Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **30** (2004), 505–512

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ESSENTIAL COVER AND CLOSURE

R. R. Andruszkiewicz

Communicated by V. Drensky

Abstract. We construct some new examples showing that Heyman and Roos construction of the essential closure in the class of associative rings can terminate at any finite or the first infinite ordinal.

1. Introduction. All rings considered in this paper are associative, but do not necessarily have a unity. We say that a subring A of a ring R is *essential* in R if $A \cap I \neq 0$ for every non-zero ideal I of R. We use the notation $I \triangleleft R$ to mean that I is an ideal of a ring R. If an ideal A of a ring R is essential, then R is said to be an *essential extension* of A. The two-sided annihilator of a subset X of a ring R is defined to be the subring $a_R(X) = \{r \in R : rX = Xr = 0\}$.

All considered classes of rings are assumed to be closed under isomorphisms. For a given ring R, we denote by $\{R\}$ the class of all isomorphic copies of R. Now let $\mathcal{M}$ be an arbitrary class of rings. The class $\mathcal{EM}$ consisting of all essential extensions of rings belonging to $\mathcal{M}$, shall be called the *essential cover* of the class $\mathcal{M}$. The class $\mathcal{M}$ will be called *essentially closed* if $\mathcal{M} = \mathcal{EM}$. Each

2000 *Mathematics Subject Classification:* 16N80, 16S70, 16D25, 13G05.
Key words: essential extension, essential cover, essential closure, accessible subring.

class $\mathcal{M}$ can be embedded in an essentially closed class, namely as follows (cf. [8]): define $\mathcal{M}^{(0)} = \mathcal{M}$ and $\mathcal{M}^{(i+1)} = \mathcal{E}\mathcal{M}^{(i)}$ for each integer $i \geq 0$. Then the class $\mathcal{M}^c = \bigcup_{i=1}^{\infty} \mathcal{M}^{(i)}$ is essentially closed, it contains the class $\mathcal{M}$, and moreover, it is contained in every other essentially closed class containing $\mathcal{M}$. The class $\mathcal{M}^c$ shall be called the *essential closure* of the class $\mathcal{M}$. Many authors have used the notions of essential cover and essential closure in their characterizations of the supernilpotent and special radicals. Naturally, one is interested in conditions on $\mathcal{M}$ so that $\mathcal{M}^c = \mathcal{M}^{(n)}$, for some integer $n \geq 0$. There has been considerable interest in the following problems (cf. [7, Problem D]):

PROBLEM 1. For every integer $n \geq 0$ determine necessary and/or sufficient conditions on a class $\mathcal{M}$ such that $\mathcal{M}^c = \mathcal{M}^{(n+1)} \neq \mathcal{M}^{(n)}$.

PROBLEM 2. Determine necessary and/or sufficient conditions on a class $\mathcal{M}$ such that $\mathcal{M}^{(n+1)} \neq \mathcal{M}^{(n)}$, for every integer $n \geq 0$.

In this paper we study these problems for classes of rings with trivial two-sided annihilators. It is convenient to describe the essential closures of such classes in terms of accessible subrings. Namely, we say that a subring A of a ring R is *n-accessible* in R if there exists a chain of subrings $A_0 = A \subseteq A_1 \subseteq \cdots \subseteq A_n = R$ such that $A_i \lhd A_{i+1}$ for $i = 0, 1, \ldots, n-1$, and A is said to be *precisely n-accessible* if it is not k-accessible in R for any non-negative integer $k < n$. We say that A is *accessible* in R if there exists a positive integer n such that A is n-accessible. It is well known that a subring A of a commutative ring R is n-accessible in R if and only if $RA^n \subseteq A$.

Lemma 1.1 [2]. *Let A be an accessible subring of a ring R such that $a_A(A) = 0$. Then A is essential in R if and only if $a_R(A) = 0$. In particular, if A is essential in R, then A is essential in every subring of R containing A.*

2. Some general results.

By easy induction and Lemma 1.1 we obtain the following

Proposition 2.1. *Let $\mathcal{M}$ be a class of rings A such that $a_A(A) = 0$. Then*

$$\mathcal{M}^{(n)} = \{R : R \text{ has an essential } n\text{-accessible subring } A \in \mathcal{M}\}$$

for every integer $n \geq 0$.

