Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **36** (2010), 285–298

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE CHARACTER OF GROWTH OF A NON-CONTRACTING SEMIGROUP

O. V. Rozumenko

Communicated by S. Troyanski

Abstract. An estimation of the growth of a non-contracting semigroup $Z_t = \exp itA$ where A is a non-dissipative operator with a two-dimensional imaginary component is given. Estimation is given in terms of the functional model in de Branges space.

Contracting semigroup $Z_t = \exp\{itA\}$ generated by a dissipative operator A has a well-studied functional model [6]. In the case of non-dissipativity of operator A, construction of the corresponding functional model is based on the use of the L. de Branges technique [6, 7]. In this case, the semigroup Z_t is not dissipative and its character of growth is exponential [2, 4]. Problem of calculation of the growth index of the semigroup Z_t in terms of functional model seems to be natural. The present paper is dedicated to the solution of this problem. An explicit estimation of the character of the growth of Z_t in terms of channel elements of functional model realized in L. de Branges space is obtained.

2010 *Mathematics Subject Classification:* 47A45.
Key words: non-contracting semigroup, non-dissipative operator.

1. Preliminary information.

I. Recall [6] that set of bounded linear operators acting from Hilbert H into space G is standardly denoted by $[H, G]$.

Family of Hilbert spaces H, E and operators $A \in [H, H]$, $\varphi \in [H, E]$, $J \in [E, E]$, where J is an involution, $J = J^* = J^{-1}$, is said to be [1] a local colligation

(1) $$\Delta = (A, H, \varphi, E, J),$$

if condition

(2) $$A - A^* = i\varphi^* J\varphi$$

is true. Operator A is said to be the main operator of colligation, φ – the channel operator, and J – the metric operator of colligation Δ [3]. Space H is said to be the inner and E – the outer spaces of colligation Δ.

Suppose that the outer space E of the colligation Δ (1) is finite-dimensional, $\dim E = r < \infty$. And let $\{f_\alpha\}_1^r$ be an orthonormal basis in E, then the vectors

$$g_\alpha = \varphi^* f_\alpha \quad (1 \le \alpha \le r)$$

in H is said to be the channel vectors [6], and the colligation relation (2) can be written as follows:

(3) $$\frac{A - A^*}{i} = \sum_{\alpha,\beta=1}^{r} \langle \cdot, g_\alpha \rangle J_{\alpha,\beta} g_\beta,$$

where $J_{\alpha,\beta} = \langle J f_\alpha, f_\beta \rangle$ are matrix elements of the matrix J corresponding to the operator J in the basis $\{f_\alpha\}_1^r$.

Family

(4) $$\Delta = (A, H, \{g_\alpha\}_1^r, J)$$

is said to be an operator complex [3] if condition (3) holds where $J = J^* = J^{-1}$.

Complex (4) is said to be simple [6] if $H_1 = H$, where

$$H_1 = \operatorname{span}\{A^n g_\alpha : 1 \le \alpha \le r \text{ and } n \ge 0\}.$$

On the linear manifold of continuous on $[0, l]$ vector-functions $f(x) = (f_1(x), \ldots, f_r(x))$ with values in the Euclid space, define the hermitian non-negative bilinear form

(5) $$\langle f, g \rangle_F = \int_0^l f(x) dF_x g^*(x),$$

where F_t is a matrix-valued non-decreasing function on $[0,l]$ for which $\operatorname{tr} F_t \equiv t$. Denote by $L^2_{r,l}(F_x)$ the Hilbert space obtained as a result of the closure of the introduced linear manifold of vector-functions $f(x)$ with regard to metric (5) for which $\langle f,f\rangle_F < \infty$ with due factorization by the kernel of metric (5). Define in $L^2_{r,l}(F_x)$ the operator

$$\left(\overset{\circ}{A}_c f\right)(x) = \alpha_x f(x) + i\int_x^l f(t)dF_t J, \tag{6}$$

where α_t is a real non-decreasing bounded on $[0,l]$, $0 \le l < \infty$, function.

