1)g(V, W) + (n - \frac{r}{n-1})\eta(V)\eta(W). \tag{5.14}$$

From (5.14) we can state the following theorem:

Theorem 2. *If a K-contact manifold whose characteristics vector field ξ belongs to the k-nullity distribution satisfies the condition $R(\xi, X).C = 0$, then the manifold is η-Einstein.*

Again using (5.14) and (2.2) from (5.3) it follows that

$$\eta(C(X, Y)Z) = (k - 1)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{5.15}$$

Thus using (5.15) from (5.13) we have

$$C(U, V, W, X) = (k - 1)[g(U, X)\eta(V) - g(V, X)\eta(U)]\eta(W). \tag{5.16}$$

Putting $X = \xi$ in (5.16) we have $C(U, V, W, \xi) = 0$. That is, $C(\xi, X)Y = 0$. Thus we have the following theorem:

Theorem 3. *If a K-contact manifold whose characteristics vector field ξ belongs to the k-nullity distribution satisfies the condition $R(\xi, X).C = 0$, then $C(\xi, X)Y = 0$.*

Since for $k = 1$ the manifold becomes Sasakian, from (5.16) we have the following corollary:

Corollary. *If a Sasakian manifold satisfies the condition $R(\xi, X).C = 0$, then it is locally isometric with a unit sphere $S^n(1)$.*

The above corollary has been proved by M. C. Chaki and Tarafdar in [1].

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Received 17.09.92