Definition 2.2 [6]. *We say that a ring R is an* iterated maximal essential extension *of a ring A and write $R = \mathrm{IME}(A)$ if A is an essential accessible subring* *of R and, for every ring S in which A is accessible, there exists a homomorphism of S into R which is the identity map on A.*

As an immediate consequence of [2, Theorem 2.1, Proposition 2.3], and Lemma 1.1, we obtain the following

Theorem 2.3. *Let $R = \mathrm{IME}(A)$. Suppose A is an n-accessible subring of R. Then $\{A\}^c = \{A\}^{(n)}$ is the class of all isomorphic copies of all subrings of R containing A.*

Let P be an integral domain with K the field of fractions. We say that P is a *completely normal ring* (cf. [4]) if for any $x \in K$ and $0 \neq a \in P$, we have that $x^n a \in P$ for $n = 1, 2, \ldots$, implies $x \in P$. It is well known that a noetherian integral domain P is a completely normal ring if and only if P is an integrally closed.

Corollary 2.4. *Suppose A is a non-zero precisely n-accessible subring of a completely normal ring P. Then the class $\{A\}^{(n)}$ is essentially closed, $\{A\}^{(n)} \neq \{A\}^{(n-1)}$ and $\{A\}^{(n)}$ is the class of all isomorphic copies of all subrings of P containing A.*

Proof. From Proposition 2.1 we obtain that $P \in \{A\}^{(n)}$. Assume that $P \in \{A\}^{(n-1)}$. By Proposition 2.1, there exists an $(n-1)$-accessible subring $B \cong A$ in P, which contradicts [3, Corollary 3.5]. Thus, $P \in \{A\}^{(n)} \setminus \{A\}^{(n-1)}$ and $\{A\}^{(n)} \neq \{A\}^{(n-1)}$. By [2, Proposition 2.18] it follows that $P = \mathrm{IME}(A)$. Therefore, by Theorem 2.3, $\{A\}^c = \{A\}^{(n)}$ is the class of all isomorphic copies of all subrings of P containing A. $\square$

For a semiprime ring A, we denote by $Q(A)$ the ring of right Martindale fractions determined by the filter of all essential ideals of A.

Proposition 2.5. *Let A be a semiprime ring. Then $a_{Q(A)}(A) = 0$. If A is an essential accessible subring in a ring S, then there exists an embedding of rings $f : S \longrightarrow Q(A)$ such that $f(a) = a$ for all $a \in A$.*

Proof. If $q \in Q(A)$ and $qA = 0$, then $q = 0$ and so $a_{Q(A)}(A) = 0$. Assume that A is an essential accessible subring of a ring S. Denote by A_S the ideal of S generated by A. It is well known that there exists a positive integer m such that $A_S^m \subseteq A$. On the other hand, A^m is an essential ideal of A, because A is a semiprime ring. Hence $I = A_S^m$ is an essential ideal of A and $I \triangleleft S$. Let $l_s(a) = sa$ for $s \in S$ and $a \in I$. Then l_s is a homomorphism of the right A-module I into A. It is easily seen that the function $F : S \longrightarrow Q(A)$ defined by $F(s) = [l_s, I]$ is an embedding of rings such that $F(a) = a$ for all $a \in A$. $\square$

From Propositions 2.1 and 2.5 and Lemma 1.1 we obtain the following

Theorem 2.6. *Let A be a semiprime ring. Then $\{A\}^c$ is the class of all isomorphic copies of all subrings S of $Q(A)$, in which A is accessible.*

By [3, Proposition 3.1] and Theorem 2.6 we have immediately the following corollary.

