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# GENERATING FUNCTION AND SEMI-ORTHOGONAL PROPERTIES OF A NEW CLASS OF POLYNOMIALS

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ABSTRACT. We define a new class of polynomials by means of a generating function and establish its semi-orthogonal properties. We further employ one of our semi-orthogonal properties to generate a theory concerning a finite series expansion involving the $Z$-polynomials.

**1. Introduction.** Recently [2], we have defined the $B$-polynomials:

$$B_m(x) = \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m} \; {}_{r+q+1}F_{s+p}\left[\begin{matrix} c_r, 1-b_q-m, -m; \\ d_s, 1-a_p-m \end{matrix} \; \frac{\beta}{\alpha}x(-1)^{p-q-1}\right](\alpha)^m, \tag{1.1}$$

by means of the generating function:

$$\begin{aligned} &{}_pF_q\left[\begin{matrix} a_p; \\ b_q \end{matrix} \; t\right] \; {}_rF_s\left[\begin{matrix} c_r; \\ d_s \end{matrix} \; xt\right] = \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m} \\ &x \; {}_{r+q+1}F_{s+p}\left[\begin{matrix} c_r, 1-b_q-m, -m; \\ d_s, 1-a_p-m \end{matrix} \; \frac{\beta}{\alpha}x(-1)^{p-q-1}\right], \end{aligned} \tag{1.2}$$

where the symbol $a_p$ represents the set of parameters $a_1, \cdots, a_p$.

It is interesting to note that the $B$-polynomials lead to the generalization ofsome classical polynomials and yield some new polynomials.

In this paper, we obtain a generating function and two semi-orthogonal properties of the polynomials:

$$Z_n(x; a, y) = \frac{1}{(a)_n} \; {}_2F_0\left[\begin{matrix} -n, 1-a-n; \\ - \end{matrix} \; -\frac{x}{y}\right], \tag{1.3}$$

which we shall call the $Z$-polynomials.

The $Z$-polynomials are related to the Bessel polynomials [6], the Laguerre polynomials and the Hermite polynomials by the following relations:

$$Z_n(x; 2-a-2n, b) = \frac{y_n(x; a, b)}{(2-a-2n)_n}, \tag{1.4}$$

$$Z_n(x; a, b) = \frac{y_n(x; 2-a-2n, b)}{(a)_n}, \tag{1.5}$$

$$Z_n(1; 1+a, x) = \frac{(-1)^n n! x^{-n} L_n^a(x)}{(1+a)_n}, \tag{1.6}$$

$$Z_n(1; 1/2, x^2) = \frac{(2x)^{-2n} H_{2n}(x)}{(1/2)_n}, \tag{1.7}$$

$$Z_n(1; 3/2, x^2) = \frac{(2x)^{-2n-1} H_{2n+1}(x)}{(3/2)_n}. \tag{1.8}$$

Although $Z$-polynomials are related to the above polynomials, but their basic properties would be different from those of the above polynomials, in view of their unique generating function given in Section 2.

In Section 4, we obtain a new generating function and two known orthogonality properties as particular cases of our results. We further employ one of the semi-orthogonal properties to develop a theory regarding the finite series expansion involving the $Z$-polynomials.

**2. The generating function.** We define the polynomials $Z_n(x; a, b)$ by means of the generating function

$$t^{(1-a)/2}\Gamma(a)J_{a-1}(2i\sqrt{t})e^{-xt/y} = \sum_{n=0}^{\infty} \frac{t^n}{n!} Z_n(x; a, y). \tag{2.1}$$

Proof. In (2.1), taking $\alpha = \beta = 1$, $p = r = s = 0$, $q = 1, b_1 = a$, and setting $-x/y$ for $x$, we have

$${}_0F_1\left[\begin{matrix} -; \\ a \end{matrix}\; t\right] {}_0F_0\left[\begin{matrix} -; \\ - \end{matrix}\; -\frac{xt}{y}\right] = \sum_{m=0}^{\infty} \frac{t^m}{m!(a)_m}\, {}_2F_0\left[\begin{matrix} -m, 1-a-m; \\ - \end{matrix}\; -\frac{x}{y}\right]. \tag{2.2}$$

In (2.2), putting ${}_0F_1\begin{bmatrix} -; & \\ & t \\ a & \end{bmatrix} = t^{(1-a)/2}\Gamma(a)J_{a-1}(2i\sqrt{t})$, ${}_0F_0\begin{bmatrix} -; & \\ & -\frac{xt}{y} \\ - & \end{bmatrix} = e^{-xt/y}$, and using (1.3), we obtain the generating function (2.1).

