SYMMETRIC AND ASYMMETRIC GAPS IN SOME FIELDS OF FORMAL POWER SERIES

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ABSTRACT. We consider non-archimedean real closed fields of cardinality $\aleph_1$ that have special type of symmetric gaps and compare these fields with well known $\eta_1$-fields (Hausdorff), semi-$\eta_1$-fields, and some super-real fields (Dales, Woodin). All these fields are realized as fields of formal power series. We describe all symmetric Dedekind and non-Dedekind gaps of semi-$\eta_1$-fields (in particular, for a nonstandard real line). We consider a construction of fields with symmetric gaps that are not semi-$\eta_1$. By this construction we give examples of fields with different asymmetric gaps.

1. Introduction. Throughout this paper we consider non-archimedean real closed totally ordered fields of cardinality $\aleph_1$. One of the directions for investigation of the totally ordered fields is gap (cut) theory. We will follow [6]. A pair $(A, B)$ of non-empty subsets $A, B$ of a field $(F, +, \cdot, <)$ is called a gap if

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A < B (i.e., ∀a ∈ A∀b ∈ B : a < b) and A ∪ B = F. The set A is called a short-shore of a gap (A, B) in F if there exists a₀ ∈ A such that for all a ∈ A, we have a + (a − a₀) ∈ A (the “distance” between a₀ and every a ∈ A is much less than “distance” between a₀ and B). If a shore is not short then it is called a long shore. If both A and B are long then (A, B) is called a symmetric gap. If one of the shores is long and the other one is short then (A, B) is called an asymmetric gap. Note that (short, short)-gap is impossibly (see [6] for details).

A subset H ⊂ L is said to be cofinal (coinitial) in a totally ordered set L if ∀l ∈ L∃h ∈ H such that l ≤ h(l ≥ h). min{card(H)} H is cofinal (coinitial) in L} is called a cofinality (coinitiality) of L and is denoted cf(L) (coi(L)). A gap (A, B) of F is said to have (α, β)-type if cf(A) = α and coi(B) = β (see [1]).

Note that if (A, B) is a symmetric gap then cf(A) = coi(B); the cardinal cf(A) is called cofinality of (A, B) and is denoted by cf(A, B) [6].

For x, y in a field F \ {0}, let x ~ y if ∃n ∈ N such that |x| ≤ n|y| and |y| ≤ n|x|. Let F̂ be the set of equivalence classes of F mod ∼. Let ̂x ∈ F̂ such that x ∈ ̂x. Define x ≪ y if ∀n ∈ N n|x| < |y| and ̂x < ̂y ⇔ x ≪ y. Put ̂x · ̂y = ̂x̂y. So, we have (F̂, ·, <) is a totally ordered group. F̂ is called the group of archimedean classes of F [6] or the value group of F [1]. If x, y ∈ F \ {0} and x + y ≠ 0 then ̂x + ̂y = max{̂x, ̂y}.

Hausdorff has introduced a notion η₁-set: a totally ordered set L is called an η₁-set if ∀A, B ⊆ L such that A < B with |A ∪ B| < N₁ there exists t ∈ L with A < t < B.

There are two “isomorphism theorems” for real closed fields. A classical theorem of Erdős, Gilman and Henriksen [1] states that any two real closed fields that are η₁-sets of cardinality N₁ are ordered isomorphic. This theorem is equivalent to CH [5]. Pestov introduced a notion of symmetric gap and proved the following isomorphism theorem:

**Theorem 1.1** [6]. Let F₁ and F₂ be really closed ordered fields such that card(F₁) = card(F₂) = N₁ and cofinality of each symmetric gap in both fields is N₁. Then F₁ and F₂ are isomorphic as ordered fields iff the groups of archimedean classes of both fields are order-isomorphic.

In [4] we considered a class K of real closed fields to which Pestov’s isomorphism theorem applies. A real closed field F ∈ K if

1) card(F) = card(F̂) = N₁,
2) if (A, B) is a symmetric gap of F then cf(A, B) = N₁.