Theorem 1 [6]**.** *Simple operator complex Δ (4), when the spectrum of A is real, is unitarily equivalent to the simple part of complex*

$$\overset{\circ}{\Delta}_c = \left(\overset{\circ}{A}_c, L^2_{r,l}(F_x), \{e_\alpha\}_1^r, J\right),$$

where $e_\alpha = (0,\ \ldots,\ \underset{\alpha-1}{0}\ \underset{\alpha}{1}\ \underset{\alpha+1}{0}, \ldots,\ 0)$ is the standard basis in the Euclid space of vector-rows E^r.

Consider a local colligation

$$\Delta = (A, H, \varphi, E, J)$$

such that $\dim E = 2$ and $J = J_N = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$; and let $g_\alpha = \varphi^* e_\alpha$ $(\alpha = 1,\ 2)$ where $\{e_\alpha\}_1^2$ is the orthonormal basis in E. Then we obtain the operator complex

$$\Delta = \left(A, H, \{g_1, g_2\}, J_N = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\right). \tag{7}$$

Let the spectrum of operator A be concentrated at zero, $\sigma(A) = 0$. Then, in view of Theorem 1, the simple complex Δ (7) is unitarily equivalent to the simple part of the model operator complex

$$\overset{\circ}{\Delta}_C = \left(\overset{\circ}{A}_C, L^2_{2,l}(F_x), \{e_1, e_2\}, J_N = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\right), \tag{8}$$

where $\overset{\circ}{A}_C$ in $L^2_{2,l}(F_x)$ acts by the formula

$$\left(\overset{\circ}{A}_C f\right)(x) = i\int_x^l f(\xi)dF_\xi J_N, \tag{9}$$

and the non-decreasing on $[0,l]$ matrix-valued function

$$F_x = \begin{bmatrix} \alpha_x & \beta_x \\ \bar{\beta}_x & \gamma_x \end{bmatrix}$$

is such that $\operatorname{tr} F_x \equiv x$.

II. Denote by $M_x(\lambda)$ the matrix-function which is the solution of the integral equation

$$M_x(\lambda) + i\lambda \int_0^x M_t(\lambda) dF_t J_N = I, \tag{10}$$

where $x \in [0,l]$, $\lambda \in \mathbb{C}$, which in the case of $dF_t = a_t dt$ is equivalent to the Cauchy problem

$$\begin{cases} \dfrac{d}{dx} M_x(\lambda) + i\lambda M_x(\lambda) a_x J_N = 0; \\ M_0(\lambda) = I. \end{cases}$$

Consider the vector-row

$$L_x(\lambda) = [1,0] M_x(\lambda) = \left[A_x(\lambda), B_x(\lambda)\right],$$

which, in virtue of (10), is the solution of integral equation

$$L_x(\lambda) + i\lambda \int_0^x L_t(\lambda) dF_t J_N = [1,0]. \tag{11}$$

Let $2P_\pm = I \pm J_N$, then $P_\pm^2 = P_\pm = P_\pm^*$; $P_+ P_- = 0$; $P_+ + P_- = I$. Single out the following important properties of the vector-row $L_x(z)$:

$$L_x(\lambda) P_+ = E_x(\lambda) L_0^+, \quad L_x(\lambda) P_- = \tilde{E}_x(\lambda) L_0^-,$$

where $L_0^\pm = L_0 P_\pm$, $L_0^+ = \frac{1}{2}[1,i]$, $L_0^- = \frac{1}{2}[1,-i]$ $(L_0 = [1,0])$, and the functions $E_x(\lambda)$ and $\tilde{E}_x(\lambda)$ are given by

$$E_x(\lambda) = A_x(\lambda) - iB_x(\lambda), \quad \tilde{E}_x(\lambda) = A_x(\lambda) + iB_x(\lambda). \tag{12}$$

Function $\tilde{E}_x(\lambda)$ is said to be the adjoint function to $E_x(\lambda)$ (since in the case of the real matrix-function F_t we have $\tilde{E}_x(\lambda) = \overline{E_x\left(\bar{\lambda}\right)}$ [3, 6]).

The following theorem [6] is true.