**Note**: The generating function (2.1) for $y = 1$ can also be derived using a result given by Bailey [3, p. 327].

**3. The semi-orthogonal properties.** The semi-orthogonal properties to be established are

$$\int_0^\infty x^{-a-2n}e^{-z/x}Z_m(x;a,z)Z_n(x;a,z)dx$$

(3.1) $$= 0, \quad \text{if} \quad m < n$$

(3,1a) $$= \frac{n!\Gamma(a-1)}{(a)_n}z^{1-a-2n}, \quad \text{if} \quad m = n$$

(3,1b) $$= -\frac{2(n+1)!\Gamma(a-2)}{a(a)_n}z^{1-a-2n}, \quad \text{if} \quad m = n+1$$

where Re$a > 1 + m - n$

$$\int_{-\infty}^\infty x^{2(1/2-a-2n)}e^{-z/x^2}Z_n(x^2;a,z)Z_n(x^2;a,z)dx$$

(3,2) $$= 0, \quad \text{if} \quad m < n$$

(3,2a) $$= \frac{n!\Gamma(a-1)}{(a)_n}Z^{1-a-2n}, \quad \text{if} \quad m = n$$

(3,2b) $$= -\frac{2(n+1)!\Gamma(a-2)}{a(a)_n}, \quad \text{if} \quad m = n+1$$

where $a = 3/2 - 2n, 5/2 - 2n, 7/2 - 2n, \ldots$

**Proof**. To prove (3.1), let us consider

$$Z_m(x;a,z)Z_n(x;a,z) = \frac{1}{(a)_m(a)_n}\,{}_2F_0\begin{bmatrix} -m, 1-a-m; & \\ & -\frac{x}{z} \\ - & \end{bmatrix}\,{}_2F_0\begin{bmatrix} -n, 1-a-n; & \\ & -\frac{x}{z} \\ - & \end{bmatrix}$$

so that

$$\int_0^\infty x^{-a-2n} e^{-z/x} Z_m(x;a,z) Z_n(x;a,z) dx$$

(3.3)

$$= \frac{1}{(a)_m (a)_n} \sum_{r=0}^{m} \frac{(-m)_r (1-a-m)_r}{r!(-z)^r} \sum_{u=0}^{n} \frac{(-n)_u (1-a-n)_u}{u!(-z)^u} \int_0^\infty x^{-a-2n+r+u} e^{-z/x} dx$$

On evaluating the last integral by using the definition of gamma function, viz.

$$\int_0^\infty x^{-n-2} e^{-z/x} dx = \Gamma(n+1)/z^{n+1}, \quad \text{Re} n > -1 \tag{3.4}$$

and using [4, p. 3, (4)], the right hand side of (3.3) reduced to the form

$$\frac{z^{1-a-2n}}{(a)_m (a)_n} \sum_{r=0}^{m} \frac{(-m)_r (1-a-m)_r}{(-1)^r r!} \Gamma(a+2n-r-1)\, {}_2F_1 \left[ \begin{matrix} -n, 1-a-n; \\ 2-a-2n+r \end{matrix} \; 1 \right] \tag{3.5}$$

On applying Vandermonde's theorem:

$${}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, 2, \ldots. \tag{3.6}$$

to (3.5) and using the relation $(1-n+r)_n = (-1)^n (-r)_n$, we have

$$\frac{z^{1-a-2n}}{(a)_m (a)_n} \sum_{r=0}^{m} \frac{(-m)_r (-r)_n (1-a-m)_r \Gamma(a+2n-r-1)}{r!(2-a-2n+r)_n} (-1)^{n-r} \tag{3.7}$$

If $r < n$, the numerator of (3.7) vanishes, and since $r$ runs from 0 to $m$, it follows that (3.7) also vanishes, when $m < n$. Now, it is clear that for $m < n$ all terms of (3.7) vanish, which proves (3.1a).

