Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **30** (2004), 483–494

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

A NOTE ON THE ASYMPTOTIC BEHAVIOUR OF A PERIODIC MULTITYPE GALTON-WATSON BRANCHING PROCESS

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Communicated by N. M. Yanev

Abstract. In this work, the problem of the limiting behaviour of an irreducible Multitype Galton-Watson Branching Process with period d greater than 1 is considered. More specifically, almost sure convergence of some linear functionals depending on d consecutive generations is studied under hypothesis of non extinction. As consequence the main parameters of the model are given a convenient interpretation from a practical point of view. For a better understanding of the theoretical results, an illustrative example is provided.

1. Introduction. The Multitype Galton-Watson Branching Process (MP) is a modification of standard Galton-Watson Branching process, in which

2000 *Mathematics Subject Classification:* 60J80.

Key words: Periodic Multitype Galton-Watson Branching Process, almost sure convergence, cycle.

*Research supported by the Ministerio de Ciencia y Tecnología and the FEDER through the Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica, grant BFM2003-06074.

several types of individuals coexist in a population. These individuals reproduce independently of each other and with the same probability distribution inside a type. This process has been widely studied (see for example [7], [9]). Among the main results we can find an exhaustive classification of the process, based on the matrix of means, in positively regular, periodic and reducible cases. The problem of the limiting behaviour of such a process has been also considered in the literature (see for example [1], [3], [4], [5] and [6], among others). Some of these authors deal with the periodic MP but without taking into account the cyclic structure of this particular process. The aim of this paper is the study, under non extinction assumption, of certain cycle of the process. We start our research, in Section 2, with a short description of the MP. We focus our attention on the periodic case and on the investigations on its asymptotic behaviour developed in [4]. In Section 3 the study of some linear functionals associated to the process is considered. These functionals summarize somehow the cyclic structure of the process and allow us to provide in Section 4 adequate interpretations for the main parameters, analogous to those given for the positively regular case. Finally, in Section 5, a simulated example is proposed as an illustration of the theoretical results.

2. The probability model. Let us consider a MP with m types, i.e. a sequence of m-dimensional random vectors $\{Z(n)\}_{n\geq 0}$, valued on $\mathbb{N}_0^m$, and defined in a recursive manner as follows:

$$Z(0) = \varepsilon_i, \qquad \varepsilon_i = (\delta_{i1}, \ldots, \delta_{im}), \qquad \delta_{ij} : \text{Kronecker's Delta}$$

$$Z(n+1) = \sum_{i=1}^{m} \sum_{j=1}^{Z_i(n)} (Y_{ij}^1(n), \ldots, Y_{ij}^m(n))$$

considering the empty sum as the vector $\mathbf{0}$. The random vectors $\{Y_{ij}(n) : i = 1, \ldots, m,\ j = 1, 2, \ldots,\ n = 0, 1, 2, \ldots\}$, with $Y_{ij}(n) := (Y_{ij}^1(n), \ldots, Y_{ij}^m(n))$, are assumed independent, taking values on $\mathbb{N}_0^m$, and such that, for every fixed type i, they are distributed according to the same probability law, namely $p_i(\cdot)$, i.e. for an index i and a vector z fixed, $p_i(z) := P[Y_{ij}(n) = z]$ for $n = 0, 1, \ldots$, $j = 1, 2, \ldots$ Thus, $Y_{ij}(n)$ denotes the vector formed by the number of individuals of the different types produced by the jth individual of type i who lives in generation n, according to the reproduction law $p_i(\cdot)$. Consequently, $Z(n)$ denotes the vector of individuals of the different types who form the nth generation. It is obvious that the process $\{Z(n)\}_{n\geq 0}$ is a m-dimensional homogenous Markov Chain.

The elements of the matrix of means of the process, $M = (m_{ij})_{1\leq i,j\leq m}$ can be defined by $m_{ij} := E[Z_j(1)|Z(0) = \varepsilon_i]$, i.e. m_{ij} represents the average number of individuals of type j produced by an individual of type i. The coefficient (i,j) of the matrix M^n will be denoted as $m_{ij}^{(n)}$. We can formulate the following classification of MP $\{Z(n)\}_{n\geq 0}$, based on the structure of the matrix of means M.