Note that by the Theorem 1.1 every two fields from this class are isomorphic iff the groups of archimedean classes of the fields are isomorphic.
In section 2 we investigate asymmetric gaps of special fields from the class $\mathcal{K}$ and show that the class $\mathcal{K}$ is strictly wider than a class of all $\eta_1$-fields of cardinality $\aleph_1$. We consider also examples of fields from the class $\mathcal{K}$ that have an asymmetric $(\aleph_1, \aleph_1)$ gap (in particular, a nonstandard real line).

In section 3 we consider Dedekind and non-Dedekind symmetric gaps of fields with cofinality $\aleph_0$ from $\mathcal{K}$ (in particular, semi-$\eta_1$-fields); prove that the class $\mathcal{K}$ is wider than a class of all semi-$\eta_1$-fields of cardinality $\aleph_1$; using [1] we show that super real fields are in our class $\mathcal{K}$.

2. On asymmetric gaps in some fields of formal power series.

By definitions of symmetric gap and $\eta_1$-set, we evidently have the following

**Proposition 2.1.** $F$ is an $\eta_1$-field iff each gap $(A, B)$ of $F$ has only one of the following types $(\aleph_1, \aleph_1)$, $(\aleph_0, \aleph_1)$, $(\aleph_1, \aleph_0)$, $(1, \aleph_1)$, $(\aleph_1, 1)$ and $\text{cf} (F) = \aleph_1$.

In [4] it was shown that if $F$ is a totally ordered real closed $\eta_1$-field with $\text{card}(F) = \aleph_1$ then $F \in \mathcal{K}$.

Our aim here is to show that the class $\mathcal{K}$ is strictly wider than the class of all $\eta_1$-fields of cardinality $\aleph_1$. To this end we give examples of fields from the class $\mathcal{K}$ with $(\aleph_0, \aleph_0)$-asymmetric gaps. Note that any symmetric gap of $F \in \mathcal{K}$ has type $(\aleph_1, \aleph_1)$.

Denote by $\mathbb{R}[[G]]$ a field of formal power series $x = \sum_{g \in G} r_g g$, where $r_g \in \mathbb{R}, \text{supp}(x) = \{g \in G | r_g \neq 0\}$ is inversely well-ordered subset of a totally ordered group $G$ (i.e., each subsets of supp($x$) has a maximal element). The order in $\mathbb{R}[[G]]$ is as follows: $x > 0$ if $r_\gamma > 0$, where $\gamma = \max \text{supp}(x)$. Let $\beta$ be a regular cardinal with $\aleph_0 < \beta \leq \text{card}(G)$. By $R[[G, \beta]]$ is denoted a subfield of $\mathbb{R}[[G]]$, which consists of such formal power series $x$ that $\text{card(\text{supp}(x)}) < \beta$ (the field of bounded formal power series)[1, 2].

If $G$ is divisible group then $R[[G, \beta]]$ are real-closed fields; if $\text{card}(G) \geq c$ then $\text{card}(\mathbb{R}[[G, \aleph_1]]) = \text{card}(G)$; if $\text{card}(G) \geq \aleph_0$ then $2^{\text{cf}(G)} \leq \text{card}(\mathbb{R}[[G]]) \leq 2^{\text{card}(G)}$ (see [1]). So, if $\aleph_1 = c = \text{card}(G)$, we have $\text{card}(\mathbb{R}[[G, \aleph_1]]) = \text{card}(G) = \aleph_1$.

We assume CH for the following description of $\mathcal{K}$ by means of fields of bounded formal power series[4, 3]: the class $\mathcal{K}$ coincides with a class of all fields of bounded formal power series $\mathbb{R}[[G, \aleph_1]]$, where $G$ is a totally ordered divisible Abelian group and $\text{card}(G) = \aleph_1$. The cofinality of a field $F \in \mathcal{K}$ and the cofinality of its group of archimedean classes are the same.

Let our class $\mathcal{K} = \mathcal{K}^0 \cup \mathcal{K}^1$, where $F \in \mathcal{K}^i$ if $\text{cf}(F) = \aleph_i$ ($i \in \{0; 1\}$).
Proposition 2.2. If $F \in \mathcal{K}^1$ then $F$ has a symmetric gap.