Theorem 2. *The vector-function $L_x(\lambda) = [A_x(\lambda), B_x(\lambda)]$, which is the non-trivial ($L_x(\lambda) \not\equiv [1,0]$) solution of the integral equation (11), is such that*

1) $L_t(\lambda) \in L^2_{2,a}(F_t)$, *for all* $a \in [0,l]$ *and* $\lambda \in \mathbb{C}$;

2) *functions* $E_x(\lambda) = A_x(\lambda) - iB_x(\lambda)$ *and* $\tilde{E}_x(\lambda) = A_x(\lambda) + iB_x(\lambda)$ *do not have roots in the semiplanes* $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\lambda < 0$ *correspondingly, besides,*

$$|E_x(\lambda)| - \left|\tilde{E}_x(\lambda)\right| = \begin{cases} > 0, & \operatorname{Im}\lambda > 0; \\ = 0, & \operatorname{Im}\lambda = 0; \\ < 0, & \operatorname{Im}\lambda < 0; \end{cases}$$

and $E_x(0) = \tilde{E}_x(0) = 1$, *for all* $x \in [0,l]$.

Recall [1, 6] that function $g(\lambda)$ is said to be a function of the bounded type in $\operatorname{Im}\lambda > 0$ if it is a quotient of two holomorphic bounded in $\operatorname{Im}\lambda > 0$ functions. It is easy to see [1] that if $\operatorname{Re} g(\lambda) \geq 0$ in $\operatorname{Im}\lambda > 0$ and $g(\lambda)$ is analytic in the semiplane $\operatorname{Im}\lambda > 0$, then $g(\lambda)$ is a function of bounded type. This easily yields [1] the following representation of analytic functions $g(\lambda)$ of bounded type in $\operatorname{Im}\lambda > 0$:

$$g(\lambda) = B(\lambda)e^{-i\lambda h}G(\lambda),$$

where $B(\lambda)$ is the Blashke product corresponding to the zeroes of $g(\lambda)$; number $h \in \mathbb{R}$ is said to be the mean type of $g(\lambda)$; and $G(\lambda)$ is holomorphic function in $\operatorname{Im}\lambda > 0$ for which

$$\operatorname{Re} G(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2} \quad (\lambda = x + iy;\ y > 0);$$

besides, the real function $\mu(t)$ is such that $\mu(0) = 0$ and

$$\int_{-\infty}^{\infty} \frac{|d\mu(t)|}{1+t^2} < \infty.$$

Consider a pair of integer functions $A(\lambda)$ and $B(\lambda)$ such that functions $E(\lambda) = A(\lambda) - iB(\lambda)$ and $\tilde{E}(\lambda) = A(\lambda) + iB(\lambda)$ do not have roots in the semiplanes $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ correspondingly, besides

$$|E(\lambda)| - \left|\tilde{E}(\lambda)\right| = \begin{cases} > 0, & \operatorname{Im}\lambda > 0; \\ = 0, & \operatorname{Im}\lambda = 0; \\ < 0, & \operatorname{Im}\lambda < 0. \end{cases}$$

Associate with such pair of functions Hilbert space $\mathcal{B}(A,B)$ [1].

A linear manifold of integer functions $F(\lambda)$ is said to be an L. de Branges space $\mathcal{B}(A,B)$ [1, 6] if

a) $\dfrac{F(\lambda)}{E(\lambda)}\left(\dfrac{F(\lambda)}{\tilde{E}(\lambda)}\right)$ is the function of bounded type and non-positive mean type in the upper, $\operatorname{Im}\lambda>0$ (lower, $\operatorname{Im}\lambda<0$), semiplane;

b)

$$\int_{-\infty}^{\infty}\left|\frac{F(t)}{E(t)}\right|dt=\int_{-\infty}^{\infty}\left|\frac{F(t)}{\tilde{E}(y)}\right|dt<\infty$$

takes place.