Corollary 2.7. *Let A be a non-zero subring of a completely normal ring P. Then $\{A\}^c$ is the class of all isomorphic copies of all subrings of P in which A is accessible.*

Definition 2.8. *We say that a subring A of a ring R has a* bounded degree of accessibility *in R if there exists a positive integer n such that for every subring S of R in which A is an accessible subring, A is n-accessible in S. We say that n bounds the degree of accessibility of A in R.*

By Theorem 2.6 and Lemma 1.1 we have

Proposition 2.9. *If n bounds the degree of accessibility of a semiprime ring A in $Q(A)$, then the class $\{A\}^{(n)}$ is essentially closed.*

Example 2.10. Let P be a completely normal ring with K the field of fractions. Assume that there exists a non-zero element $a \in P$ such that $Pa \neq P$ is a semiprime ideal in P. Let $A = M_2(Pa^2)$. It follows from [2, Example 2.19] that 2 bounds the degree of accessibility of A in $Q(A)$ and $\mathrm{IME}(A)$ does not exist. Therefore, by Proposition 2.9 there exists a prime ring A for which $\mathrm{IME}(A)$ does not exist but the class $\{A\}^{(2)}$ is essentially closed.

Example 2.11. We show that there exists a prime ring A which does not have a bounded degree of accessibility in $Q(A)$, but the class $\{A\}^{(1)}$ is essentially closed. Let p be a prime number. Let in the ring $\mathbb{Q}[x]$: $P = \mathbb{Z} + \mathbb{Z}x + x^2\mathbb{Q}[x]$ and $A = p\mathbb{Z} + \mathbb{Z}x + x^2\mathbb{Q}[x]$. Then P is a subring of $\mathbb{Q}[x]$ and $A \triangleleft P$. Let $A_i = p\mathbb{Z} + \dfrac{1}{p^i}\mathbb{Z}x + x^2\mathbb{Q}[x]$ for $i = 0, 1, \ldots$. It is easily seen that $A_i \triangleleft A_{i+1}$ for $i = 0, 1, \ldots$ and $A_0 = A$. Thus, A is n-accessible in A_n for all $n = 1, 2, \ldots$. Suppose A is $(n-1)$-accessible in A_n for some positive integer n, then $A_n A^{n-1} \subseteq A$, so $\dfrac{1}{p^n}x \cdot p^{n-1} \in A$, a contradiction. Therefore A does not have a bounded degree of accessibility in $Q(A)$.

Suppose now that A is accessible in a subring S of $\mathbb{Q}[x]$. Then there exists a positive integer n such that $SA^n \subseteq A$. Hence $Sp^n \subseteq A$, that is $S \subseteq \dfrac{1}{p^n}A$. Let $f \in S$. For $k = 1, 2, \ldots$, we have $f^k p^n \in A$, which means in particular that $f(0)^k p^n \in \mathbb{Z}$. Notice that $\mathbb{Z}$ is a completely normal ring, so $f(0) \in \mathbb{Z}$. Consequently, $S \subseteq \mathbb{Z} + \frac{1}{p^n}\mathbb{Z}x + x^2\mathbb{Q}[x]$.

Suppose now that $S \not\subseteq A_n$. Then there exists an integer b_0 such that $p \not| b_0$ and there exist $b_1 \in \mathbb{Z}$ and $g \in x^2\mathbb{Q}[x]$ such that $h = b_0 + b_1\frac{1}{p^n}x + g \in S$. So $w = b_0 + \frac{b_1}{p^n}x \in S$. Since $w^p = b_0^p + \frac{b_0^{p-1}b_1}{p^{n-1}}x + g_1$ for some $g_1 \in x^2\mathbb{Q}[x]$, then $b_0^p + \frac{b_0^{p-1}b_1}{p^{n-1}}x \in S$. By easy induction we can show that $b_0^{p^n} + b_1'x \in S$, for some integer b_1', so $b_0^{p^n} \in S$. But $p \not| b_0$ and $p \in S$, then $1 \in S$. Therefore $I = \left\{k \in \mathbb{Z} : \frac{k}{p^n}x \in S\right\}$ is a non-zero ideal of $\mathbb{Z}$, there is an integer l such that $I = l\mathbb{Z}$. Consequently, $S = \mathbb{Z} + \frac{l}{p^n}\mathbb{Z}x + x^2\mathbb{Q}[x]$. Let ϕ be an automorphism of the ring $\mathbb{Q}[x]$ such that $\phi(x) = \frac{p^n}{l}x$. Thus, $\phi(S) = P$ and $S \in \{A\}^{(1)}$.