Proof. If $F \in \mathcal{K}^1$ then $F \cong \mathbb{R}[[\hat{F}, \mathcal{N}_1]] \subset \mathbb{R}[[\hat{F}]]$ and under CH, $\text{card}(\mathbb{R}[[\hat{F}, \mathcal{N}_1]]) = \aleph_1 < \text{card}(\mathbb{R}[[\hat{F}]]) = 2^{\aleph_1}$. Hence $\mathbb{R}[[\hat{F}]] \setminus \mathbb{R}[[\hat{F}, \mathcal{N}_1]] \neq \emptyset$ and $F$ has a symmetric gap (see Proposition 2.1 from [3]). □

Lemma 2.1. Let $F$ be a real closed field. $(A, B)$ is an $(\alpha, \beta)$ gap in $\hat{F}$, where $\alpha, \beta$ are infinite regular cardinals. Then there exists an asymmetric $(\alpha, \beta)$ gap in $F$.

Proof. Let $A_1 = \{x \in F| \exists g \in A, x < g\}, B_1 = \{x \in F| \exists g \in B, g < x\}$. If $x \in F$ then $x \in \hat{F} = A \cup B$. If $x \in A$ then $\exists g \in A$ (there is no the last element in $A_1$ because of $\text{cf}(A) = \alpha$ is infinite) such that $x < g$. Thus $x \in A_1$. By the same argument, if $x \in B$ then $\exists g \in B$, such that $g < \hat{x}$ and $x \in B_1$. It is obvious that $\text{cf}(A) = \text{cf}(A_1) = \alpha$, $\text{coi}(B) = \text{coi}(B_1) = \beta$. Hence $(A_1, B_1)$ is a $(\alpha, \beta)$ gap in $F$. Let $x_0 \in A_1 \subset F$ and $x_0 < x, x \in A$. Consider $x + (x - x_0) = 2x - x_0$. We have $2x - x_0 = \max\{\hat{x}, \hat{x_0}\} = \hat{x} \in A$. Therefore $x + (x - x_0) \in A_1$ and $(A_1, B_1)$ is asymmetric. □

Now we remain the construction of a group $(G(L, P), \cdot, <)$ with $\mathbb{R}[[G(L, P), \mathcal{N}_1]] \in \mathcal{K} [4]$. Let $L$ be a totally ordered set and $\text{cf}(L) \geq \aleph_0$. Let $P$ be a totally ordered infinite field and $\text{max}\{|L|, |P|\} = \aleph_1$. The totally ordered Abelian divisible group $(G(L, P), \cdot, <)$ is as follows: $G(L, P) = \{(t_{i_1}^{r_{i_1}} t_{i_2}^{r_{i_2}} \cdots t_{i_n}^{r_{i_n}})| t_{i_j} \in L, n, n \in \mathbb{N}\}$. We suppose that $t_{i_1} > t_{i_2} > \cdots > t_{i_n}$ and for given element $t_1 \in L$, let $t_1^r = 1 \forall r \in P$. Put $(t_{i_1}^{r_{i_1}}) \cdot (t_{i_2}^{r_{i_2}}) = (t_{i_1}^{r_{i_1}} t_{i_2}^{r_{i_2}})$ and $(t_{i_1}^{r_{i_1}}) \cdot (t_{i_2}^{r_{i_2}}) = (t_{i_1}^{r_{i_1}} t_{i_2}^{r_{i_2}})$. For example, $(t_{i_1}^{1/2} t_{i_2}^{-1}) \cdot (t_{i_1} t_{i_2} t_{i_3}) = (t_{i_1}^{3/2} t_{i_3})$. Let $g_1 = (t_{i_1}^{1} \cdots t_{i_k}^{1})$ by definition, put $g_1 < g_2 \iff g_1 g_2^{-1} < 1$ and $g_1 < 1 \iff r_{i_k} < 0$. For example, we compare $(t_{i_1}^{3/2} t_{i_2}^{-2} t_{i_3}^{5})$ and $(t_{i_1}^{3} t_{i_3})$. We have $(t_{i_1}^{3/2} t_{i_2}^{-2} t_{i_3}^{5}) \cdot (t_{i_1}^{3} t_{i_3}) = t_{i_2}^{-2} t_{i_3}^{5}, t_{i_2} > t_{i_3}, -2 < 0$ hence $(t_{i_1}^{3/2} t_{i_3}) < (t_{i_1}^{3} t_{i_3})$.