The space $\mathcal{B}(A,B)$ is Hilbert [1]. Scalar product in $\mathcal{B}(A,B)$ is specified in the natural way:

$$\langle F(\lambda),G(\lambda)\rangle_{\mathcal{B}(A,B)}=\int_{-\infty}^{\infty}F(t)\bar{G}(t)\frac{dt}{|E(t)|^2}.$$

The L. de Branges Theorem 3 [1]**.** *Consider the family of L. de Branges Hilbert spaces* $\mathcal{B}\left(A_x(\lambda),B_x(\lambda)\right)$ *where the vector-row* $L_x(\lambda)=[A_x(\lambda),$ $B_x(\lambda)]$ *is the solution of the integral equation* (11) *on the segment* $x\in[0,l]$ *for some matrix-valued measure* F_t*. Correlate the function*

$$F(\lambda)=\frac{1}{\pi}\int_0^a[f(t),g(t)]dF_tL_t^*\left(\bar{\lambda}\right), \tag{13}$$

with each row $[f(t),g(t)]\in L^2_{2,l}\left(F_t\right)$ *where* a *is an inner point of the segment* $[0,l]$, $0<a<l$*. Then* $F(\lambda)\in\mathcal{B}\left(A_a(\lambda),\ B_a(\lambda)\right)$*, besides, the "Parseval equality"*

$$\pi\int_{-\infty}^{\infty}\frac{|F(t)|^2}{|E_a(t)|^2}dt=\int_0^a[f(t),g(t)]dF_t\left[\begin{array}{c}\tilde{f}(t)\\ \tilde{g}(t)\end{array}\right]$$

is true. For any function $G(\lambda)\in\mathcal{B}\left(A_a(\lambda),B_a(\lambda)\right)$ *there exists the vector-function* $[\varphi(t),\psi(t)]\in L^2_{2,l}\left(F_t\right)$ *with support on* $[0,a]$ *such that representation* (13) *takes place for* $G(\lambda)$*.*

Theorem 4 [6]**.** *Let the spectrum* $\sigma(A)$ *of operator* A *of the local complex* Δ (7) *be concentrated at zero,* $\sigma(A)=\{0\}$*. Then, in the case of its simplicity,* *complex* Δ *is unitarily equivalent to the functional model*

$$\hat{\Delta} = \left(\hat{A}, \mathcal{B}\left(A_l(\lambda), B_l(\lambda)\right), \left\{\hat{e}_1(\lambda), \hat{e}_2(\lambda)\right\}, J_N\right) \tag{14}$$

where $\hat{A}$ *in* $\mathcal{B}\left(A_l(\lambda), B_l(\lambda)\right)$ *acts via the formula*

$$\hat{A}F(\lambda) = \frac{F(\lambda) - F(0)}{\lambda}, \quad F(\lambda) \in \mathcal{B}\left(A_l(\lambda), B_l(\lambda)\right), \tag{15}$$

and the functions $\hat{e}_\alpha(\lambda)$ *are given by*

$$\hat{e}_1(\lambda) = \frac{B_l^*(\bar{\lambda})}{\lambda}, \quad \hat{e}_2(\lambda) = \frac{1 - A_l^*(\bar{\lambda})}{\lambda}. \tag{16}$$

2. Estimation of growth of the semigroup.

I. Consider the semigroup

$$Z_t f(\xi) = e^{i\hat{A}t} f(\xi) =$$

$$= f(\xi) + it\hat{A}f(\xi) + \frac{i^2t^2}{2!}\hat{A}^2 f(\xi) + \cdots, \quad f(\xi) \in L^2_{2,l}\left(F_\xi\right) \tag{17}$$

where $\hat{A}$ is given by (15).

The explicit formula for Z_t is given by the following theorem [5].

Theorem 5. *The semigroup* $Z_t = \exp(it\hat{A})$, *where* $\hat{A}$ *is given by* (15), *on the functions* $f(\lambda) \in \mathcal{B}\left(A_l(\lambda), B_l(\lambda)\right)$ *acts in the following way:*

$$Z_t f(\lambda) = f(0) + P_+ e^{\frac{it}{\lambda}} \left(f(\lambda) - f(0)\right)$$

where P_+ *is the orthoprojector on the subspace of continuable into the upper semiplane functions.*

