INVARIANT FUNCTIONS ON NEIL PARABOLA IN $\mathbb{C}^n$

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Abstract. We present the Carathéodory and the inner Carathéodory distances and the Carathéodory-Reiffen metric on generalized Neil parabolas in $\mathbb{C}^n$. It is a generalization of the results from [4] and [5].

1. Introduction and results. In the paper [3] the authors had asked for an effective formula for the Carathéodory distance on the Neil parabola in the bidisc. Such a formula was presented by G. Knese in [4], where he also computed the formula for the Carathéodory-Reiffen pseudometric. It should be pointed out that these are the first effective formulas for the Carathéodory distance and the Carathéodory-Reiffen pseudometric of a non-trivial complex space. In [5] N. Nikolov and P. Pflug generalized Knese’s result. The authors presented formula for the inner Carathéodory distance in so called generalized

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Neil parabola (but still in bidisc) and, as a corollary, they obtained sufficient and necessary condition for the Carathéodory distance on the Neil parabola to be inner. Moreover, they presented also formula for the Carathéodory-Reifen pseudometric on the two-dimensional generalized Neil parabola.

In this paper we present next possible generalization of the definition of Neil parabola, namely we embed the unit disc in $\mathbb{C}^n$. It turns out that in such a generalized Neil parabola all the results obtained in [5] are still valid. The aim of this paper is to translate the results from the two-dimensional case onto the $n$-dimensional one. Below we present all the necessary definitions.

Let $D$ be the unit disc in $\mathbb{C}$. For $M = (m_1, \ldots, m_n) \in \mathbb{N}^n$, where $m_j$'s are relatively prime and such that $m_1 \leq \cdots \leq m_n$ define

$$D \ni \lambda \mapsto (\lambda^{m_1}, \ldots, \lambda^{m_n}) \in A := p(D) \subset \mathbb{D}^n.$$  

$A$ is called the $n$-dimensional generalized parabola. Note that $A$ is one-dimensional analytic subset of $\mathbb{D}^n$ with $\text{reg } A = A_* := A \setminus \{0\}$. Recall that G. Knese worked with $M = (3, 2)$ while N. Nikolov and P. Pflug obtained their results for $M = (n, m)$, where $n, m$ are relatively prime.

The mapping $p$ is a global bijective holomorphic parametrization for $A$. Observe that there exist $r_1, \ldots, r_n \in \mathbb{Z}$ such that $r_1m_1 + \cdots + r_nm_n = 1$.

Define $q : A \to \mathbb{C}$ with the formula

$$q(z_1, \ldots, z_n) = \begin{cases} 
  z_1^{r_1} \cdots z_n^{r_n}, & z_1 \cdots z_n \neq 0 \\
  0, & z_1 \cdots z_n = 0.
\end{cases}$$

Observe that $q = p^{-1}$. Note that $q$ is continuous on $A$ and holomorphic on $A_*$. Thus the mapping $q|_{A_*} : A_* \to \mathbb{D} := D \setminus \{0\}$ is biholomorphic.

Let

$$\mathcal{O}_M(\mathbb{D}) := \{ h \in \mathcal{O}(\mathbb{D}, \mathbb{D}) : h^{(s)}(0) = 0, s \in S \},$$

where $S := \{ s \in \mathbb{N} : s \notin m_1\mathbb{Z}_+ + \cdots + m_n\mathbb{Z}_+ \}$. Note that if $m_1 = 1$ then $S = \emptyset$ and if $m_1 > 1$ then $\max_{s \in S} =: s^* < \text{nm}_1 \cdots m_n$, where $r := \max_{j=1,\ldots,n} |r_j|$.

Observe that if $f \in \mathcal{O}(A, \mathbb{D})$, i.e. $f$ is locally the restriction of a holomorphic function on an open neighborhood of $A$ in $\mathbb{C}^n$, then $f \circ p \in \mathcal{O}_M(\mathbb{D})$. Moreover, the converse is true. Indeed, we have the following

**Lemma 1** (cf. Section 5 in [4]). If $h \in \mathcal{O}_M(\mathbb{D})$, then $h \circ q \in \mathcal{O}(A, \mathbb{D})$.

All the proof will be presented in Section 2. We will also use the following identification.
Lemma 2. $\mathcal{O}(\mathbb{D}, A) = \{ p \circ \psi : \psi \in \mathcal{O}(\mathbb{D}, \mathbb{D}) \}$.

For $a \in A$ let $T_a A$ denote the tangent space of $A$ at $a$. Recall that if $a = p(\lambda)$, $\lambda \in \mathbb{D}_*$, then $T_{p(\lambda)} A$ is spanned by the vector $p'(\lambda)$. If $a = 0$ then

$$T_0 A = \begin{cases} \lambda p'(0), & \lambda \in \mathbb{C}, \\
\mathbb{C}^n & \text{if } m_1 = 1
\end{cases}$$

We will study some invariant functions. So let us recall the objects we will deal with in this paper. For details we refer the Reader to [2] and [3]. For $z, w \in A$ and $X \in T_z A$ we define

$$c_A(z, w) := \sup\{ p_D(f(z), f(w)) : f \in \mathcal{O}(A, \mathbb{D}) \},$$

$$m_A(z, w) := \sup\{ m_D(f(z), f(w)) : f \in \mathcal{O}(A, \mathbb{D}) \},$$

$$\gamma_A(z; X) := \max\{|f'(z)X| : f \in \mathcal{O}(A, \mathbb{D})\},$$

$$\tilde{k}_A(z, w) := \inf\{ p_D(\zeta, \xi) : \exists \varphi \in \mathcal{O}(\mathbb{D}, A) : \varphi(\zeta) = z, \ \varphi(\xi) = w\},$$

$$k_A := \text{the largest distance on } A \text{ below of } \tilde{k}_A,$$

$$k_A(z; X) := \inf\{ \alpha > 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, A) : \varphi(0) = z, \ \alpha \varphi'(0) = X\}.$$ 

where $p_D := \tanh^{-1} m_D$ denotes the Poincaré distance and $m_D(a, b) := \frac{|a - b|}{1 - ab}$, $a, b \in \mathbb{D}$, is the Möbius distance on $\mathbb{D}$. We set $\tilde{k}_A(z, w) := \infty$ or $k_A(z; X) := \infty$ if there are no respective discs $\varphi$. We call $c_A$ the Carathéodory distance, $m_A$ is the Möbius distance, $\gamma_A$ is the Carathéodory-Reiffen metric, $k_A$ is the Lempert function, $k_A$ is the Kobayashi distance and $\kappa_A$ is the Kobayashi-Royden metric for $A$.