Note that the group $G(L, P)$ is isomorphic to the subgroup of finite sums $(P[[L, \mathcal{N}_1]], +, <)$ of the group of formal power series $P[[L, \mathcal{N}_1]]$. We show here that the group $P[[L, \mathcal{N}_0]]$ has an $(\aleph_0, \aleph_0)$ gap and so it is not a $\eta_1$-set.

Theorem 2.1. The group $G(L, P) \cong P[[L, \mathcal{N}_0]]$ has an $(\aleph_0, \aleph_0)$ gap.

Proof. Since $\aleph_0 \leq \text{cf}(L) \leq \aleph_1$ there exists a sequence $\{q_n\}_{n \in \mathbb{N}} \subset P[[L, \mathcal{N}_0]]$ such that $q_1 \gg q_2 \gg \cdots q_n \gg \cdots$ i.e. $\forall n \in N \forall i \in \mathbb{N} q_{i+1} \cdot n < q_i$. Let $\forall k \in \mathbb{N}$

$$a_k = q_1 + q_2 + \cdots + q_k; b_k = q_1 + q_2 + \cdots + q_{k-1} + \frac{2}{3}q_k.$$ 

$$A := \{g \in P[[L, \mathcal{N}_0]]| \exists n \in N \ g < a_n\}, \quad B := \{g \in P[[L, \mathcal{N}_0]]| \exists n \in N \ g > b_n\}.$$
Let us show that \((A, B)\) is a gap in \(P[[L, \aleph_0]]\). Suppose that there exists \(c \in P[[L, \aleph_0]]\) such that \(\forall k \in N \ a_k < c < b_k\). Let
\[
c = \gamma_1 h_1 + \gamma_2 h_2 + \cdots + \gamma_k h_k; h_i \in L, \gamma_i \in P.
\]
We claim that \(c = a_k\). Indeed for \(k = 1\), we have
\[
a_1 < c < b_1; \ q_1 < \gamma_1 h_1 + \gamma_2 h_2 + \cdots + \gamma_k h_k < \frac{2}{3} q_1 \Rightarrow h_1 = q_1.
\]
For \(k = 2\), we have
\[
a_2 < c < b_2; \ q_1 + q_2 < \gamma_1 q_1 + \gamma_2 h_2 + \cdots + \gamma_k h_k < q_1 + \frac{2}{3} q_2 \Rightarrow
\]
\[
\Rightarrow q_2 < (\gamma_1 - 1)q_1 + \gamma_2 h_2 + \cdots + \gamma_k h_k < \frac{1}{2} q_2 \Rightarrow
\]
\[
\Rightarrow \gamma_1 = 1, \ h_2 = q_2 \Rightarrow c = q_1 + \gamma_2 q_2 + \gamma_3 h_3 + \cdots + \gamma_k h_k.
\]
If for \(k = n\)
\[
c = q_1 + q_2 + \cdots + q_{n-1} + \gamma_n q_n + \gamma_{n+1} h_{n+1} + \cdots + \gamma_k h_k
\]
is true then for \(k = n+1\), we will have
\[
a_{n+1} < c < b_{n+1}; \ q_1 + q_2 + \cdots + q_{n+1} <
\]
\[
< q_1 + q_2 + \cdots + q_{n-1} + \gamma_n q_n + \gamma_{n+1} h_{n+1} + \cdots + \gamma_k h_k < q_1 + q_2 + \cdots + \frac{1}{2} q_{n+1} \Rightarrow
\]
\[
\Rightarrow q_{n+1} < (\gamma_n - 1)q_n + \gamma_{n+1} h_{n+1} + \cdots + \gamma_k h_k < \frac{1}{2} q_{n+1} \Rightarrow
\]
\[
\Rightarrow \gamma_n = 1, \ h_{n+1} = q_{n+1} \Rightarrow c = q_1 + q_2 + \cdots + q_n + \gamma_{n+1} q_{n+1} + \gamma_{n+2} h_{n+2} + \cdots + \gamma_k h_k.
\]
So, by induction, \(c = a_k\). It is a contradiction. \(\Box\)