Consider the local complex $\hat{\Delta}$ (14) and denote by $\mathcal{M}$ the linear span of vector-functions of the type

$$f(\xi) = \left(u_+(\xi), h(\lambda), u_-(\xi)\right) \tag{18}$$

where $u_{\pm}(\xi)$ is a vector-function from the space of vector-rows $E^2 = E$ such that $\operatorname{supp} u_{\pm}(\xi) \in \mathbb{R}_{\mp}$, and $h(\lambda) \in \mathcal{B}\left(A_l(\lambda), B_l(\lambda)\right)$. Specify on $\mathcal{M}$ the norm

$$\|f\|^2 = \int_{-\infty}^{0} \|u_+(\xi)\|_E^2 d\xi + \|h(\lambda)\|^2 + \int_{0}^{\infty} \|u_-(\xi)\|_E^2 d\xi < \infty. \tag{19}$$

Closure of the manifold $\mathcal{M}$ in this metric forms the Hilbert space, we denote it by $\mathcal{H}$. Denote by P_M [6] the operator of contraction on the set M, namely:

$$(P_M f)(\xi) = f(\xi)\chi_{M}(\xi)$$

where $\chi_M(\xi)$ is the characteristic function of set M ($M \subset \mathbb{R}$) ($\chi_M(\xi) = 1$ as $\xi \in M$, and $\chi_M(\xi) = 0$ as $\xi \notin M$). Specify in the space $\mathcal{H}$ the semigroup U_t,

$$\left(U_t f\right)(\lambda, \xi) = \left(u_+(t,\xi), h_t(\lambda), u_-(t,\xi)\right) \quad (t \geq 0). \tag{20}$$

The vector-function $u_-(t,\xi)$ is given by

$$u_-(t,\xi) = P_{\mathbb{R}_+} u_-(\xi + t). \tag{21}$$

Consider the Cauchy problem

$$\begin{cases} i\dfrac{d}{d\xi}y_t(\lambda,\xi) + \dfrac{y_t(\lambda,\xi) - y_t(0,\xi)}{\lambda} = \displaystyle\sum_{\alpha,\beta=1}^{2} \left\langle P_{(-t,0)} u_-(\xi+t), \hat{e}_\alpha \right\rangle J_{\alpha\beta}\hat{e}_\beta; \\ y_t(\lambda,-t) = h(\lambda), \quad \xi \in (-t,0), \end{cases} \tag{22}$$

where $\hat{e}_\alpha$ is given by (16), and let

$$h_t(\lambda) = y_t(\lambda, 0).$$

Finally,

$$u_+(t,\xi) = u_+(\xi+t) + P_{(-t,0)} \left\{ u_-(\xi+t) - i \sum_{\alpha,\beta=1}^{2} \left\langle y_t(\lambda,\xi), \hat{e}_\alpha \right\rangle \hat{e}_\beta \right\}, \tag{23}$$

where $y_t(\lambda,\xi)$ is the solution of the Cauchy problem (22), it is easy to see that

$$y_t(\lambda,\xi) = h(0) + P_+ e^{\frac{i(\xi+t)}{\lambda}} (h(\lambda) - h(0)) - \\ - i \int_{-t}^{\xi} e^{i\hat{A}(\xi-\theta)} \sum_{\alpha,\beta=1}^{2} \left\langle u_-(\theta+t), \hat{e}_\alpha(\lambda) \right\rangle_{\mathcal{B}} J_{\alpha\beta}\hat{e}_\beta d\theta.$$

Specify in $\mathcal{H}$ an indefinite metric

$$(24)\quad \langle f,f\rangle_J=\int\limits_{-\infty}^{0}\langle J_N u_+(\xi),u_+(\xi)\rangle_E\,d\xi+\|h(\lambda)\|^2_{\mathcal{B}}+\int\limits_{0}^{\infty}\langle J_N u_-(\xi),u_-(\xi)\rangle_E\,d\xi.$$

It is easy to ascertain [6] that $\langle U_t f,U_t f\rangle_J=\langle f,f\rangle_J$ and so the semigroup U_t (20) is a J-isometry.

A semigroup U_t is said to be J-unitary [6] if U_t is unitary in the J-metric (24),

$$U_t^*JU_t=J,\quad U_tJU_t^*=J\quad(\forall t\in\mathbb{R}_+).$$

It is easy to see [7] that U_t is a J-unitary dilation.