If $S \subseteq A_n$, then a similar argument shows that $S = p\mathbb{Z} + \frac{l}{p^n}\mathbb{Z}x + x^2\mathbb{Q}[x]$ for some positive integer l. Consequently, $S \cong A$ and $S \in \{A\}^{(0)}$. Therefore, $\{A\}^c = \{A\}^{(1)} = \{A\} \cup \{P\}$, by Corollary 2.7.

Example 2.12. Let (p_n) be an ascending sequence of prime numbers. In the field $\mathbb{Q}$ of rational numbers we define the subrings $S_i = \left\langle \frac{1}{p_1}, \dots, \frac{1}{p_i} \right\rangle$ for $i = 1, 2, \dots$. Consider A_n in the ring $\mathbb{Q}[x]$ of the form:

$$A_n = S_{n+1}x + S_{n+2}x^2 + S_{n+3}x^3 + \cdots,$$

for $n = 0, 1, \dots$. Observe that

$$(S_{n+i}x^i)(S_{n+1+j}x^j) \subseteq S_{n+i+j}x^{i+j} \subseteq A_n,$$

for $n = 0, 1, \dots$, $i, j = 1, 2, \dots$. Thus, A_n is a subring of $\mathbb{Q}[x]$ and $A_n \lhd A_{n+1}$ for $n = 0, 1, \dots$. Therefore $A = A_0$ is an n-accessible subring in A_n for $n = 1, 2, \dots$. Then $A_n \in \{A\}^{(n)}$ for $n = 1, 2, \dots$, by Proposition 2.1. Suppose now that there exists a positive integer n such that $A_n \in \{A\}^{(n-1)}$. By Proposition 2.1, A_n contains a $(n-1)$-accessible subring $B \cong A$. Let $f : A \longrightarrow B$ be a ring isomorphism. Then

$$f(a_1x + a_2x^2 + \cdots + a_kx^k) = a_1f(x) + a_2f(x)^2 + \cdots + a_kf(x)^k,$$

for all $a_i \in S_i$, $i = 1, 2, \dots, k$, $k = 1, 2, \dots$. Moreover, B is $(n-1)$-accessible in A_n, so $A_n B^{n-1} \subseteq B$ and in particular $A_n f(x)^{n-1} \subseteq B$. But $f(x) = b_{n+1}x + b_{n+2}x^2 + \cdots + b_{n+s}x^{n+s}$ for some $b_{n+i} \in S_{n+i}$, $i = 1, 2, \dots, s$, and $b = b_{n+s} \neq 0$. Now $xf(x)^{n-1} \in B$, and there exist $a_i \in S_i$, $i = 1, 2, \dots, k$, such that $a_k \neq 0$ and $xf(x)^{n-1} = a_1 f(x) + a_2 f(x)^2 + \cdots + a_k f(x)^k$. Therefore, $1 + (n-1)s = sk$ which implies $s = 1$, $n = k$ and $b^{n-1} = a_n b^n$, that is $a_n b = 1$. But $a_n \in S_n \subseteq S_{n+1}$ and $b \in S_{n+1}$, so $S_{n+1}b = S_{n+1}$. Moreover, $f(x) = bx$, thus $B = f(A) = S_1 bx + S_2 b^2 x^2 + \cdots$ and $A_n f(x)^{n-1} \subseteq B$. Hence $S_{n+1} x b^{n-1} x^{n-1} \subseteq B$, and then we get $S_{n+1} b^{n-1} \subseteq S_n b^n$ and $S_{n+1} \subseteq S_n b^n \subseteq S_{n+1} b = S_{n+1}$. This means $S_n b = S_{n+1} b$, that is $S_{n+1} = S_n$, a contradiction. Thus $A_n \notin \{A\}^{(n-1)}$ for $n = 1, 2, \dots$. Therefore $\{A\}^{(n)} \neq \{A\}^{(n+1)}$ for all $n = 0, 1, \dots$.