Recall that the associated inner Carathéodory distance $c^i_A$ is given by

$$c^i_A(z, w) := \inf\{ L_{c_A}(\alpha) : \alpha \text{ is a } \| \cdot \|\text{-rectifiable curve in } A \text{ connecting } z, w\}, \quad z, w \in A,$$

where $L_{c_A}$ denotes the $c_A$-length. We say that the curve $\alpha$ is $\| \cdot \|\text{-rectifiable}$ if its Euclidean length is finite. Obviously, $c_A \leqslant c^i_A$.

Theorem 3 (cf. Theorem 3 in [5]). Let $\lambda \in \mathbb{D}$. Then

$$\gamma_A(p(\lambda); p'(\lambda)) = \frac{m_1 |\lambda|^{m_1 - 1}}{1 - |\lambda|^{2m_1}}.$$
Theorem 4 (cf. Theorem 1 in [5]). Let $\lambda, \mu \in \mathbb{D}$. Then
\[
c_A(p(\lambda), p(\mu)) = \begin{cases} 
p_D(\lambda^{m_1}, \mu^{m_1}) & \text{if } \text{Re}(\lambda \bar{\mu}) \geq \cos(\pi/m_1)|\lambda \mu| \\
p_D(\lambda^{m_1}, 0) + p_D(0, \mu^{m_1}) & \text{otherwise} \end{cases}
\]

Theorem 5 (cf. Theorem 4.1 in [4]). Let $\lambda, \mu \in \mathbb{D}$.

(a) If $S = \emptyset$, i.e. $m_1 = 1$, then
\[
c_A(p(\lambda), p(\mu)) = p_D(\lambda, \mu).
\]

(b) If $S = \{1\}$, i.e. $m_1 = 2, m_j = 3$ for some $1 < j \leq n$, then
\[
c_A(p(\lambda), p(\mu)) = \begin{cases} 
p_D(\lambda^2, \mu^2) & \text{if } |a| \geq 1 \\
p_D\left(\frac{a - \lambda}{1 - a\lambda}, \frac{a - \mu}{1 - a\mu}\right) & \text{if } |a| < 1 \end{cases}
\]
where $a = a_{\lambda, \mu} := \frac{1}{2} \left(\lambda + \frac{1}{\lambda} + \mu + \frac{1}{\mu}\right)$. In the case when $\lambda \mu = 0$ the formula should be read as in the case $|a| \geq 1$.

Due to the results above we have the following correspondence between the Carathéodory distance and its associated inner one.

Corollary 6 (cf. Corollary 2 in [5]). Let $\lambda, \mu \in \mathbb{D}$.

(a) If $\text{Re}(\lambda \bar{\mu}) \geq \cos(\pi/m_1)|\lambda \mu|$ then
\[
c_A'(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu)).
\]

(b) If $\text{Re}(\lambda \bar{\mu}) < \cos(\pi/m_1)|\lambda \mu|$ then
\[
c_A(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu)) \iff (\lambda \bar{\mu})^{m_1} < 0.
\]

Thus, the following conditions are equivalent
- $c_A'(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu))$;
- $c_A'(p(\lambda), p(\mu)) = p_D(\lambda^{m_1}, \mu^{m_1})$;
- $\text{Re}(\lambda \bar{\mu}) \geq \cos(\pi/m_1)|\lambda \mu|$ or $(\lambda \bar{\mu})^{m_1} < 0$.

In particular, $c_A$ is inner iff $m_1 = 1$. 
It turns out that (as in the case of domains in \( \mathbb{C}^n \)) \( \gamma_A \) is the infinitesimal form of \( c_A \) outside the origin.

**Corollary 7** (cf. [5]). Let \( \lambda \in \mathbb{D} \) (if \( m_1 = 1 \) we may take \( \lambda \in \mathbb{D} \)). Then

\[
\lim_{\mu \to \lambda} \frac{c_A(p(\lambda), p(\mu))}{|\lambda - \mu|} = \gamma_A(p(\lambda); p'(\lambda)).
\]

Now assume that \( m_1 > 1 \). Let \( X \in T_0 A = \mathbb{C}^n \). Observe that

\[
\gamma_A(0; X) = \max \{ |f'(0)X| : f \in \mathcal{O}(A, \mathbb{D}), f(0) = 0 \}.
\]

Then for such an \( f \) we have \((f \circ p)(\lambda) = \lambda^{m_1} h(\lambda), \ \lambda \in \mathbb{D}, \) where \( h \in \mathcal{O}(\mathbb{D}, \mathbb{\bar{D}}) \). Observe that

\[
\frac{\partial f}{\partial z_j}(0) = \frac{h^{(m_j - m_1)}(0)}{(m_j - m_1)!}, \quad j = 1, \ldots, n.
\]

Thus, for \( X = (X_1, \ldots, X_n) \in \mathbb{C}^n \) we have

\[
\gamma_A(0; X) = \max \left\{ \sum_{j=1}^{n} \frac{h^{(m_j - m_1)}(0)}{m_j!} X_j : h \in \mathcal{O}_M(\mathbb{D}), h(0) = 0 \right\}
\]

\[
= \max \left\{ \sum_{j=1}^{n} \frac{h^{(m_j - m_1)}(0)}{m_j - m_1!} X_j : h \in \mathcal{O}(\mathbb{D}, \mathbb{\bar{D}}), h^{(j)}(0) = 0, j + m_1 \in S \right\}.
\]

In particular, \( \gamma_A(0; X) = \|X\| \) if \( n - 1 \) coordinates of \( X \) is equal 0. Using the first equality above, we will prove the following infinitesimal result at the origin.