Obviously,

$$\|U_tf\|^2=\int\limits_{-\infty}^{0}\|u_+(t,\xi)\|^2_E d\xi+\|h_t(\lambda)\|^2_{\mathcal{B}}+\int\limits_{0}^{\infty}\|u_-(t,\xi)\|^2_E d\xi,$$

where

$$\|h_t(\lambda)\|^2_{\mathcal{B}}=\int\limits_{-\infty}^{\infty}h_t(z)\overline{h_t(z)}\frac{dz}{|E(z)|^2}=\int\limits_{-\infty}^{\infty}|h_t(z)|^2\frac{dz}{|E(z)|^2}.$$

Note that in the case of dissipativity of operator $\hat{A}$ the dilation U_t is unitary. In the studied case, the operator $\hat{A}$ (15) is not dissipative.

As is known [2, 4], for the semigroup U_t, when $|t|\gg 1$, the estimation $\|U_t\|\le e^{\beta_\pm t}$ takes place where $\beta_\pm\ge 0$, besides,

$$\beta_+=\lim_{t\to\infty}\frac{\ln\|U_t\|}{t};\quad \beta_-=\lim_{t\to\infty}\frac{\ln\|U_{-t}\|}{t}.$$

Taking into account (23), we have

$$\|U_tf\|^2=\int\limits_{-\infty}^{-t}\|u_+(\xi+t)\|^2_E d\xi+\int\limits_{-t}^{0}\|u_-(\xi+t)-i\sum_{\alpha,\beta=1}^{2}\langle y_t(\lambda.\xi),\hat{e}_\alpha\rangle\,\hat{e}_\beta\|^2_E d\xi+$$

$$+\|h_t(\lambda)\|^2_B+\int\limits_{0}^{\infty}\|u_-(\xi+t)\|^2_E d\xi=\int\limits_{-\infty}^{0}\|u_+(\xi)\|^2_E d\xi+\int\limits_{t}^{\infty}\|u_-(\xi)\|^2_E d\xi+$$

$$+\int_{-t}^{0}\|u_-(\xi+t)-i\sum_{\alpha,\beta=1}^{2}\langle y_t(\lambda,\xi),\hat{e}_\alpha\rangle\hat{e}_\beta\|_E^2 d\xi+\|h_t(\lambda)\|_B^2$$

Denote

$$u_-(\xi+t)-i\sum_{\alpha,\beta=1}^{2}\langle y_t(\lambda,\xi),\hat{e}_\alpha\rangle\,\hat{e}_\beta=v. \tag{25}$$

The following equality [6] is true,

$$\int_{-t}^{0}\left\langle J_N\left[u_-(\xi+t)-i\sum_{\alpha,\beta=1}^{2}\langle y_t(\lambda,\xi),\hat{e}_\alpha\rangle\,\hat{e}_\beta\right],u_-(\xi+t)-\right.$$
$$\left.-i\sum_{\alpha,\beta=1}^{2}\langle y_t(\lambda,\xi),\hat{e}_\alpha\rangle\,\hat{e}_\beta\right\rangle d\xi+$$
$$+\|h_t(\lambda)\|^2=\int_{0}^{t}\langle J_N u_-(\xi),u_-(\xi)\rangle\,d\xi+\|h(\lambda)\|^2,$$

or, taking into account (25), we have

$$\int_{-t}^{0}\langle J_N v,v\rangle d\xi+\|h_t(\lambda)\|^2=\int_{0}^{t}\langle J_N u_-(\xi),u_-(\xi)\rangle d\xi+\|h(\lambda)\|^2.$$

Let $J_N=Q_+-Q_-$ where $Q_\pm$ are such orthoproectors that $Q_++Q_-=I$ and $Q_+Q_-=0$, then

$$\int_{-t}^{0}\|v\|^2 d\xi+\|h_t(\lambda)\|^2=\int_{-t}^{0}\langle Jv,v\rangle d\xi+\|h_t(\lambda)\|^2+2\int_{-t}^{0}\langle Q_-v,v\rangle d\xi=$$
$$=\int_{0}^{t}\langle Ju_-(\xi),u_-(\xi)\rangle d\xi+\|h(\lambda)\|^2+2\int_{-t}^{0}\langle Q_-v,v\rangle d\xi=$$
$$=\int_{0}^{t}\|u_-(\xi)\|d\xi+\|h(\lambda)\|^2-2\int_{0}^{t}\langle Q_-u_-(\xi),u_-(\xi)\rangle d\xi+2\int_{-t}^{0}\langle Q_-v,v\rangle d\xi.$$

