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COINCIDENCE OF VIETORIS AND WIJSMAN TOPOLOGIES: A NEW PROOF

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Abstract. Let (X, d) be a metric space and $CL(X)$ the family of all nonempty closed subsets of X. We provide a new proof of the fact that the coincidence of the Vietoris and Wijsman topologies induced by the metric d forces X to be a compact space. In the literature only a more involved and indirect proof using the proximal topology is known. Here we do not need this intermediate step. Moreover we prove that (X, d) is boundedly compact if and only if the bounded Vietoris and Wijsman topologies on $CL(X)$ coincide.

1. Introduction. The Vietoris topology was introduced in the early 1920's by Vietoris in [6, 7] and its basic topological properties were identified by Michael [5] in a 1951 paper that set the agenda for the study of hyperspaces for the next thirty years. The Wijsman topology, defined at first for some applications in statistics [8], and successively used in many applications related to variational problems, is now recognized as a fundamental tool in the construction of the lattice of the hypertopologies [1]. For, most of the known topologies can be defined in terms of the Wijsman topologies: for instance, the Vietoris topology is the supremum of the Wijsman topologies generated by the distances on X which are equivalent to d [3].

In [3] the following theorem was proved:

Theorem. *Let (X, d) be a metric space. The following are equivalent:*

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(1) *X is compact;*

(2) *The Vietoris topology and the Wijsman topology induced by the metric d on $CL(X)$ coincide.*

The importance of the mentioned topologies entitled us to think that a new more direct proof of this fact could be useful.

Let us introduce first our notation: throughout the paper (X, d) will be a metric space with a metric d and $CL(X)$ the hyperspace of all closed non-empty subsets of X. As usual, by $S(x, \epsilon)$ we mean the open ball with the center x and radius ϵ. If E is a subset of X we put:

$$E^- = \{A \in CL(X) : A \cap E \neq \emptyset\},$$
$$E^+ = \{A \in CL(X) : A \subset E\}.$$

The Vietoris topology τ_V on $CL(X)$ [5, 6, 7] has as a subbase all sets of the form V^- and $(B^c)^+$, where V runs over all open sets in X, B runs over all closed sets in X and B^c stands for the complement of B.

If $A \in CL(X)$ and $x \in X$, then the distance from x to A is given by the familiar formula

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

The weakest topology on $CL(X)$ such that $A \to d(x, A)$ is continuous for each $x \in X$ is usually called the Wijsman topology [1]. We denote this topology by $\tau_{W(d)}$.

Remark. The coincidence of the Wijsman topology $\tau_{W(d)}$ and the Vietoris topology on $CL(X)$ forces d to be a bounded metric. (If d is not bounded, there is a sequence $\{b_n\}$ in X with $d(b_1, b_n) \to \infty$. Put $B_n = \{b_1, b_n\}$ for every $n \in Z^+$. Then $\{B_n\}$ $\tau_{W(d)}$ -converges to $\{b_1\}$, but fails to τ_V converge to $\{b_1\}$.)

Proof of Theorem. (1) $\Rightarrow$ (2) This part is standard, and is left to the reader.

(2) $\Rightarrow$ (1) Suppose that (X, d) is not compact. Let $\{x_n\}$ be a sequence in X without a cluster point. Put $A = \{x_n : n \in Z^+\}$. Without loss of generality we can suppose that all points of A are distinct. Every subset of A is closed.

If A is a totally bounded set, there is a cauchy subsequence $\{y_n\}$ of $\{x_n\}$. Put $H = \{y_{2n} : n \in Z^+\}$ and $H_n = H \cup \{y_{2n-1}\}$ for every $n \in Z^+$. Then it is easy to verify that $\{H_n\}$ $\tau_{W(d)}$-converges to H, but fails to τ_V-converge to H ($H \in (B^c)^+$, where $B = \{y_{2n-1} : n \in Z^+\}$ is a closed set in X; but $H_n \notin (B^c)^+$ for every $n \in Z^+$).

So we can suppose that A is not totally bounded. There is $\epsilon > 0$ and a subsequence $\{y_n\}$ of $\{x_n\}$ with $d(y_n, y_m) > 2\epsilon$ for every $n, m \in Z^+$ and $n \neq m$.