**Proposition 8** (cf. Prop. 4 in [5]). Let \( X_{\lambda, \mu} := (\lambda^{m_1} - \mu^{m_1}, \ldots, \lambda^{m_n} - \mu^{m_n}) \). Then

\[
\lim_{\lambda, \mu \to 0} \frac{c_A(p(\lambda), p(\mu))}{\gamma_A(0; X_{\lambda, \mu})} = 1.
\]

**Corollary 9** (cf. Corollary 5 in [5]). Let \( m_1 > 1 \). For any \( j \in \{2, \ldots, n\} \) there are points \( \lambda, \mu \in \mathbb{D} \) such that

\[
c_A(p(\lambda), p(\mu)) > \max \{ p_\mathbb{D}(\lambda^{m_1} \mu^{m_1}), p_\mathbb{D}(\lambda^{m_j} \mu^{m_j}) \}.
\]
In the proof of Proposition 8 we use the following

**Lemma 10** (cf. [5]). There exists a constant $c > 0$ such that for any $\lambda, \mu \in \mathbb{D}$

\[
(3) \quad c_{\lambda}(p(\lambda), p(\mu)) \geq \max\{p_D(\lambda^{m_j}, \mu^{m_j}) : j = 1, \ldots, n\} \geq c\|X_{\lambda, \mu}\|,
\]

\[
(4) \quad \max\{|\lambda|^{k-m_n}, |\mu|^{k-m_n}\} \cdot \|X_{\lambda, \mu}\| \geq \frac{c}{k}|\lambda^k - \mu^k|, \quad m_n < k,
\]

\[
(5) \quad \gamma_{\lambda}(0; X_{\lambda, \mu}) \geq c\|X_{\lambda, \mu}\|.
\]

Moreover,

\[
(6) \quad \lim_{\lambda, \mu \to 0} \sum_{j=m+1}^{\infty} \frac{|\lambda^j - \mu^j|}{\|X_{\lambda, \mu}\|} = 0.
\]

**Proposition 11** (cf. Proposition 7 in [5]). Let $M = (m_1, \ldots, m_n)$ be such that $m_1 = \cdots = m_j = 2$, $m_{j+1} = 2k + 1$ for some $1 \leq j \leq n - 1$ and $k \in \mathbb{N}$. Then

\[
m_{\lambda}(p(\lambda), p(-\lambda)) = \frac{2|\lambda|^{2k+1}}{1 + |\lambda|^{2k+2}}, \quad \lambda \in \mathbb{D}.
\]

Finally, we discuss the Kobayashi distance and Kobayashi-Royden metric on $A$. Due to Lemma 2, we have the following result.

**Proposition 12** (cf. Proposition 8 in [5]). (a) Let $\lambda, \mu \in \mathbb{D}$. Then

\[
k_{\lambda}(p(\lambda), p(\mu)) = \tilde{k}_{\lambda}(p(\lambda), p(\mu)) = p_D(\lambda, \mu).
\]

(b) If $\lambda \in \mathbb{D}$, then

\[
\kappa_{\lambda}(p(\lambda); p'(\lambda)) = \gamma_D(\lambda; 1).
\]

If $\lambda = 0$ and $X = (X_1, \ldots, X_n) \in T_0 A$, $X \neq 0$ then

\[
\kappa_{\lambda}(0; X) = \begin{cases} |X_1| & \text{if } m_1 = 1 \\ \infty & \text{if } m_1 > 1. \end{cases}
\]
We conclude this note by generalizing the example of the coordinate cross discussed in [5]. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^n$, $j = 1, \ldots, n$. Put

$$V_1 := \bigcup_{j=1}^{n} \mathbb{D} e_j.$$  

**Proposition 13** (cf. Remark in [5]). (a) Let $\lambda, \mu \in \mathbb{D}$. Then

$$c_{V_1}(\lambda e_j, \mu e_k) = k_{V_1}(\lambda e_j, \mu e_k) = \begin{cases} p_\mathbb{D}(\lambda, \mu) & \text{if } j = k, \\ p_\mathbb{D}(\lambda, 0) + p_\mathbb{D}(0, \mu) & \text{if } j \neq k \end{cases}.$$  

$$\tilde{k}_{V_1}(\lambda e_j, \mu e_k) = \begin{cases} p_\mathbb{D}(\lambda, \mu) & \text{if } j = k, \\ \infty & \text{if } j \neq k, \lambda \mu \neq 0 \end{cases}.$$  

(b) If $\lambda \in \mathbb{D}_*$ then

$$\gamma_{V_1}(\lambda e_j; e_j) = \kappa_{V_1}(\lambda e_j; e_j) = \gamma_\mathbb{D}(\lambda; 1).$$  

If $\lambda = 0$ and $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$ then

$$\gamma_{V_1}(0; X) = \sum_{j=1}^{n} |X_j|,$$

$$\kappa_{V_1}(0; X) = \begin{cases} |X_j| & \text{if } X = X_j e_j, \ j = 1, \ldots, n \\ \infty & \text{otherwise} \end{cases}.$$  

2. Proofs.

**Proof of Lemma 1.** $h \circ q$ is holomorphic on $A_*$ because it may be extended to a holomorphic function on the set

$$\Omega := \left\{ (z_1, \ldots, z_n) \in \mathbb{D}^n : \prod_{j \in M^+} |z_j|^{r_j} < \prod_{k \notin M^+} |z_k|^{-r_k} \right\},$$

where $M^+ := \{j \in \{1, \ldots, n\} : r_j \in \mathbb{Z}_+\}$, and $\Omega$ is an open neighborhood of $A_*$. 
To prove that $h \circ q$ is holomorphic at the origin observe that

\begin{equation}
(12) \quad h(\lambda) = \sum_{j \in \mathbb{Z}_+ \setminus S} a_j \lambda^j, \quad \lambda \in \mathbb{D}.
\end{equation}

Moreover, the following identities hold

\[ z_j^{m_k} = z_k^{m_j}, \quad j, k \in \{1, \ldots, n\}, \quad (z_1, \ldots, z_n) \in A. \]

Hence for any $j = m_1 b_{j,1} + \cdots + m_n b_{j,n} \in \mathbb{Z}_+ \setminus S$ and $z = (z_1, \ldots, z_n) \in A$ we obtain

\[ q^j(z) = z_1^{r_1 m_1 b_{j,1}} \cdots z_1^{r_1 m_n b_{j,n}} \cdots z_n^{r_n m_1 b_{j,1}} \cdots z_n^{r_n m_n b_{j,n}} = z_1^{r_1 m_1 b_{j,1}} \cdots z_1^{r_1 m_n b_{j,n}} \cdots z_n^{r_n m_1 b_{j,1}} \cdots z_n^{r_n m_n b_{j,n}} = z_1^{r_1 m_1 + \cdots + r_n m_n b_{j,1}} \cdots z_n^{r_1 m_1 + \cdots + r_n m_n b_{j,n}} = z_1^{b_{j,1}} \cdots z_n^{b_{j,n}}. \]

Using the equality above and (12) we get

\begin{equation}
(13) \quad (h \circ q)(z) = \sum_{j \in \mathbb{Z}_+ \setminus S} a_j z_1^{b_{j,1}} \cdots z_n^{b_{j,n}}, \quad z = (z_1, \ldots, z_n) \in A,
\end{equation}

where $b_{j,k} \in \mathbb{Z}_+$ for all $j \in \mathbb{Z}_+ \setminus S$ and $k = 1, \ldots, n$.