Thus,

$$\|U_t f\|^2 = \int\limits_{-\infty}^{0} \|u_+(\xi)\|_E^2 d\xi + \int\limits_{t}^{\infty} \|u_-(\xi)\|_E^2 d\xi + \int\limits_{0}^{t} \|u_-(\xi)\|_E^2 d\xi +$$

$$+\|h(\lambda)\|_B^2 - 2\int\limits_{-t}^{0} \langle Q_- u_-(\xi+t), u_-(\xi+t)\rangle d\xi + 2\int\limits_{-t}^{0} \langle Q_+ v, v\rangle d\xi =$$

$$= \|f\|^2 + 2\int\limits_{-t}^{0} \left[\langle Q_+ v, v\rangle_E - \langle Q_- u_-(\xi+t), u_-(\xi+t)\rangle_E\right] d\xi$$

II. Let

$$A_+ = \hat{A} + i\varphi^* Q_- \varphi,$$

then

$$\hat{A} = A_+ - i\varphi^* Q_- \varphi,$$

where A_+ is a dissipative operator [6]. Denote

(26) $$V_t = e^{-itA_+}, \quad Z_t = e^{it\hat{A}}.$$

Then

$$\frac{d}{dt}(V_t Z_t) = -iA_+ V_t Z_t + V_t(iA)Z_t = V_t(-iA_+)Z_t + V_t(i\hat{A})Z_t =$$

$$= iV_t\left(\hat{A} - A_+\right)Z_t = iV_t(-i\varphi^* Q_-\varphi)Z_t = V_t\varphi^* Q_-\varphi Z_t$$

Consequently,

$$V_t Z_t - I = \int\limits_{0}^{t} V_s \varphi^* Q_- \varphi Z_s ds,$$

multiply both parts of equality by V_{-t}, obtain

(27) $$Z_t = V_{-t} + \int\limits_{0}^{t} V_{s-t}\varphi^* Q_- \varphi Z_s ds,$$

then

$$Z_t h = V_{-t} h + \int\limits_0^t V_{s-t} \varphi^* Q_- \varphi Z_s h ds$$

consequently,

$$\|Z_t\| \leq \|V_{-t}\| + \left\| \int\limits_0^t V_{s-t} \varphi^* Q_- \varphi Z_s ds \right\|.$$

Rewrite (27) as follows,

$$Z_t - \int\limits_0^t V_{s-t} T Z_s ds = V_{-t},$$

where $T = \varphi^* Q_- \varphi$. By the mean theorem,

$$Z_t - t V_{\xi - t} T Z_\xi = V_{-t}$$

where $\xi = \xi(t) \in (0, t)$. Let $s = 0$, then

$$\|Z_t\| \leq \|V_{-t}\| + \|t V_{\xi - t} T Z_\xi\|.$$

Since V_t is a contraction semigroup, then $\|V_t\| \leq 1$ and thus

$$\|Z_t\| \leq 1 + \|t V_{\xi - t} T Z_\xi\|$$

or

$$\|Z_t\| \leq 1 + t \|V_{\xi - t} T\| e^{\alpha \theta_t t},$$

where $0 < \theta_t < 1$,

$$\alpha = \lim_{t \to \infty} \frac{\ln \|Z_t\|}{t}.$$

Then

$$\frac{\|Z_t\| - 1}{t} \leq \|V_{\xi - t} T\| e^{\alpha \theta_t t} \leq \|T\| e^{\alpha \theta_t t}$$

Since

$$\|Z_t\| \leq e^{\alpha t},$$

then

$$\frac{\|Z_t\| - 1}{t} \leq \frac{e^{\alpha t} - 1}{t} \to \alpha$$

when $t \to 0$. From the other side,

$$\frac{\|Z_t\| - 1}{t} \le \|T\| e^{\alpha\theta_t t} \to \|T\|$$

as $t \to 0$. Thus,

$$\alpha \le \|T\|.$$

Let us estimate $\|T\|$.