Definition 2.1. *A MP $\{Z(n)\}_{n\geq 0}$ is said to be irreducible, if for each pair $i,j \in \{1,\dots,m\}$, there exists a generation $n = n(i,j) \geq 1$ such that $m_{ij}^{(n(i,j))} > 0$. Otherwise the process is said to be reducible.*

If $\{Z(n)\}_{n\geq 0}$ is irreducible, then for every type $i \in \{1,\dots,m\}$ it can be defined the constant $d := g.c.d.\{n \geq 1 : m_{ii}^{(n)} > 0\}$ (*g.c.d.* denotes the greatest common divisor), which does not depend on the type i (see [2]). Moreover, the m types can be reordered and grouped in d disjoint groups $\{D_{[a]_d} : 1 \leq a \leq d\}$, where $[x]_d$ denotes the rest of dividing x by d. Moreover $m_{ij} = 0$ except if $i \in D_{[a]_d}$ and $j \in D_{[a+1]_d}$, so that the matrix of means M can be written in the following way:

$$M = \begin{pmatrix} 0 & M(1,2) & 0 & \cdots & 0 \\ 0 & 0 & M(2,3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M(d-1,d) \\ M(d,1) & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $M([a]_d,[a+1]_d) = (m_{ij})_{i\in D_{[a]_d},\, j\in D_{[a+1]_d}}$ for $a = 1,\dots,d$. Intuitively that means that if an individual's type belongs to the group $D_{[a]_d}$, this individual will generate with probability one individuals whose types belong to the group $D_{[a+1]_d}$. Thus, only after a number of generations multiple of d it can be found again in the population individuals of a type belonging to the group $D_{[a]_d}$.

Definition 2.2. *An irreducible MP $\{Z(n)\}_{n\geq 0}$ is said to be positively regular if $d = 1$. Otherwise, it will be called periodic, and $d > 1$ will be referred as the period of the process.*

Since the matrix M is supposed non negative and finite, Perron-Frobenius' Theory establishes (see [8]) that if $\{Z(n)\}_{n\geq 0}$ is periodic, then there exists an eigenvalue of M, ρ, real positive with maximum modulus and left and right associated eigenvectors, ν and μ respectively, which can be normalized as follows:

$$\nu\mathbf{1} = \sum_{i=1}^{m} \nu_i = 1, \quad \nu(a)\mu(a) = \sum_{j\in D_{[a]_d}} \mu_j\nu_j = 1,\ a = 1,\dots,d$$

where $\mu_i > 0$, $\nu_i > 0$ for all $i \in \{1,\dots,n\}$, being $\mathbf{1}$ the m-dimensional vector with all the coordinates equal to 1, $\mu(a) := (\mu_j : j \in D_{[a]_d})$ and $\nu(a) := (\mu_j : j \in D_{[a]_d})$.

From now on we will assume ν and μ subject to such restrictions.

In order to study the asymptotic behaviour of the periodic MP, let us examine first some results related to the extinction of this process. With the following definition we avoid trivialities.

Definition 2.3. *The irreducible MP $\{Z(n)\}_{n\geq 0}$ is called singular if $\sum_{j=1}^{m} p_i(\varepsilon_j) = 1$ for all $i \in \{1, \ldots, m\}$.*

The magnitude of the maximum modulus eigenvalue ρ plays a crucial role in the study of the extinction of the process. Sevast'yanov (see [10]) obtains that an irreducible, non singular process becomes extinct almost surely if and only if $\rho \leq 1$, independently of the type which the process starts with. Consequently, since every non null vector is a transient state of the Markov Chain $\{Z(n)\}_{n\geq 0}$, if $\rho > 1$ then the total number of individuals in the process approaches to infinity along the generations with a positive probability. So, in order to study the asymptotic behaviour of the process in the periodic case we consider $\rho > 1$. In this sense we have the following result due to Kesten and Stigum (see [4]).