The series (13) is convergent for $z = (\lambda^m_1, \ldots, \lambda^m_n)$, $|\lambda| \leq R < 1$. Thus it converges for $z \in R^m_1 \mathbb{D} \times \cdots \times R^m_n \mathbb{D}$ which gives us holomorphicity of the extension of $h \circ q$ in some neighborhood of the origin. □

**Proof of Lemma 2.** Since $p \in \mathcal{O}(\mathbb{D}, A)$, we have that $p \circ \psi \in \mathcal{O}(\mathbb{D}, A)$. Now assume that $f \in \mathcal{O}(\mathbb{D}, A)$. Since $f = p \circ q \circ f$ it suffices to show that $q \circ f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$.

Fix $\lambda \in \mathbb{D}$. If $f(\lambda) \neq 0$ then $q \circ f$ is holomorphic in some neighborhood of $\lambda$. If $f(\lambda) = 0$, i.e., $f_1(\lambda) = \cdots = f_n(\lambda) = 0$, where $f = (f_1, \ldots, f_n)$, then $f_j(\zeta) = (\zeta - \lambda)^{s_j} \tilde{f}_j(\zeta)$ for some $s_j \in \mathbb{N}$ and $\tilde{f}_j \in \mathcal{O}(U_\lambda)$, $\tilde{f}_j(\zeta) \neq 0$, $\zeta \in U_\lambda$, $j = 1, \ldots, n$, where $U_\lambda \subset \mathbb{D}$ is some neighborhood of $\lambda$. Since

\begin{equation}
(14) \quad (\zeta - \lambda)^{s_k m_k} \tilde{f}_j^{m_k}(\zeta) = (\zeta - \lambda)^{s_k m_j} \tilde{f}_k^{m_j}(\zeta), \quad \zeta \in U_\lambda, \quad j, k \in \{1, \ldots, n\},
\end{equation}

there exists $l \in \mathbb{N}$ such that $s_j = lm_j$, $j = 1, \ldots, n$. Indeed, from (14) it follows that

\begin{equation}
(15) \quad s_j m_k = s_k m_j, \quad j, k \in \{1, \ldots, n\}.
\end{equation}
Invariant functions on Neil parabola in $\mathbb{C}^n$

Fix $j \in \{1, \ldots, n\}$. Observe that $m_j = p_{j,1} \ldots p_{j,s(j)}$, where $p_{j,s}$'s are prime numbers. Since $m_1, \ldots, m_n$ are relatively prime, for any $1 \leq s \leq s(j)$ there exists $1 \leq k \leq n$ such that $p_{j,s} \nmid m_k$. Then (15) implies that $s_j = p_{j,1} \ldots p_{j,s(j)}l_j$ for some $l_j \in \mathbb{N}$. Using (15) again, we conclude that $l_j = l_k =: l$ for all $j, k \in \{1, \ldots, n\}$.

Hence

$$(q \circ f)(\zeta) = f_1^{\tau_1}(\zeta) \ldots f_n^{\tau_n}(\zeta) = (\zeta - \lambda)^l \tilde{f}_1^{\tau_1}(\zeta) \ldots \tilde{f}_n^{\tau_n}(\zeta), \quad \zeta \in U_\lambda.$$  

Thus $q \circ f \in \mathcal{O}(U_\lambda)$ and the proof is complete. \hfill \Box

Proof of Theorem 3. Recall that

$$\gamma_A(p(\lambda); p'(\lambda)) = \max \left\{ \frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} : h \in \mathcal{O}_M(\mathbb{D}) \right\}.$$  

Observe that if $\phi \in \text{Aut}(\mathbb{D})$ and $h \in \mathcal{O}_M(\mathbb{D})$ then $\phi \circ h \in \mathcal{O}_M(\mathbb{D})$ and

$$\frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} = \frac{|(\phi \circ h)'(\lambda)|}{1 - |(\phi \circ h)(\lambda)|^2}.$$  

Thus

$$\gamma_A(p(\lambda); p'(\lambda))$$

$$= \max \left\{ \frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} : h \in \mathcal{O}_M(\mathbb{D}), h(0) = 0 \right\}$$

$$= \max \left\{ \frac{|(\lambda^{m_j} \hat{h}(\lambda))'|}{1 - |\lambda^{m_j} \hat{h}(\lambda)|^2} : \hat{h} \in \mathcal{O}(\mathbb{D}, \pi), \hat{h}^{(j)}(0) = 0, j + m_1 \in S \right\}$$

$$= |\lambda|^{m_1 - 1} \max \left\{ \frac{|m_1 h(\lambda) + \lambda h'(\lambda)|}{1 - |\lambda^{m_1} h(\lambda)|^2} : h \in \mathcal{O}(\mathbb{D}, \pi), h^{(j)}(0) = 0, j + m_1 \in S \right\} = \frac{m_1 |\lambda|^{m_1 - 1}}{1 - |\lambda|^{2m_1}}.$$  

The last equality may be proved exactly as in the proof of Theorem 3 in [5] with $m_1$ instead of $m$. \hfill \Box

Proof of Theorem 4. The proof follows the proof of Theorem 1 in [5] with $m_1$ instead of $m$. \hfill \Box

Proof of Theorem 5. Ad (a). It is a consequence of Theorem 4, since $m_1 = 1.$
Ad (b). Since $S = \{1\}$, the proof of Theorem 4.1 from [4] may be repeated. □