**Corollary 2.1.** Field \(\mathbb{R}[[G(L, P), \aleph_1]]\) has an \((\aleph_0, \aleph_0)\) asymmetric gap.

**Proof.** By Lemma 2.1, the gap \((A, B)\) of \(G(L, P)\) from the proof of Theorem 2.1 generates the gap \((\hat{A}, \hat{B})\) in the field \(\mathbb{R}[[G(L, P), \aleph_1]]\), where
\[
\hat{A} = \{ x \in \mathbb{R}[[G(L, P), \aleph_1]] \mid \exists g \in A, x < 1 \cdot g \},
\]
\[
\hat{B} = \{ x \in \mathbb{R}[[G(L, P), \aleph_1]] \mid \exists g \in B, 1 \cdot g < x \}
\]
and this gap also has type \((\aleph_0, \aleph_0)\). \(\Box\)

**Corollary 2.2.** Group \(G(L, P)\) and field \(\mathbb{R}[[G(L, P), \aleph_1]]\) are not \(\eta_1\)-sets.
We consider also examples of fields from the class $\mathcal{K}$ that have an asymmetric $(\aleph_1, \aleph_1)$ gap.

Let $S$ and $T$ be two totally ordered sets. Then $S \odot T$ denotes the totally ordered set which is ‘$S$ followed by $T$‘: $\forall s \in S \forall t \in T \ s < t$ [1].

1) Let $L = \omega_1 \odot \omega_1^* \odot \omega_1^*$, where $\omega_1$ and $\omega_1^*$ are two copies of the ordinal $\omega_1$; $\omega_1^*$ is the ordinal $\omega_1$ with the inverse of the usual order. So the set $\omega_1 \odot \omega_1^*$ give us the $(\aleph_1, \aleph_1)$ gap in $L$, which generates the $(\aleph_1, \aleph_1)$ asymmetric gap in the field $\mathbb{R}[[G(L, P), \aleph_1]] \in \mathcal{K}$.

2) Now we consider a non-standard real line $^*\mathbb{R}$, which is an ultrapower of $\mathbb{R}$ by an $\aleph_1$-good ultrafilter over $\aleph_1$. It is known [1] that $^*\mathbb{R}$ is $\eta_1$-field and it is order-isomorphic to the field $\mathbb{R}[[G, \aleph_1]]$ of bounded formal power series with $G = \mathbb{R}[[Q, \aleph_1]]$ and $Q$ is Sierpinski’s set. $Q$ consists of dyadic sequences $\alpha = (\alpha_\tau)_{\tau < \omega_1}$ with lexicographic order such that $\{ \tau < \omega_1 : \alpha_\tau = 1 \}$ is non-empty and has a largest member.

Describe a $(\aleph_1, \aleph_1)$ gap in $Q$. Let $(a^\alpha)_{\sigma < \omega_1}$ be a sequence in $Q$ such that

\[ a^\alpha(\tau) = \begin{cases} 0, & \tau > \sigma \lor \tau \text{ is "even"}; \\ 1, & \tau \text{ is "odd"} \lor \tau \text{ is limit}. \end{cases} \]

So, we have $a^1 = (100\ldots0\ldots), a^2 = (101000\ldots0\ldots), a^3 = (10101000\ldots0\ldots), \ldots$, $a^\omega = (10101000\ldots|_{\omega}1000\ldots), a^{\omega+1} = (10101000\ldots|_{\omega}101000\ldots), \ldots$. It is an increasing sequence.

Let $(b^\sigma)_{\sigma < \omega_1}$ in $Q$ such that

\[ b^\sigma(\tau) = \begin{cases} 0, & \tau > 2\sigma + 2 \lor (\tau < 2\sigma + 2 \text{ and } \tau \text{ is "even"}); \\ 1, & \tau = 2\sigma + 2 \lor (\tau < 2\sigma + 2 \text{ and } \tau \text{ is "odd"}) \lor \tau \text{ is limit}. \end{cases} \]

That is $b^1 = (1011000\ldots0\ldots), b^2 = (101011000\ldots0\ldots), b^3 = (10101011000\ldots0\ldots), \ldots$, $b^{\omega} = (10101000\ldots|_{\omega}11000\ldots), b^{\omega+1} = (10101000\ldots|_{\omega}1011000\ldots), \ldots$. It is a decreasing sequence.