When $m = n$, using the standart result:

$$(-r)_n = \begin{cases} \frac{(-1)^n r!}{(r-n)!}, & 0 \le n \le r \\ 0, & n > r \end{cases} \tag{3.8}$$

and simlifying with the help of [4, p. 3, (4)], we have

$$\int_0^\infty x^{-a-2n} e^{-z/x} \{z_n(x;a,z)\}^2 dx = \frac{n!\Gamma(a-1)}{(a)_n} Z^{1-a-2n} \tag{3.9}$$

which proves (3.1a)

In (3.7), putting $m = n+1$, adding the resulting two terms ($r = n, n+1$) and simplifying with the help of (3.8) and [4, p. 3, (4)], we obtain

$$\int_0^\infty x^{-a-2n} e^{-z/x} Z_{n+1}(x;a,z) Z_n(x;a,z)\,dx$$

$$= -\frac{2(n+1)!\Gamma(a-2)}{a(a)_n} z^{1-a-2n}$$

which proves (3.1c).

To prove (3.2), we employ the same method of proof as in (3.1), but instead of (3.4), we use the following formula:

$$\text{(3.10)} \qquad \int_{-\infty}^\infty x^{-2n} e^{-z/x^2}\,dx = \frac{\Gamma(n-1/2)}{2^{n-1/2}}, \quad n = 1,2,3,\ldots.$$

**Note**: The integral (3.1) and (3.2) also exist for $m = n+2, n+3, n+4, \ldots$, and yeld a series of interesting results.

## 4. Particular cases.

(a) **Generating functions.** On applying (1.5) to (2.1), we obtain the following generating function (but somewhat less general) for the Bessel polynomials:

$$\text{(4.1)} \qquad t^{(1-a)/2}\Gamma(a) J_{a-1}(2i\sqrt{t}) e^{-xt/z} = \sum_{n=0}^\infty \frac{t^n}{n!} \frac{y_n(x;2-a-2n,z)}{(a)_n}$$

(b) **Semi-orthogonal properties.**

(i) In (3.9), putting $z = 1$, setting $2-a-2n$ for $a$ in $Z_n$ and the index of $x$, we have

$$\int_0^\infty x^{a-2} e^{-1/x} \{Z_n(x;2-a-2n,1)\}^2 dx$$

$$\text{(4.2)} \qquad = \frac{n!\pi}{\Gamma(2-a-n)\sin\pi(2-a-2n)} \mathrm{Re}\, a < 1-m-n$$

Now, using (1.4) and simplifying, we obtain the orthogonality property for the Bessel polinomials given by Exton [5, p. 215, (14)].

(ii) In (3.9), putting $z = 1$, replacing $x$ by $1/x$, setting $2-a-2n$ for $a$ in $Z_n$ and in the index of $x$, and using the relation $Z_n(1/x;a,1) = Z_n(1;a,x)$ we obtain

$$\int_0^\infty x^{-a} e^{-x} \{Z_n(1;2-a-2n,x)\}^2 dx = \frac{n!\Gamma(1-a-2n)}{(2-a-2n)_n}; \; Re\, a < 1-m-n.$$

Now, using (1.4) and simplifying, we get the orthogonality property obtained by the author [1, p.78, (2.1)]:

**5. Finite series expansions involving $Z$-polynomials.** Based on the relations (3.1a) and (3.1b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in finite series expansion of the $Z$-polinomials. Specially if $f(x)$ is a suitable function defined for all $x$, we consider for expansions of the general form

$$f(x) = \sum_{m=0}^{n} A_m x^{-2m} Z_m(x; a, z), \quad 0 < x < \infty, \quad m \leq n \tag{5.1}$$

where the expansion coefficients are given by

$$A_m = \frac{z^{a+2m-1}}{m!\Gamma(1-a)} \int_0^\infty f(x) x^{-a} e^{-x/z} Z_m(x; a, z) dx \tag{5.2}$$

valid under the conditions of (3.1).

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