Theorem 2.1. *Let $\{Z(n)\}_{n\geq 0}$ be a periodic, not singular MP with period d and $\rho > 1$. If $Z(0) = \varepsilon_i$ with $i \in D_{[a]_d}$, then there exists a random variable W such that, for every $j \in D_{[b]_d}$, the following equality holds almost surely*

$$\lim_{n\to\infty} \frac{1}{\rho^{nd+b-a}} Z_j(nd + b - a) = W\nu_j \tag{1}$$

Moreover if the following logarithm condition holds for all $i \in D_{[a]_d}$, $j \in D_{[a+1]_d}$ and $a \in \{1, \ldots, d\}$

$$E[Z_j(1) \log^+ Z_j(1) | Z(0) = \varepsilon_i] < \infty \tag{2}$$

then $E[W|Z(0) = \varepsilon_i] = \mu_i(a)$. Otherwise $W = 0$ almost surely.
If (2) holds and there exist indices a and $i' \in D_{[a]_d}$ such that

$$\sum_{j\in D_{[a+1]_d}} Z_j(1)\mu_j(a + 1) \tag{3}$$

can take on at least two values with positive probability given that $Z(0) = \varepsilon_{i'}$, then the distribution of W has a jump of magnitude $q_i := P(Z(n)\mathbf{1} \to 0|Z(0) = \varepsilon_i)$ at the origin and continuous density function on the set of the positive real numbers, given that $Z(0) = \varepsilon_i$. If (3) does not hold for any pair (i', a) with $1 \leq i' \leq m$ and $1 \leq a \leq d$, then the distribution of W is degenerate at one point.

In the same way it can be obtained the asymptotic behaviour of every linear functional $bZ(n)$, with $b \in \mathbb{R}^m$.

3. Asymptotic behaviour of a cycle of the periodic process $\{Z(n)\}_{n\geq 0}$. Until now, the study of the asymptotic behaviour of the periodic process $\{Z(n)\}_{n\geq 0}$, summarized in Theorem 2.1, is considered generation by generation for every group, because once the process has started with one individual of type i belonging to the group $D_{[a]_d}$, then, with probability one, individuals of the types belonging to the group $D_{[a+l]_d}$ only can be found in the generations $nd + l$, and no individuals of other types exist in these generations. So, only in generations multiple of d can be found individuals of the types belonging to group $D_{[a]_d}$. Taking into account this cyclic structure, it is interesting the study of functionals depending on d consecutive generations of the process, which we refer as *cycle*. The main property of a cycle is the possibility of recording individuals of all the types.

Denote by $|Z_{D_{[a]_d}}(n)| := Z_{D_{[a]_d}}(n)\mathbf{1} = \sum_{i\in D_{[a]_d}} Z_i(n)$ and suppose $Z(0) = \varepsilon_i$ with $i \in D_{[a]_d}$. Let us consider the cycle:

$$\rho^{d-1}|Z_{D_{[a]_d}}(nd)|+\rho^{d-2}|Z_{D_{[a+1]_d}}(nd+1)|+\ldots+|Z_{D_{[a-1]_d}}(nd+d-1)|$$

where individuals from all types, weighted in a convenient way, in d consecutive generations of the process are considered.

In relation with the asymptotic behaviour of the previous cycle we can provide the following result.

Theorem 3.1. *Let $\{Z(n)\}_{n\geq 0}$ be a periodic, non singular MP with period d and $\rho > 1$. If $Z(0) = \varepsilon_i$ with $i \in D_{[a]_d}$ then there exists a random variable W such that:*

$$\lim_{n\to\infty} \frac{\rho^{d-1}|Z_{D_{[a]_d}}(nd)| + \ldots + |Z_{D_{[a-1]_d}}(nd+d-1)|}{\rho^{nd+d-1}} = W \ \ a.s.$$

Moreover the logarithm condition

$$\sum_{a=1}^{d} \sum_{i\in D_{[a]_d}} \sum_{j\in D_{[a+1]_d}} E[Z_j(1)\log^+ Z_j(1)|Z(0) = \varepsilon_i] < \infty$$

is equivalent to $E[W|Z(0) = \varepsilon_i] = \mu_i > 0$.