**Proof of Corollary 6.** The proof follows the proof of Corollary 2 in [5] with $m_1$ instead of $m$. □

**Remark 14** (cf. Remark (a) in [5]). In [5] for $m_2 \in \mathbb{N}$ the following distance was introduced

$$p_{\mathbb{D}}^{(m_1)}(\lambda, \mu) := \max\{p_{\mathbb{D}}(\lambda^{m_1}h(\lambda), \mu^{m_1}h(\mu)) : h \in \mathcal{O}(\mathbb{D}, \mathbb{D})\}.$$  

Note that

$$\lim_{\varepsilon \to 0} \frac{p_{\mathbb{D}}^{(m_1)}(\lambda, \lambda + \varepsilon)}{|\varepsilon|} = |\lambda|^{m_1 - 1} \max \left\{ \frac{|m_1 h(\lambda) + \lambda h'(\lambda)|}{1 - |\lambda^{m_1}h(\lambda)|^2} : h \in \mathcal{O}(\mathbb{D}, \mathbb{D}) \right\}$$

$$= \gamma_A(p(\lambda); p'(\lambda))$$

by the proof of Theorem 3. So it follows that the associated inner distance $f_{\mathbb{D}}^{(m_1)}$ of $p_{\mathbb{D}}^{(m_1)}$ equals to $c_A^i(p(\cdot), p(\cdot))$. Then

$$c_A^i(p(\lambda), p(\mu)) \geq p_{\mathbb{D}}^{(m_1)}(\lambda, \mu)$$

$$\geq c_A(p(\lambda), p(\mu)) \geq p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1}).$$

Moreover, the proof of Corollary 6 shows that the following conditions are equivalent

- $c_A^i(p(\lambda), p(\mu)) = p_{\mathbb{D}}^{(m_1)}(\lambda, \mu)$;
- $c_A^i(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu))$;
- $c_A^i(p(\lambda), p(\mu)) = p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1})$;
- $\text{Re}(\lambda \bar{\mu}) \geq \cos(\pi/m_1)|\lambda \mu|$ or $(\lambda \bar{\mu})^{m_1} < 0$.

**Proof of Corollary 7.** Since

$$c_A^i(p(\lambda), p(\mu)) \geq c_A(p(\lambda), p(\mu)) \geq p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1}), \quad \lambda \in \mathbb{D},$$

for $\lambda \in \mathbb{D}_+$ (if $m_1 = 1$ we may take $\lambda \in \mathbb{D}$) we have

$$\lim_{\mu \to \lambda} \frac{c_A(p(\lambda), p(\mu))}{|\lambda - \mu|} = \lim_{\mu \to \lambda} \frac{p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1})}{|\lambda - \mu|} = \lim_{\mu \to \lambda} \frac{m_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1})}{|\lambda - \mu|}$$

$$= \frac{m_1|\lambda|^{m_1 - 1}}{1 - |\lambda|^{2m_1}} = \gamma_A(p(\lambda); p'(\lambda)).$$

□
Proof of Lemma 10. Fix \( \lambda, \mu \in \mathbb{D} \). Without loss of generality we may assume that \( \lambda \neq \mu \) and \( |\mu| \leq |\lambda| \). Moreover, it suffices to obtain each inequality with different constant, since minimum of these constants will do the job.

Ad (3). The first inequality in (3) we obtain with help of the projection from \( A \) onto its \( m_j \)-th coordinate, while the second one is a consequence of the equivalence of norms in \( \mathbb{C}^n \).

Ad (4). Let \( \sqrt{1} = \{\varepsilon_{m_j,0}, \ldots, \varepsilon_{m_j, m_j-1}\} \) and let \( R_{m_j,s} := \varepsilon_{m_j,s}[0,1] \), \( s = 0, \ldots, m_j - 1 \). Observe that there is a constant \( \delta = \delta(M) > 0 \) such that

\[
\Lambda_{m_j,s,\delta} \cap \Lambda_{m_l,t,\delta} = \emptyset \quad \text{if} \quad R_{m_j,s} \neq R_{m_l,t},
\]

where \( \Lambda_{m_j,s,\delta} := \{re^{i\varphi} : r \in R_{m_j,s}, \varphi \in (-\delta, \delta)\} \), \( s = 0, \ldots, m_j - 1 \), \( j = 1, \ldots, n \).

Observe that \( \mu/\lambda \in \mathbb{D} \). Since \( m_j \)'s are relatively prime, one of the following two cases holds:

1° There exists \( j \in \{1, \ldots, n\} \) such that \( \mu/\lambda \notin \bigcup_{s=0}^{m_j-1} \Lambda_{m_j,s,\delta} \); 2° \( \mu/\lambda \in \Lambda_{m_n,0,\delta} \).

Ad 1°. Then there is a constant \( c = c(\delta) > 0 \) such that \( |1-(\mu/\lambda)^{m_j}| \geq 2c \).

Therefore

\[
|\lambda|^{k-m_n} \|X_{\lambda,\mu}\| \geq |\lambda|^{k-m_j}|\lambda^{m_j} - \mu^{m_j}|
\]

\[
= |\lambda|^k|1-(\mu/\lambda)^{m_j}| \geq 2c|\lambda|^k \geq \frac{c}{k}|\lambda^k - \mu^k|.
\]

Ad 2°. To obtain (4) in this case it suffices to prove that there exists \( c > 0 \) such that

\[
\left| \frac{\frac{1}{k} \left| 1 - \frac{(\mu/\lambda)^k}{(\mu/\lambda)^{m_n}} \right|}{1 - \frac{(\mu/\lambda)^{m_n}}{\mu^{m_n}}} \right| \leq 1, \quad k > m_n.
\]

Since

\[
\lim_{\mu/\lambda \to 1} \left| \frac{1 - \frac{(\mu/\lambda)^k}{(\mu/\lambda)^{m_n}}}{1 - \frac{(\mu/\lambda)^{m_n}}{\mu^{m_n}}} \right| = \frac{k}{m_n},
\]

there is a constant \( r > 0 \) such that

\[
\left| \frac{1 - \frac{(\mu/\lambda)^k}{(\mu/\lambda)^{m_n}}}{1 - \frac{(\mu/\lambda)^{m_n}}{\mu^{m_n}}} \right| \leq \frac{2k}{m_n}, \quad |1 - \mu/\lambda| < r, \quad k > m_n.
\]

Hence in case \( |1 - \mu/\lambda| < r \), a constant \( c_1 := \frac{m_n}{2} \) will do the job.