We have two possibilities:

a) there is $\eta > 0$ and a subsequence $\{z_n\}$ of $\{y_n\}$ with $S(z_n, \eta) = \{z_n\}$;

b) the negation of a) which implies that there is a subsequence $\{v_n\}$ of $\{y_n\}$ with $S(v_n, \frac{1}{n}) \neq \{v_n\}$.

Consider first b). For every $n \in Z^+$ let $u_n \in S(v_n, \frac{1}{n})$ be a different point from v_n. Put $H = \{v_n : n \in Z^+\}$ and $H_n = H \cup \{u_n\}$ for every $n \in Z^+$. Again $\{H_n\}$ $\tau_{W(d)}$ -converges to H, but fails to τ_V-converge to H (the same reason as above).

Now consider a). So there are $\eta > 0$ and a subsequence $\{z_n\}$ of $\{y_n\}$ with $S(z_n, \eta) = \{z_n\}$. Put $z_n^1 = z_n$ for any $n \in Z^+$. The sequence $\{z_n^1\}$ is bounded (see Remark); so there is a subsequence $\{z_n^2\}$ such that $\{d(z_1^1, z_n^2)\}$ converges. Again there is a subsequence $\{z_n^3\}$ from $\{z_n^2\}$ such that $\{d(z_2^2, z_n^3)\}$ converges, and so on. Now take the diagonal subsequence $\{z_n^n\}$. It is easy to verify that we can select subsequences in such a way that the resulting sequence $\{z_n^n\}$ consists of distinct points. Set $w_n = z_n^n$. Put

$$B = \{w_{2n} \ : n \in Z^+\}, \ L = X \setminus B$$

and

$$L_n = L \cup \{w_{2n}\} \ \text{ for any } \ n \in Z^+.$$

Then B, L and L_n $(n \in Z^+)$ are closed sets and $L \in (B^c)^+$. Since $L_n \in B^-$ for every $n \in Z^+$, $\{L_n\}$ fails to τ_V-converge to L. We show that $\{L_n\}$ τ_{W_d}-converges to L.

Take $x \in X$ and we show that $\{d(x, L_n)\} \to d(x, L)$. If $x \in L$ we are done. Suppose $x \notin L$. Then there is $n_0 \in Z^+$ such that $x = w_{2n_0}$. Suppose $\{d(x, L_n)\}$ does not converge to $d(x, L)$. There is $\delta > 0$ such that $\mid d(x, L_n) - d(x, L) \mid > \delta$ for infinitely many $n \in J \subset Z^+$. So $d(x, L_n) < d(x, L) - \delta$ for any $n \in J$. There is $y \in L_n$ such that $d(x, y) < d(x, L) - \delta$ for any $n \in J$, so y must be w_{2n}.

Thus we have

$$d(w_{2n_0}, w_{2n}) < d(w_{2n_0}, L) - \delta$$

for any $n \in J$ and also

$$d(w_{2n_0}, w_{2n-1}) \leq d(w_{2n_0}, L) - \frac{\delta}{2}$$

for infinitely many n. So

$$d(w_{2n_0}, L) \leq d(w_{2n_0}, L) - \frac{\delta}{2},$$

a contradiction. □

By using of the same idea we can prove a new result concerning the coincidence of the Wijsman topology and the bounded Vietoris topology. The bounded Vietoris topology $\tau_{bV(d)}$ (induced by the metric d) on $CL(X)$ [1, 2] has as a subbase all sets of the form V^-, where V runs over all open sets in X and

$(B^c)^+$, where B runs over all closed d-bounded sets in X. Of course, if (X, d) is a bounded metric space, then τ_V and $\tau_{bV(d)}$ coincide on $CL(X)$.

Proposition. *Let (X, d) be a metric space. The following are equivalent:*

(1) *Every closed d-bounded set is compact;*

(2) *The Wijsman topology $\tau_{W(d)}$ and the bounded Vietoris topology $\tau_{bV(d)}$ coincide on $CL(X)$.*

Proof. (1) $\Rightarrow$ (2) This part is standard, and is left to the reader.

(2) $\Rightarrow$ (1) Suppose there is a closed d-bounded set C which is not compact. So there is a sequence $\{x_n\}$ in C without a cluster point in X. Now we will proceed in the same way as above. □

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