P r o o f. Since $Z(0) = \varepsilon_i$ with $i \in D_{[a]_d}$, applying Theorem 2.1, we have that

$$\lim_{n\to\infty} \frac{Z_j(nd+l)}{\rho^{nd+l}} = W\nu_j \text{ a.s. } \; j \in D_{[a+l]_d},\; l = 0,\ldots,d-1$$

The sum extended to the group $D_{[a+l]_d}$ is

$$\lim_{n\to\infty} \frac{|Z_{D_{[a+l]_d}}(nd+l)|}{\rho^{nd+l}} = W \sum_{j\in D_{[a+l]_d}} \nu_j \text{ a.s.}$$

Note that, since $\nu\mathbf{1} = 1$, summing over all the groups we obtain:

$$\lim_{n\to\infty} \left(\frac{|Z_{D_{[a]_d}}(nd)|}{\rho^{nd}} + \ldots + \frac{|Z_{D_{[a+d-1]_d}}(nd+d-1)|}{\rho^{nd+d-1}} \right) = W \quad \text{a.s.}$$

and the proof is concluded. The properties of the random variable W can be directly deduced from Theorem 2.1. □

The following results, specially interesting from a practical point of view, are consequence of the previous Theorem.

Corollary 3.1. *Let $\{Z(n)\}_{n\geq 0}$ be a periodic, non singular MP with period d and $\rho > 1$. If $Z(0) = \varepsilon_i$ with $i \in D_{[a]_d}$ and logarithm condition (2) holds, then, for every group $D_{[a+l]_d}$ with $l = 0,\ldots,d-1$, we have that on $\{Z(n)\mathbf{1} \to \infty\}$:*

(i) For each $k \in D_{[a+l]_d}$

$$\lim_{n\to\infty} \frac{\rho^{d-1-l} Z_k(nd+l)}{\rho^{d-1}|Z_{D_{[a]_d}}(nd)| + \ldots + |Z_{D_{[a-1]_d}}(nd+d-1)|} = \nu_k \quad a.s.$$

(ii)

$$\lim_{n\to\infty} \frac{\rho^{d-1-l} |Z_{D_{[a+l]_d}}(nd+l)|}{\rho^{d-1}|Z_{D_{[a]_d}}(nd)| + \ldots + |Z_{D_{[a-1]_d}}(nd+d-1)|} = \sum_{j\in D_{[a+l]_d}} \nu_j \;\; a.s.$$

P r o o f. Applying Theorem 3.1, we deduce that if $Z(0) = \varepsilon_i$ with $i \in D_{[a]_d}$, then:

$$\lim_{n\to\infty} \frac{\rho^{d-1}|Z_{D_{[a]_d}}(nd)| + \ldots + |Z_{D_{[a-1]_d}}(nd+d-1)|}{\rho^{nd+d-1}} = W \text{ a.s.}$$

and $P(W>0)>0$, because logarithm condition holds. Moreover, due to the fact that $\{W>0\}=\{Z_{D_{[a]_d}}(dn)\to\infty\}$ a.s., the result follows from the application of Theorem 2.1 for $k\in D_{[a+l]_d}$. $\square$

Moreover, for every group $D_{[a]_d}$, the following relation is verified.

Corollary 3.2. *Let $\{Z(n)\}_{n\geq 0}$ be a periodic, non singular MP with period d and $\rho>1$. If $Z(0)=\varepsilon_i$ with $i\in D_{[a]_d}$ and logarithm condition (2) holds, then, for every group $D_{[a+l]_d}$ with $l=0,\dots,d-1$, we have for each $k\in D_{[a+l]_d}$, that on $\{Z(n)\mathbf{1}\to\infty\}$:*

$$\lim_{n\to\infty}\frac{Z_k(nd+l)}{|Z_{D_{[a+l]_d}}(nd+l)|}=\frac{\nu_k}{\sum_{j\in D_{[a+l]_d}}\nu_j}\quad a.s.$$

Remark 3.1. The previous results are also valid for $d=1$. In this case we obtain the asymptotic behaviour for the positively regular process given by [3].

Remark 3.2. Taking into account the additive property of the MP $\{Z(n)\}_{n\geq 0}$, and the previous results, it is easy to obtain the asymptotic behaviour of the kth coordinate of the cycle

$$\rho^{d-1}Z_k(nd)+\ldots+Z_k(nd+d-1),\ k\in D_{[a]_d}$$

supposed the process to start with any vector $Z(0)$. The results we get for this cycle are analogous to the Corollaries above.