We see that $\forall \sigma < \omega_1 \forall \delta < \omega_1 \ a_\sigma < b_\delta$. Between these sequences there is the only dyadic sequence $(10101010\ldots1010\ldots)$ of length $\omega_1$. Both our sequences “converge” to $(101010\ldots1010\ldots) \notin Q$. Therefore the sequences generate a gap in $Q$. It is clearly, that the gap has type $(\aleph_1, \aleph_1)$. This gap generates the $(\aleph_1, \aleph_1)$ asymmetric gap in the field $\mathbb{R}[[G, \aleph_1]]$.

Remark 2.1. $^*\mathbb{R}$ has symmetric $(\aleph_1, \aleph_1)$ gaps [3] and it has asymmetric $(\aleph_1, \aleph_1)$ gaps as well.

3. Semi-$\eta_1$-super-real fields from the class $\mathcal{K}$. Dales and Woodin in [1] introduced a semi-$\eta_1$-field, which is generalization of $\eta_1$-field: a totally ordered field $F$ is called a semi-$\eta_1$-field if for each strictly increasing sequence
(s_n)_{n \in \mathbb{N}}$ and strictly decreasing sequence $(t_n)_{n \in \mathbb{N}}$ with $s_n < t_m \forall m, n \in \mathbb{N}$ there exists $x \in F$ such that $s_n < x < t_m \forall n, m \in \mathbb{N}$. It is easy to see that

**Proposition 3.1.** $F$ is a semi-$\eta_1$-field iff each gap $(A, B)$ of $F$ has only one of the following types $(\aleph_1, \aleph_1)$, $(\aleph_0, \aleph_1)$, $(\aleph_1, \aleph_0)$, $(\aleph_1, 1)$, $(1, \aleph_1)$, $(1, \aleph_0)$, $(\aleph_0, 1)$. Clearly, each $\eta_1$-field is a semi-$\eta_1$-field.

By Proposition 3.1 and Corollary 2.1, we obtain the following

**Proposition 3.2.** $\mathbb{R}[[G(L, P), \aleph_1]]$ (see section 2 of this paper) is not a semi-$\eta_1$-field.

**Note 3.1.** Each $\eta_1$-field of cardinality $\aleph_1$ belongs to $\mathcal{K}^1$.

A gap $(A, B)$ of a field $F$ is called a Dedekind gap or a fundamental gap if $\forall \varepsilon \in F^+$ there exist $x \in A, y \in B$ such that $|y - x| < \varepsilon$ [1, 6]. It is easy to see by the definition that each Dedekind gap without the first and the last elements is symmetric.

**Proposition 3.3** [3]. Let $F$ be a $\eta_1$-field with $\text{card}(F) = \aleph_1$. Then

(a) there exist $2^{\aleph_1}$ symmetric Dedekind gaps;

(b) there exist $2^{\aleph_1}$ symmetric non-Dedekind gaps;

(c) if $(A, B)$ is symmetric gap then $\text{cf}(A, B) = \aleph_1$.

**Theorem 3.1.** Let $F \in \mathcal{K}^0$ and $\mathbb{R}[[\widehat{F}]] \setminus \mathbb{R}[[\aleph_1]] \neq \emptyset$. Then

(a) there is no symmetric Dedekind gap in $F$;

(b) $F$ has $2^{\aleph_1}$ symmetric non-Dedekind gaps.

**Proof.** (a). Since $\mathbb{R}[[\widehat{F}]] \setminus \mathbb{R}[[\aleph_1]] \neq \emptyset$, by Proposition 2.1. from [3], $F$ has a symmetric gap. A symmetric gap $(A, B)$ is Dedekind iff (see Proposition 2.2. from [3]) $\exists x_0 \in R[[\widehat{F}]] \setminus R[[\aleph_1]] A < x_0 < B$ such that

(*) $\text{supp}(x_0)$ is inversely order-isomorphic to $\aleph_1$ and coinitial in $\widehat{F}$.