Recall that a class $\mathcal{M}$ is called *hereditary* if $A \lhd R \in \mathcal{M}$ implies $A \in \mathcal{M}$. Heyman and Roos in [8] proved that the essential cover of a hereditary class $\mathcal{M}$ of semiprime rings is again hereditary and essentially closed. In [1] it is shown that if $\mathcal{M}$ is a class of semiprime rings and regular (that is, if $R \in \mathcal{M}$ and $0 \neq A \lhd R$, then A has a non-zero image in $\mathcal{M}$) rather than hereditary, then so is $\mathcal{EM}$, but the authors comment that they do not know a regular class $\mathcal{M}$ of semiprime rings for which $\mathcal{EM}$ is not essentially closed. Watters in [9] constructed such a class. However Beidar in [5] proved that for each integer $n \geq 0$ there exists a regular class $\mathcal{M}$ of prime rings such that the process of constructing the essential closure of $\mathcal{M}$ terminates at precisely the n-th step. Now we construct new general examples of such classes. It also allows us to construct essential closure which satisfies some extra properties.

Example 2.13. We show that there exists a regular class $\mathcal{M}$ of commutative domains such that $\mathcal{M}^{(n)} \neq \mathcal{M}^{(n+1)}$ for every $n = 0, 1, \dots$. Let (p_n) and (q_n) be ascending sequences of prime numbers such that $p_i \neq q_j$ for all i, j. Let A be a ring as in Example 2.12 and let $\mathcal{P}$ be the class of all commutative domains of non-zero characteristic. It is easily seen that the class $\mathcal{P}$ is hereditary and essentially closed. Let $\mathcal{M} = \{A\} \cup \mathcal{M}$ and let $0 \neq I \lhd A$. Then there exists a positive integer t such that $I \not\subseteq q_t A$, because $\bigcap_{i=1}^{\infty} q_i A = 0$. Then $\overline{I} = I/(I \cap (q_t A)) \neq 0$ and $q_t \overline{I} = 0$. Suppose that $\overline{I}$ is not a domain. Thus, there exist $f, g \in I \setminus (q_t A)$ such that $fg \in q_t A$. Therefore there exists a positive integer n such that $f, g \in S_n[x]$ and $fg \in q_t S_n[x]$. But q_t is a prime element in a Euclidean domain S_n, then $f \in q_t S_n[x]$ or $g \in q_t S_n[x]$, by Gauss' Lemma. There is no loss of generality in assuming that $f \in q_t S_n[x]$ and $f = a_1 x + a_2 x^2 + \cdots + a^n x^n$ for some $a_i \in S_i$, $i = 1, 2, \dots, n$. Then $a_i = q_t b_i$ for some $b_i \in S_n$, $i = 1, 2, \dots, n$. But S_i is a Euclidean domain and $S_i \subseteq S_n$, then $b_i \in S_i$ for $i = 1, 2, \dots, n$. Therefore $f \in q_t A$, a contradiction. Thus $\mathcal{M}$ is a regular class of commutative domains. It follows from Example 2.12 that $A_n \in \mathcal{M}^{(n)}$ for $n = 0, 1, \ldots$. If $A_n \in \mathcal{M}^{(n-1)}$ for some positive integer n, then there exists an $(n-1)$-accessible subring $B \in \mathcal{M}$ in A_n, by Proposition 2.1. But the additive group of the ring A_n is torsion-free, then $B \cong A$, contrary to Example 2.12. Therefore $\mathcal{M}^{(n)} \neq \mathcal{M}^{(n+1)}$ for all $n = 0, 1, \ldots$.

Lemma 2.14. *Let P be an integral domain and let $a \in P$ be a non-zero element such that $P \neq \langle 1\rangle + Pa$. Let S be a subring of P containing a non-zero accessible subring A of P. If $0 \neq I \triangleleft S$, then there exists an element $b \in I$ such that $Pb \subseteq I$ and $\langle b\rangle + Pb^n$ is precisely an n-accessible subring of P for every positive integer n.*

Proof. There is a positive integer m such that $PA^m \subseteq A$. Then $0 \neq B = IA^m \subseteq A \cap I$ and $B \triangleleft A$. Thus B is accessible in P and $PB^{m+1} \subseteq B$. Let $0 \neq c \in B^{m+1}$. Then $0 \neq Pac \subseteq Pa$. Since $P \neq \langle 1\rangle + Pa$, we obtain $P \neq \langle 1\rangle + Pac$. Hence, by [3, Proposition 2.2], there exists $b \in Pac$ such that the subring $\langle b\rangle + Pb^n$ is precisely n-accessible in P for every positive integer n. But $Pb \subseteq Pac \subseteq PB^{m+1} \subseteq B \subseteq I$, and the result follows. $\square$