4. Interpretation of the main parameters. Mode (see [7]) provides adequate interpretations for the main parameters of the positively regular process from its asymptotic behaviour. In this sense we propose convenient interpretations, with an specially practical interest, for the parameters ρ, μ and ν associated to the periodic process $\{Z(n)\}_{n\geq 0}$, making use of the asymptotic behaviour of the cycle

$$\rho^{d-1}|Z(nd)|+\ldots+|Z(nd+d-1)|$$

First note that, since under the logarithm condition this cycle normalized by ρ^{nd+d-1} converges almost surely to a non null random variable, then ρ^d represents the rate of growth of the cycle in d consecutive generations, independently of the initial type and number of individuals.

On the other hand, the convergence of kth coordinate of the cycle

$$\rho^{d-1}Z_k(nd)+\ldots+Z_k(nd+d-1)$$

indicates that the proportion of individuals of type k that one can expect in a large enough generation of the cycle is ν_k, independently of the type of individual which started the process. Moreover, taking into account the behaviour of the cycle

$$\rho^{d-1}|Z_{D_{[a]_d}}(nd)| + \ldots + |Z_{D_{[a]_d}}(nd+d-1)|,$$

$\nu_k / \sum_{j\in D_{[a]_d}} \nu_j$ represents the expected proportion of individuals of a type $k \in D_{[a]_d}$ in large generations, with respect to the total of individuals of the cycle with types belonging to the group $D_{[a]_d}$.

Finally, taking into account that $E[W|Z(0) = \varepsilon_i] = \mu_i$ if logarithm condition holds, the parameter μ_i represents the reproductive mean value of the individuals of type i.

5. Illustrative example. In order to illustrate the results obtained and the interpretations proposed for the parameters ρ, μ and ν, let us model the polyphase birth of certain biological specie by means of a MP. More specifically, suppose three phases in the life of individuals of such specie: birth (1), growth (2) and reproduction (3), after which individuals disappear. Denote by p and q the probabilities of going from phase (1) to (2) and from phase (2) to (3) respectively. If we assume there exist two forms of reproduction respectively, namely A and B, which happen with probabilities p_A and p_B, with $p_A + p_B = 1$, then we have that $q_A := qp_A$, is the probability of entering the reproductions phase according to type A and $q_B := qp_B$, according to type B. The average number of descendants per individual belonging to the reproduction phase is m_A and m_B, depending on the type A or B, respectively. This model can describe, for example, the growth of animal species with two forms of reproduction: sexual (A) or hermaphrodite (B).

A possible model for the previous situation is a periodic MP with 4 types, where the matrix of means M is

$$M = \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & 0 & q_A & q_B \\ m_A & 0 & 0 & 0 \\ m_B & 0 & 0 & 0 \end{pmatrix}$$

We have simulated 29 generations of this model, starting with $Z(0) = (1, 1, 1, 0)$, and supposing that $p = 0.8$, $q_A = 0.3$, $q_B = 0.6$. Both forms of reproduction produce new descendants according Poisson distributions of parameters $m_A = 3$ and $m_B = 6$. From this values we obtain $\rho = 1.533 > 1$, the reproductive value $\mu = (1.828, 3.503, 3.579, 7.158)$ and the vector of proportions

$\nu = (0.548, 0.285, 0.059, 0.118)$. The data corresponding to the simulation made are summarized in the following table:

n	$Z_1(n)$	$Z_2(n)$	$Z_3(n)$	$Z_4(n)$
0	1	1	1	0
1	14	1	0	1
2	3	7	1	0
3	8	3	1	5
4	18	8	1	2
⋮	⋮	⋮	⋮	⋮
25	23884	6313	1584	3239
26	19296	19220	1875	3818
27	22553	15500	5700	11589
28	69033	18071	4650	9243
29	55257	55251	5440	10819