$$\langle Tf, f\rangle = \langle \varphi^* Q_- \varphi f, f\rangle = \langle Q_- \varphi f, \varphi f\rangle$$

$$|\langle Q_- \varphi f, \varphi f\rangle| \le \|Q_-\| \langle \varphi f, \varphi f\rangle \le \langle \varphi f, \varphi f\rangle = \|\varphi f\|^2$$

As is well-known,

$$\varphi f = \begin{pmatrix} \langle f, \hat{e}_1\rangle \\ \langle f, \hat{e}_2\rangle \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

then

$$\|\varphi f\|^2 = |f_1|^2 + |f_2|^2$$

$$\int\limits_{-\infty}^{\infty} f(x) \frac{B_l(x)}{x} \frac{dx}{|E_l(x)|^2} = f_1; \quad \int\limits_{-\infty}^{\infty} f(x) \frac{1 - A_l(x)}{x} \frac{dx}{|E_l(x)|^2} = f_2$$

$$|f_1| \le \|f\| \cdot \|\hat{e}_1\|; \quad |f_2| \le \|f\| \cdot \|\hat{e}_2\|$$

then

$$\|\varphi f\|^2 \le \|f\|^2 \|\hat{e}_1\|^2 + \|f\|^2 \|\hat{e}_2\|^2 = (\|\hat{e}_1\|^2 + \|\hat{e}_2\|^2)\|f\|^2$$

So,

$$\|\varphi f\| \le \sqrt{\|\hat{e}_1\|^2 + \|\hat{e}_2\|^2}\,\|f\|$$

or

$$\|\varphi\| \le \sqrt{\|\hat{e}_1\|^2 + \|\hat{e}_2\|^2},$$

where

$$\|\hat{e}_1\|^2 = \int\limits_{-\infty}^{\infty} \frac{|B_l(x)|^2}{|x|^2} \frac{dx}{|E_l(x)|^2}; \quad \|\hat{e}_2\|^2 = \int\limits_{-\infty}^{\infty} \frac{|1 - A_l(x)|^2}{|x|^2} \frac{dx}{|E_l(x)|^2}.$$

Thus,

$$\|\varphi^* Q_- \varphi\| = \|T\| \le \sqrt{\|\hat{e}_1\|^2 + \|\hat{e}_2\|^2}.$$

So, we have proved the following theorem.

Theorem 6. *For the semigroup Z_t (17), an estimation $\|Z_t\| \leq e^{\alpha t}$; $(t \gg 1)$ is true, where α is estimated in the following way:*

$$\alpha \leq \sqrt{\|\hat{e}_1\|^2 + \|\hat{e}_2\|^2},$$

besides, $\hat{e}_1$, $\hat{e}_2$ are given by (16).

Thus, for the semigroup Z_t (17) an explicit estimation of character of the growth of semigroup Z_t is given in terms of parameters of the colligation Δ (14).

REFERENCES

[1] L. de Branges. Hilbert spaces of entire functions. London, Prentice-Hall, 1968.

[2] S. G. Krein. Linear differential equations in Banach space. Nauka, Moskow, 1967, 464 pp (in Russian).

[3] M. S. Livšic, A. A. Yantsevich. Theory of operator colligation in Hilbert space. J. Wiley, N. Y., 1979.

[4] Yu. Lyubich. Linear Functional Analysis, Enc. of Math. Sci., Vol. 19, Berlin: Springer Verlag, 1992, 210 pp.

[5] O. V. Rozumenko. On the scattering problem for the L. de Branges functional model. *Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh.* **826**, *58* (2008), 100–114 (in Russian, English summary).

[6] V. A. Zolotarev. Analitic methods of spectral representations of non-selfadjoint and nonunitary operators. Kharkiv, KhNU, 2003, 342 pp (in Russian).

[7] V. A. Zolotarev, O. V. Rozumenko. Pavlov functional model of bounded non-selfadjoint operator. *Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh.* **749**, *56* (2006), 30–49 (in Russian, English summary).

Olga V. Rozumenko
Dept. of Higher Mathematics and Information Science
Kharkiv National University
Kharkiv, Ukraine
e-mail: `ovrozumenko@gmail.com`

Received October 15, 2010