Example 2.15. For every positive integer n there exists a regular and not hereditary class $\mathcal{M}$ of commutative domains such that $\mathcal{M}^c = \mathcal{M}^{(n)} \neq \mathcal{M}^{(n-1)}$ and the class $\mathcal{M}^c$ is hereditary. Let P be a completely normal and Jacobson semisimple ring. Assume there exists a non-zero element $a \in P$ such that $P \neq \langle 1\rangle + Pa$. Denote by $\mathcal{K}$ the class of all fields and let $\mathcal{P} = \{0\} \cup \mathcal{K}$. Then $\mathcal{P}$ is a hereditary and essentially closed class of commutative domains. Moreover, $B \notin \mathcal{P}$ for all non-zero accessible subrings B of P, because P is not a field. Let n be an arbitrary positive integer. Denote by $\mathcal{A}$ the set of all precisely n-accessible subrings of P. Then $\mathcal{A}$ is non-empty, by Lemma 2.14. Let $0 \neq I \triangleleft A \in \mathcal{A}$. Then there exists a maximal ideal M of P such that $I \not\subseteq M$. So, $0 \neq I/(I \cap M) \cong (I+M)/M$ and $(I+M)/M$ is a non-zero accessible subring of the field P/M. Thus $I/(I \cap M) \in \mathcal{A}$. Therefore $\mathcal{M} = \{\mathcal{A}\} \cup \mathcal{P}$ is a regular class of commutative domains. But $\langle b\rangle + PB^{n+1} \triangleleft \langle b\rangle + PB^n$ for $b \in P$, then the class $\mathcal{M}$ is not hereditary, by Lemma 2.14 and [3, Corollary 3.5]. As an immediate consequence of Proposition 2.1 and Corollary 2.4, we obtain that the class $\mathcal{M}^{(n)}$ is essentially closed and $P \notin \mathcal{M}^{(n-1)}$. Finally, the class $\mathcal{M}^{(n)}$ is hereditary, by Lemma 2.14, Proposition 2.1 and Corollary 2.4.

REFERENCES

[1] T. Anderson, R. Wiegandt. On essentially closed classes of rings. *Ann. Univ. Sci. Budapest, Sec. Math.* **24** (1981), 107–111.

[2] R. R. Andruszkiewicz. On iterated maximal essential extensions of rings. *Algebra Colloq.* **9**, *3* (2002), 241–258.

[3] R. R. Andruszkiewicz, E. R. Puczyłowski. Accessible subrings and Kurosh's chains of associative rings. *Algebra Colloq.* **4**, *1* (1997), 79–88.

[4] S. Balcerzyk, T. Józefiak. Commutative Noetherian and Krull Rings. Horwod-PWN, Warszawa, 1989.

[5] K. I. Beidar. Examples of rings and radicals. In: Radical Theory (Proc. Conf. Eger, 1982), Colloq. Math. Soc. J. Bolyai, Vol. **38**, North-Holland, Amsterdam, 1985, 19–46.

[6] K. I. Beidar. On essential extensions, maximal essential extensions and iterated maximal essential extensions in radical theory. In: Theory of Radicals (Proc. Conf. Szekszard, 1991), Colloq. Math. Soc. J. Bolyai, Vol. **61**, Nort-Holland, Amsterdam, 1993, 17–26.

[7] G. F. Birkenmeier. Essential covers of radical classes. *Quaest. Math.* **22**, *3* (1999), 337–351.

[8] G. A. P. Heymann, C. Roos. Essential extensions in radical theory for rings. *J. Austral. Math. Soc.* **23** (Series A) (1977), 340–347.

[9] J. F. Watters. Essential cover and closure. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **25** (1982), 279–280.

Institute of Mathematics
University of Białystok
15-267 Białystok, Akademicka 2
Poland
e-mail: `randrusz@math.uwb.edu.pl`

Received March 26, 2004
Revised July 6, 2004