First consider the branch $\{Z^{(1)}(n)\}_{n\geq 0}$ of the process generated by the individual belonging to the birth phase in the initial generation:

n	$Z_1^{(1)}(3n)$	$Z_2^{(1)}(3n+1)$	$Z_3^{(1)}(3n+2)$	$Z_4^{(1)}(3n+2)$
0	1	1	1	0
1	8	8	0	7
2	21	17	3	9
3	48	37	6	25
4	114	93	25	61
5	326	267	79	160
6	924	745	211	474
7	2785	2221	651	1363
8	7894	6313	1875	3818
9	22553	18071	5440	10819

Next, in figure 1, the number of individuals of each type is represented in front of the total number of individuals in each period (left graphic). Their

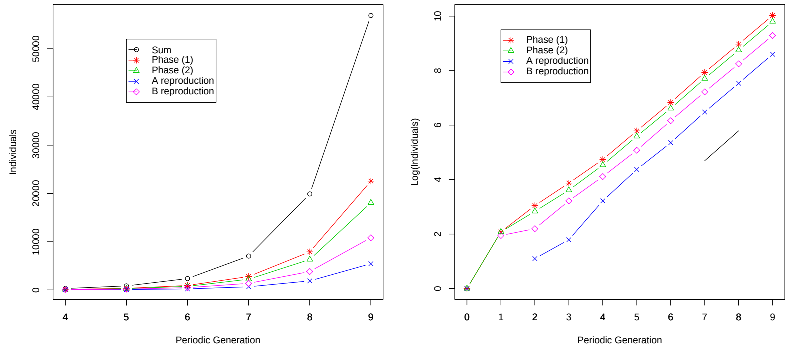

Fig. 1. Left: $(Z_1^{(1)}(nd), Z_2^{(1)}(nd+1), Z_3^{(1)}(nd+2), Z_4^{(1)}(nd+2))$.
Right: $(\log(Z_1^{(1)}(nd)), \log(Z_2^{(1)}(nd+1)), \log(Z_3^{(1)}(nd+2)), \log(Z_4^{(1)}(nd+2)))$

respective logarithms are represented in front of the rate of growth $3\log\rho$ (solid line in right graphic).

As it can be noticed, the total number of individuals is not distributed among the different types according to the vector ν in each period. However if we represent the cycle $\rho^2 Z_1^{(1)}(nd) + \rho Z_2^{(1)}(nd+1) + Z_3^{(1)}(nd+2) + Z_4^{(1)}(nd+2)$ the proportion of individuals of each type agrees with the values in ν (left graphic figure 2). In the right graphic the cycle of reproduction $Z_3(nd+2) + Z_4(nd+2)$ is represented.

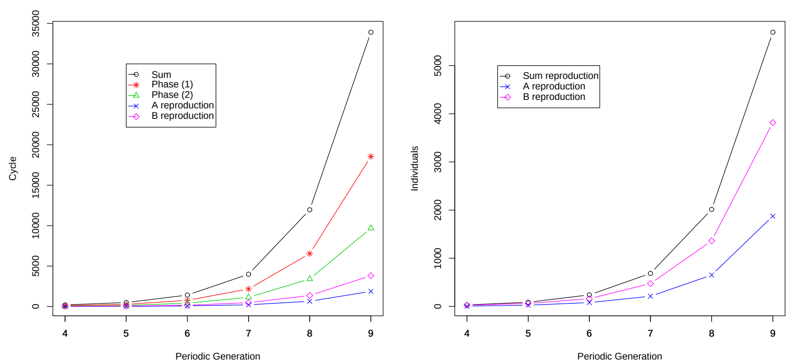

Fig. 2. Left: $(\rho^2 Z_1^{(1)}(nd), \rho Z_2^{(1)}(nd+1), Z_3^{(1)}(nd+2), Z_4^{(1)}(nd+2))$.
Right: $(Z_3^{(1)}(nd+2), Z_4^{(1)}(nd+2))$

Finally, by considering the process $\{Z(n)\}_{n\geq 0}$ and the cycle $\rho^2|Z(nd)| + \rho|Z(nd+1)|+|Z(nd+2)|$ the results obtained are analogous to the previous ones, as it is shown in figures 3 and 4.