On the other hand, if \( |1 - \mu/\lambda| \geq r \) then there is a constant \( c_2 = c_2(r) > 0 \) such that \( |1 - (\mu/\lambda)^{m_n}| \geq 2c_2 \). Therefore

\[
\frac{c_2}{k} \left| \frac{1 - \frac{(\mu/\lambda)^k}{(\mu/\lambda)^{m_n}}}{1 - \frac{(\mu/\lambda)^{m_n}}{\mu^{m_n}}} \right| \leq \frac{2c_2}{2c_2k} \leq 1.
\]
Finally we take \( c := \min\{c_1, c_2\} \).

Ad (5). Let \( l \in \{1, \ldots, n\} \) be such that \( |\lambda^{m_l} - \mu^{m_l}| = \max\{|\lambda^{m_j} - \mu^{m_j}| : j = 1, \ldots, n\} \). Let \( h(\zeta) = \zeta^{m_l}, \zeta \in \mathbb{D} \). Observe that \( h \in \mathcal{O}_M(\mathbb{D}) \) and \( h(0) = 0 \). Thus
\[
\gamma_A(0; X_{\lambda, \mu}) \geq \sum_{j=1}^{n} \frac{h^{(m_j)}(0)}{m_j} |(\lambda^{m_j} - \mu^{m_j})| = |\lambda^{m_l} - \mu^{m_l}| \geq c \|X_{\lambda, \mu}\|,
\]
where \( c > 0 \) is a constant from the inequality (3).

Ad (6). First assume that \( m_1 + 1 \leq j \leq m_n \). Then
\[
\frac{|\lambda^j - \mu^j|}{\|X_{\lambda, \mu}\|} \leq \left| \frac{\lambda^j - \mu^j}{\lambda^{m_1} - \mu^{m_1}} \right| = \left| \frac{\lambda^{\alpha_j} - \mu^{\alpha_j}}{\lambda - \mu} \right| \leq \alpha_j (|\lambda| + |\mu|),
\]
where \( \alpha_j > 1 \). Therefore

\[
\lim_{\lambda, \mu \to 0} \sum_{j=m_1+1}^{m_n} \frac{|\lambda^j - \mu^j|}{\|X_{\lambda, \mu}\|} = 0.
\]

Observe that, using (4), we have
\[
\sum_{j=m_n+1}^{\infty} \frac{|\lambda^j - \mu^j|}{\|X_{\lambda, \mu}\|} \leq \frac{1}{c} \sum_{j=m_n+1}^{\infty} j|\lambda|^{j-m_n} = \frac{1}{c} \sum_{j=1}^{\infty} (m_n + j)|\lambda|^j
\]
\[
\leq \frac{m_n + 1}{c} \sum_{j=1}^{\infty} j|\lambda|^j = \frac{(m_n + 1)|\lambda|}{c(1 - |\lambda|)^2}.
\]
Hence, letting \( \lambda, \mu \to 0 \), \( \lambda \neq \mu \), and using (16) we obtain (6). \( \square \)

Proof of Proposition 8. Let \( h^{+}_{\lambda, \mu} \in \mathcal{O}_M(\mathbb{D}) \) be an extremal function for \( c_A(p(\lambda), p(\mu)) \). Then
\[
h^{+}_{\lambda, \mu}(\zeta) = \sum_{j \in \mathbb{N} \setminus S} a_{\lambda, \mu, j} \zeta^j.
\]
Since \( |a_{\lambda, \mu, j}| \leq 1 \), it follows that
\[
|h^{+}_{\lambda, \mu}(\lambda) - h^{+}_{\lambda, \mu}(\mu)|
\]
\[
\leq H^{+}(\lambda, \mu) := \sum_{j=1}^{n} a_{\lambda, \mu, m_j} (\lambda^{m_j} - \mu^{m_j}) + \sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j|.
\]
Thus, using (3), (6), and (1)

\[ 1 \leq \liminf_{\lambda, \mu \to 0} \frac{H^+(\lambda, \mu)}{|h_\lambda^+(\lambda) - h_\lambda^+(\mu)|} = \liminf_{\lambda, \mu \to 0} \frac{H^+(\lambda, \mu)}{c_A(p(\lambda), p(\mu))} \]

\[ \leq \liminf_{\lambda, \mu \to 0} \frac{\left| \sum_{j=1}^{n} a_{\lambda, \mu, m_j} (\lambda^{m_j} - \mu^{m_j}) \right|}{c_A(p(\lambda), p(\mu))} + \frac{\sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j|}{c\|X_{\lambda, \mu}\|} \]

\[ = \liminf_{\lambda, \mu \to 0} \frac{\left| \sum_{j=1}^{n} a_{\lambda, \mu, m_j} (\lambda^{m_j} - \mu^{m_j}) \right|}{c_A(p(\lambda), p(\mu))} \leq \liminf_{\lambda, \mu \to 0} \frac{\gamma_A(0; X_{\lambda, \mu})}{c_A(p(\lambda), p(\mu))} \]

Let now \( h_\lambda^- \in \mathcal{A}(\mathbb{D}) \) be an extremal function for \( \gamma_A(0; X_{\lambda, \mu}) \). Then

\[ h_\lambda^-(\zeta) = \sum_{j \in \mathbb{Z} \setminus S} a_{\lambda, \mu, j} \zeta^j. \]

Since \( |a_{\lambda, \mu, j}| \leq 1 \), it follows that

\[ |h_\lambda^-(\lambda) - h_\lambda^-(\mu)| \geq \left| \sum_{j=1}^{n} a_{\lambda, \mu, m_j} (\lambda^{m_j} - \mu^{m_j}) \right| - \sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j|. \]