Since $\text{cf}(F) = \aleph_0$ then $\text{cf}(\widehat{F}) = \aleph_0$. Therefore if $x_0 \in R[[\widehat{F}]] \setminus R[[\aleph_1]]$ and $\text{supp}(x_0)$ is coinitial in $\widehat{F}$ then $\text{coi}(\text{supp}(x_0)) = \aleph_0$. Whence $\text{supp}(x_0)$ is not inversely order-isomorphic to $\aleph_1$. So by (*), $(A, B)$ is not Dedekind.

(b). Let $(A, B)$ be a symmetric non-Dedekind gap. Then (see Proposition 2.1. from [3]) $\exists x_0 \in R[[\widehat{F}]] \setminus R[[\aleph_1]]$ and (*) holds. Let $x_0 = \sum_{g \in \mathbb{F}} r_g g$. Put $r_g = x_0(g)$. Since $\text{supp}(x_0) = \{ g \in \widehat{F} \mid x_0(g) \neq 0 \}$ is inversely well-ordered subset of $\widehat{F}$ and $\text{card}(\text{supp}(x_0)) = \aleph_1$, there exists $\Gamma \subset \widehat{F}$ with $\text{card}(\Gamma) = \aleph_1$ and $\Gamma$ is inversely order-isomorphic to $\aleph_1$. By (*) and $\text{coi}(\widehat{F}) = \aleph_0$, $\Gamma$ is not coinitial in $\widehat{F}$. 


Denote by $S$ the set of all $x \in \mathbb{R}[[F]] \setminus \mathbb{R}[[\hat{F}, \aleph_1]]$ such that $\text{supp } x = \Gamma$. Each $x \in S$ generates a symmetric non-Dedekind gap in $\mathbb{R}[[\hat{F}, \aleph_1]]$. Let $x_1, x_2 \in S$ and $x_1 < x_2$. Denote by $(A_i, B_i)$ gaps in $\mathbb{R}[[\hat{F}, \aleph_1]]$, which produced by $x_i$ $(i = 1, 2)$. $A_i = \{ x \in \mathbb{R}[[\hat{F}]] \mid x < x_i \}$, $B_i = \{ x \in \mathbb{R}[[\hat{F}]] \mid x > x_i \}$. Prove that the gaps $(A_1, B_1), (A_2, B_2)$ are different.

There exists $g_0 = \max\{ g \in \hat{F} \mid x_1(g) \neq x_2(g) \}$. Define $x_3 \in \mathbb{R}[[\hat{F}, \aleph_1]]$ such that $\text{supp}(x_3) = \{ g \in \Gamma \mid g \geq g_0 \}$ and

1) $x_3(g) = x_1(g) = x_2(g)$ if $g > g_0$,

2) $x_3(g_0) = \frac{1}{2}(x_1(g_0) + x_2(g_0))$.

Since $\Gamma$ is inversely order-isomorphic to $\aleph_1$, we have $\text{card} (\text{supp}(x_3)) = \aleph_0$. Hence $x_3 \in \mathbb{R}[[\hat{F}, \aleph_1]]$. Evidently $x_1 < x_3 < x_2$. Therefore $x_3 \in B_1$ and $x_3 \in A_2$. So, $A_1 \neq A_2$. Thus $x_1, x_2$ produce different gaps in $\mathbb{R}[[\hat{F}, \aleph_1]]$.

The cardinality of the set of all formal power series with support $\Gamma$ equals $\mathbb{R}^{\text{card} \Gamma} = 2^{\text{card} \Gamma} = 2^{\aleph_1}$. So, the cardinality of the set of all symmetric non-Dedekind gaps is not less then $2^{\aleph_1}$. On the other hand, this is not greater then $2^{\aleph_1}$ since $\mathbb{R}[[\hat{F}, \aleph_1]]$ has at most $2^{\aleph_1}$ gaps. $\square$

**Proposition 3.4.** Let $F$ be a semi-$\eta_1$-field, which is not $\eta_1$-field with $\text{card}(F) = \aleph_1$. Then

a) $\text{coi}\{ y \in \hat{F} : y > 1 \} = \aleph_1$;

b) $F \in \mathcal{K}^0$.