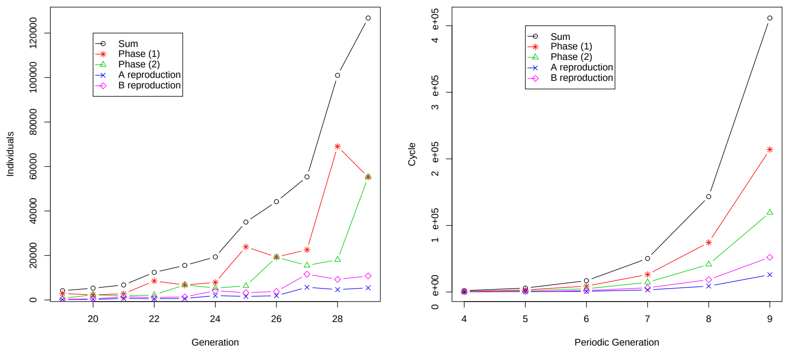

Fig. 3. Left: $(Z_1(n), Z_2(n), Z_3(n), Z_4(n))$.
Right: $\rho^2 Z_i(nd) + \rho Z_i(nd+1) + Z_i(nd+2),\ i = 1,\ldots,4$

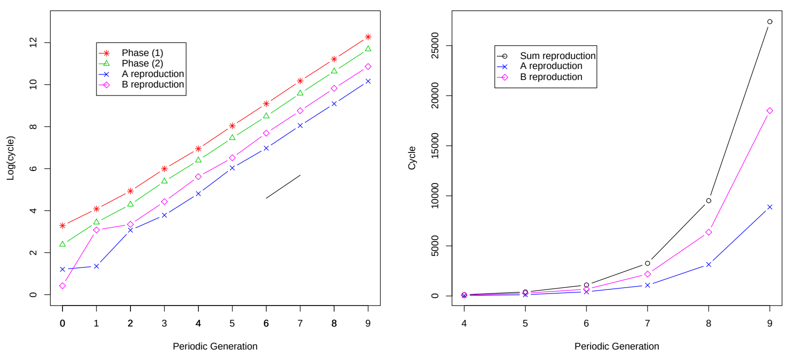

Fig. 4. Left: $\log(\rho^2 Z_i(nd) + \rho Z_i(nd+1) + Z_i(nd+2)),\ i = 1,\ldots,4$.
Right: $\rho^2 Z_i(nd) + \rho Z_i(nd+1) + Z_i(nd+2),\ i = 3,4$

Notice that the proportion of individuals belonging to both reproduction types in the cycle $\rho^2(Z_3(nd) + Z_4(nd)) + \rho(Z_3(nd+1) + Z_4(nd+1)) + Z_3(nd+2) + Z_4(nd+2)$ is controlled by the vector $\left(\dfrac{\nu_3}{\nu_3+\nu_4}, \dfrac{\nu_4}{\nu_3+\nu_4}\right) = (1/3, 2/3)$.

REFERENCES

[1] S. Asmussen. Almost sure behavior of linear functionals of supercritical branching process. *Trans. Amer. Math. Soc.* **231** (1977), 233–248.

[2] J. L. Doob. Stochastic Process. Wiley, 1953.

[3] H. Kesten, B. P. Stigum. A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.* **37** (1966), 1211–1223.

[4] H. Kesten, B. P. Stigum. Additional limit theorems for indecomposable multidimensional Galton-Watson process. *Ann. Math. Statist.* **37** (1966), 1463–1481.

[5] H. Kesten, B. P. Stigum. Limit theorems for decomposable multi-dimensional Galton-Watson process. *J. Math. Anal. Appl.* **17** (1967), 309–338.

[6] A. Joffe, F. Spitzer. On multitype branching processes with $\rho \leq 1$. *J. Math. Anal. Appl.* **19** (1967), 409–430.

[7] C. J. Mode. Multitype Branching Processes. Elsevier, 1971.

[8] E. Seneta. Non-negative matrices and Markov chains. Springer-Verlag, 1981.

[9] B. A. Sevast'yanov. Branching Processes. Mir, Moscow, 1971.

[10] B. A. Sevast'yanov. On the theory of branching random processes. *Doklady Akad. Nauk. SSSR.* **59** (1948), 1407–1410.

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Received January 13, 2004