Then, using (5) and (6), we have

\[ \lim_{\lambda, \mu \to 0} \frac{\sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j|}{\gamma_A(0; X_{\lambda, \mu})} \leq \lim_{\lambda, \mu \to 0} \frac{\gamma_A(0; X_{\lambda, \mu})}{c\|X_{\lambda, \mu}\|} = 0, \]

and, consequently,

\[ \lim_{\lambda, \mu \to 0} \frac{\sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j|}{|h_\lambda^-(\lambda) - h_\lambda^-(\mu)|} \leq \lim_{\lambda, \mu \to 0} \frac{\gamma_A(0; X_{\lambda, \mu})}{\gamma_A(0; X_{\lambda, \mu}) - \sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j|} = 0. \]
Thus, using (3), (6), and the last equality,

\[
1 \geq \limsup_{\lambda, \mu \to 0, \lambda \neq \mu} \left( \gamma_A(0; X_{\lambda, \mu}) - \sum_{j=m_1+1}^{\infty} |\lambda^j - \mu^j| / |h_{\lambda, \mu}(\lambda) - h_{\lambda, \mu}(\mu)| \right)
\]

\[
\geq \limsup_{\lambda, \mu \to 0, \lambda \neq \mu} \left( \frac{\gamma_A(0; X_{\lambda, \mu})}{c_A(p(\lambda), p(\mu))} - \sum_{j=m_1+1}^{\infty} \frac{|\lambda^j - \mu^j|}{|h_{\lambda, \mu}(\lambda) - h_{\lambda, \mu}(\mu)|} \right)
\]

\[
= \limsup_{\lambda, \mu \to 0, \lambda \neq \mu} \frac{\gamma_A(0; X_{\lambda, \mu})}{c_A(p(\lambda), p(\mu))}
\]

\[
\square
\]

Proof of Corollary 9. Observe that for any neighborhood $U$ of $0$ one may find points $\lambda, \mu \in U$ such that $\lambda^{m_1} - \mu^{m_1} = \lambda^{m_2} - \mu^{m_2} \neq 0$. Then, by Proposition 8, it suffices to show that

\[(17) \quad \gamma_A(0; X_0) > 1, \quad X_0 := (X_1, \ldots, X_n), \quad X_1 = X_j = 1.
\]

Indeed, having (17) and using the equality (cf. Corollary 1.13 (d) in [2])

\[
\lim_{\lambda', \lambda'' \to 0, \lambda' \neq \lambda''} \frac{p_D(\lambda', \lambda'')}{|\lambda' - \lambda''|} = 1
\]

we obtain the required result.

By the second equality in (1) and the fact that $\max_{s \in S} s = s^* < \infty$,

\[
\gamma_A(0; X_0) \geq \max\{|a + b| : (a, b) \in T_{m_j - m_1}\},
\]

where $T_{m_j - m_1} := \{(a, b) \in \mathbb{C}^2 : \exists h \in O(\mathbb{D}, \mathbb{D}) : h(\zeta) = a + b\zeta^{m_j - m_1} + o(\zeta^{s^* - m_1})\}$.

Let $k \in \mathbb{N}$ be such that $k(m_j - m_1) \geq s^* - m_1$. We shall show that there is a function $f \in O(\mathbb{D}, \mathbb{D})$ of the form $f(\zeta) = a + b\zeta + o(\zeta^k)$, where $a, b > 0$ and $a + b > 1$, which will imply (17).

From now on the rest of the proof of Corollary 5 in [5] may be repeated. For convenience of the Reader we recall that proof.

Note that by Shur’s theorem (cf. [1]) such a function $f$ exists if and only if

\[(18) \quad (1 - a^2 - b^2) \sum_{j=1}^{k} X_j^2 \geq 2ab \sum_{j=2}^{k} X_{j-1}X_j, \quad (X_1, \ldots, X_k) \in \mathbb{R}^k.
\]
Since \( \cos \frac{\pi}{k+1} \) is the maximal eigenvalue of the quadratic form defined by
\
\sum_{j=2}^{k} X_{j-1}X_j,
\]

it follows that
\
\cos \frac{\pi}{k+1} \sum_{j=1}^{k} X_j^2 \geq \sum_{j=2}^{k} X_{j-1}X_j, \quad (X_1, \ldots, X_k) \in \mathbb{R}^k.
\]

Then all pairs \((a,b) \in \mathbb{R}^2\) for which \(2ab \cos \frac{\pi}{k+1} \leq 1 - a^2 - b^2\) satisfy (18); in particular, we may choose \(a, b > 0\) such that \(2ab \cos \frac{\pi}{k+1} \leq 1 - a^2 - b^2 < 2ab\), i.e. \(a + b > 1\).

\(\square\)

\begin{proof}

\textbf{Proof of Proposition 11.} Observe that in this case \(S = \{2j - 1 : j = 1, 2, \ldots, k\}\) and the proof of Proposition 7 from [5] may be repeated. \(\square\)

\textbf{Proof of Proposition 12.} Ad (a). \(\tilde{k}_A(p(\lambda), p(\mu)) \leq p_\mathbb{D}(\lambda, \mu)\), since \(p \in \mathcal{O}(\mathbb{D}, A)\). From Lemma 2 we already know that for any \(\varphi \in \mathcal{O}(\mathbb{D}, A)\) with \(\varphi(\lambda) = p(\lambda)\) and \(\varphi(\tilde{\mu}) = p(\mu)\) there exists some \(\psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})\) such that \(\psi(\lambda) = \lambda\) and \(\psi(\tilde{\mu}) = \mu\). Hence \(p_\mathbb{D}(\lambda, \mu) \leq \tilde{k}_A(p(\lambda), p(\mu))\). Hence, \(p_\mathbb{D}(\lambda, \mu) = \tilde{k}_A(p(\lambda), p(\mu))\). In particular, \(\tilde{k}_A\) is a distance and, consequently, \(k_A = k_A\).

Ad (b). Again, using Lemma 2, we obtain

\[
\kappa_A(p(\lambda); p'(\lambda))
\]

\[
= \inf\{\alpha > 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, A) : \varphi(0) = p(\lambda), \alpha \varphi'(0) = p'(\lambda)\}
\]

\[
\geq \inf\{\alpha > 0 : \exists \psi \in \mathcal{O}(\mathbb{D}, \mathbb{D}) : \psi(0) = \lambda, \alpha \psi'(0) = 1\}
\]

\[
= \kappa_\mathbb{D}(\lambda; 1) = \gamma_\mathbb{D}(\lambda; 1).
\]

On the other hand, for \(\varphi := p \circ \psi\), where \(\psi \in \text{Aut}(\mathbb{D})\) is such that \(\psi(0) = \lambda\), we have that \(\varphi \in \mathcal{O}(\mathbb{D}, A), \varphi(0) = p(\lambda), \) and \(\gamma_\mathbb{D}(\lambda; 1) \varphi'(0) = p'(\lambda)\). Therefore \(\kappa_A(p(\lambda); p'(\lambda)) \leq \gamma_\mathbb{D}(\lambda; 1)\).
\end{proof}
It remains to prove formula for $\lambda = 0$. Observe that

$$\kappa_A(0; X) = \inf \{ \alpha > 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, A) : \varphi(0) = 0, \ \alpha \varphi'(0) = X \}$$

$$\geq \inf \{ \alpha > 0 : \exists \psi \in \mathcal{O}(\mathbb{D}, \mathbb{D}) : \psi(0) = 0, \ \alpha p'(0)\psi'(0) = X \}$$

$$= \begin{cases} |X_1| & \text{if } m_1 = 1, \\ \infty & \text{if } m_1 > 1. \end{cases}$$