**Proof.** a) If $y \in \hat{F}$ then the set $y$ is an archimedean class and $\text{cf}(y) = \aleph_0$. Consider a gap $(A, B)$ in $F$ with $A = \{ x \in F \mid \widehat{x} \leq \widehat{1} \}$, $B = F \setminus A$. We have $\text{cf}(A) = \text{cf}(\widehat{1}) = \aleph_0$. Since $F$ is a semi-$\eta_1$-field, $\text{coi}(B) = \aleph_1$. Since there is no the first element in $B$, $\text{coi}(B) = \text{coi}(\widehat{B}) = \aleph_1$. Thus $\text{coi}(\widehat{B}) = \text{coi}\{ y \in \hat{F} : y > 1 \} = \aleph_1$.

b) Let $(A, B)$ be a symmetric gap in $F$. Put $\text{cf}(A, B) = \alpha$ then $(A, B)$ has type $(\alpha, \alpha)$. Since $F$ is a semi-$\eta_1$-field, $\alpha = \aleph_1$. By a), we have $\text{card}(\hat{F}) = \aleph_1$. Therefore $F \in \mathcal{K}$. Suppose that $\text{cf}(F) = \aleph_1$ then (by Proposition 3.1. from [4]) there are no $\{ (1, \aleph_0), (\aleph_0, 1) \}$ gaps in $F$ and hence $F$ is $\eta_1$-field. It is a contradiction. Thus $F \in \mathcal{K}^0$. $\square$

**Corollary 3.1.** Let $F$ be a semi-$\eta_1$-field, which is not $\eta_1$-field with $\text{card}(F) = \aleph_1$. Then

a) there is no symmetric Dedekind gap in $F$;

b) there exist $2^{\aleph_1}$ symmetric non-Dedekind gaps;

c) if $(A, B)$ is symmetric gap then $\text{cf}(A, B) = \aleph_1$.

**Proof.** (a) and (b) are consequences of Proposition 3.4 and Theorem 3.1. $\square$
Remark 3.1. Let us compare our results with the following. [1, p.56, Corollary 2.35] Let $F$ be a real-closed semi-$\eta_1$-field, which is a $\beta_1$-field. Then exactly one of the following occurs:

(I). $w(F) = \aleph_0$, and $F \cong \mathbb{R}$;

(II). $\text{cf}(F) = \aleph_1$, and $F \cong \mathbb{R}$ ($\eta_1$-field);

(III). $\text{cf}(\hat{F}) = (\aleph_0, 1)$, and then $F \cong \mathbb{R}[[\mathbb{R} \times G \times \mathbb{R}, \aleph_1]]$;

(IV). $\text{cf}(\hat{F}) = (\aleph_0, \aleph_0)$, and then $F \cong \mathbb{R}[[\mathbb{C}_0(G^N), \aleph_1]]$. (See [1] for details).

Under CH, $F$ is $\beta_1$-field iff $\text{card}(F) = \aleph_1$.

Note that at the present paper we describe all symmetric gaps of the fields (II)--(IV).

Now we consider question about existence a super-real field in our class $\mathcal{K}$.

Let $X$ be a completely regular topological space and $C(X)$ be an algebra of continuous functions on $X$. Let $P$ be a prime ideal in $C(X)$. $C(X)/P := A_P$ is a totally ordered commutative algebra. The quotient map from $C(X)$ onto $A_P$ is denoted by $\pi_P$. Since $f = f^+ + f^-$ and $f^+ \cdot f^- = 0 \in P$, we have $a = \pi_P(f) \geq 0$ if $f \in f^+ + P$, $f^- \in P$.

The quotient fields of $A_P$ is denoted by $K_P$ and is called a super-real field (it is not equal to $\mathbb{R})[1]$. It is known (see [1], p.96-98) that each of possibilities (II)--(IV) from the Remark 3.1 actually occurs in the class of super-real fields (ZFC+CH). Hence, we have

Corollary 3.2. There are semi-$\eta_1 + \beta_1$-super-real fields that belong to the class $\mathcal{K}$.

Question. Is there a $\beta_1$-super-real field which is not semi-$\eta_1$ in the class $\mathcal{K}$?

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