It suffices to prove the opposite inequality in case $m_1 = 1$. Fix $X \in (T_0 A)_+$. Then there exists $k \in \mathbb{N}$ such that $X_1 = \cdots = X_k \neq 0$ and $X_{k+1} = \cdots = X_n = 0$. We define $\varphi(\lambda) := p(X_1|X_1|^{-1}\lambda)$, $\lambda \in \mathbb{D}$. Observe that $\varphi \in \mathcal{O}(\mathbb{D}, A)$, $\varphi(0) = 0$, and $|X_1|\varphi'(0) = X$. Hence $\kappa_A(0; X) \leq |X_1|$ which ends the proof. \(\square\)

Proof of Proposition 13. Ad (7). Let $\varphi_j(z) := z_j$, $z = (z_1, \ldots, z_n) \in V_1$, and $\psi_j(\zeta) := \zeta e_j$, $\zeta \in \mathbb{D}$, for $j = 1, \ldots, n$. Since $\varphi_j \in \mathcal{O}(V_1, \mathbb{D})$ and $\psi_j \in \mathcal{O}(\mathbb{D}, V_1)$, then

$$\tag{19} p_D(\lambda, \mu) \leq c_{V_1}(\lambda e_j, \mu e_j) \leq \tilde{k}_{V_1}(\lambda e_j, \mu e_j) \leq p_D(\lambda, \mu).$$

Now assume that $j \neq k$. Since $\varphi := \sum_{j=1}^n \varphi_j \in \mathcal{O}(V_1, \mathbb{D})$, then

$$p_D(\lambda, 0) + p_D(0, \mu) = p_D(|\lambda|, -|\mu|) \leq c_{V_1}(|\lambda|e_j, -|\mu|e_k) = c_{V_1}(\lambda e_j, \mu e_k).$$

Moreover, using (19),

$$k_{V_1}(\lambda e_j, \mu e_k) \leq \tilde{k}_{V_1}(\lambda e_j, 0) + \tilde{k}_{V_1}(0, \mu e_k) = p_D(\lambda, 0) + p_D(0, \mu).$$

Ad (8). It remains to consider the case $j \neq k$, $\lambda \mu \neq 0$. Suppose there is a disc $\psi \in \mathcal{O}(\mathbb{D}, V_1)$ such that $\psi(\zeta) = \lambda e_j$ and $\psi(\xi) = \mu e_k$ for some $\zeta, \xi \in \mathbb{D}$. However, these equalities imply, together with the identity principle, that $\psi \equiv 0$; a contradiction, since $\lambda \mu \neq 0$.

Ad (9). Using again the functions $\varphi_j$ and $\psi_j$, $j = 1, \ldots, n$, defined in the part of the proof of (7), we obtain

$$\gamma_D(\lambda; 1) \leq \gamma_{V_1}(\lambda e_j; e_j) \leq \kappa_{V_1}(\lambda e_j; e_j) \leq \gamma_D(\lambda; 1).$$

Ad (10). For $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$ let $\varphi_X(z) := \sum_{j=1}^n z_j e^{-i \text{Arg} X_j}$, where $z = (z_1, \ldots, z_n) \in V_1$. Since $\varphi \in \mathcal{O}(V_1, \mathbb{D})$, then

$$\sum_{j=1}^n |X_j| = \gamma_D(\varphi_X(0); \varphi_X'(0)X) \leq \gamma_{V_1}(0; X).$$
Recall now that
\[ O(V_1, \mathbb{D}) = \left\{ \sum_{j=1}^{n} f_j - (n - 1)f_1(0) : \right. \]
\[ f_j \in O(\mathbb{D} e_j, \mathbb{D}), \quad f_j(0) = f_k(0), \quad j, k = 1, \ldots, n \} . \]
Therefore
\[ \gamma_{V_1}(0; X) = \sup \{ \gamma_D(F(0); F'(0)X) : F \in O(V_1, \mathbb{D}) \} \]
\[ \leq \sum_{j=1}^{n} \sup \{ \gamma_D(f_j(0); f_j'(0)X_j) : f_j \in O(\mathbb{D}, \mathbb{D}) \} = \sum_{j=1}^{n} |X_j| . \]

Ad (11). Assume that \( X = X_j e_j \). Define \( \psi_{j,X}(\zeta) = \zeta e_j e^{i \arg X_j}, \zeta \in \mathbb{D} \). Observe that \( \psi_{j,X} \in O(\mathbb{D}, V_1), \psi_{j,X}(0) = 0 \) and \( |X_j| \psi_{j,X}'(0) = X \). Hence \( \gamma_{V_1}(0; X) \leq |X_j| \).

To prove the opposite inequality observe that for any \( \psi \in O(\mathbb{D}, V_1) \) there exist \( j \) and \( f \in O(\mathbb{D}, \mathbb{D}) \) such that \( \psi = fe_j \). Hence
\[ \gamma_{V_1}(0; X) = \inf \{ \alpha > 0 : \exists \psi \in O(\mathbb{D}, V_1) : \psi(0) = 0, \quad \alpha \psi'(0) = X \} \]
\[ \geq \inf \{ \alpha > 0 : \exists f \in O(\mathbb{D}, \mathbb{D}) : f(0) = 0, \quad \alpha f'(0) = X_j \} = |X_j| . \]

Now assume that \( X \) is not of the form \( X_j e_j \) for some \( j = 1, \ldots, n \). Then there are \( X_j \neq 0 \neq X_k \) for some \( j \neq k \). Suppose there is a disc \( \psi \in O(\mathbb{D}, V_1) \) such that \( \alpha \psi'(0) = X \) for some \( \alpha > 0 \). This, however, implies that \( \psi_j \neq \text{const} \) and \( \psi_k \neq \text{const} \); a contradiction, since \( j \neq k \). \( \square \)

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**References**


