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RESONANCES OF TWO-DIMENSIONAL SCHRÖDINGER OPERATORS WITH STRONG MAGNETIC FIELDS

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Abstract. The purpose of this paper is to study the Schrödinger operator $P(B,\omega) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 + V(x,y)$, $(x,y) \in \mathbb{R}^2$, with the magnetic field B large enough and the constant $\omega \neq 0$ is fixed and proportional to the strength of the electric field. Under certain assumptions on the potential V, we prove the existence of resonances near Landau levels as $B \to \infty$. Moreover, we show that the width of resonances is of size $\mathcal{O}(B^{-\infty})$.

1. Introduction. The model of Schrödinger operator studied in this article is the following

$$P(B,\omega) = P_0(B,\omega) + V(x,y) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 + V(x,y), \tag{1.1}$$

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defined on $L^2(\mathbb{R}^2)$, where $D_\nu = \frac{1}{i}\partial_\nu$, B is the strong magnetic field, $\omega \neq 0$ is a fixed constant and the potential V is a real smooth function decreasing at infinity.

It is well-known that the operator $P(B,\omega)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, (see [12, 23]). For $V \equiv 0$, $\omega = 0$, it was shown that the spectrum of the unperturbed Hamiltonian $P_0(B,0) := (D_x - By)^2 + D_y^2$ consists of eigenvalues $\lambda_n = (2n+1)B$, $n \in \mathbb{N}$ with infinite multiplicities called Landau levels (see [1, 7, 24, 25]). However, in the case $\omega \neq 0$, the essential spectrum of $P_0(B,\omega)$ is absolutely continuous and equal to the semi-axis $[\sqrt{B^2+\omega^2}, +\infty)$, (see [23]). On the other hand, whenever the potential V vanishes at the infinity, one can show as in [1] that $V(P_0(B,\omega)+i)^{-1}$ is a compact operator. By applying the Weyl theorem (see [15]) the essential spectrum of $P(B,\omega)$ is equal to that of $P_0(B,\omega)$. Further, the absolutely continuous spectrum of $P(B,\omega)$ was investigated in [12]. Recently, the counting function of discrete eigenvalues of $P(B,\omega)$ in $(-\infty, \sqrt{B^2+\omega^2})$ has been studied in [10].

Until now, there has been little discussion about the spectral problem of $P(B,\omega)$ which can be also regarded as the quantum hall system Hamiltonian with the unbounded edge potential (see [5, 19] and also [3]). In this work, we propose to study the existence of resonances of $P(B,\omega)$ near Landau levels when the strength of magnetic field tends to infinity. Roughly speaking, the resonances of $P(B,\omega)$ are defined by eigenvalues of some dilated operator (see below).

The past thirty years have seen increasingly rapid advances in the study of resonances of Schrödinger operators with magnetic fields (see [2, 8, 18, 27, 28] and references therein). For the two-dimensional Stark Hamiltonian with strong magnetic field, X. P. Wang proved that there exist resonances near Landau levels, (see [28]). Moreover, M. Dimassi and V. Petkov showed that there does not exist resonances in the upper-half complex plane, (see [8]). However, we notice here that the definitions of resonances in these articles are a little bit different. The resonances are defined by the complex dilation in [28] and by the complex transition in the $x-$variable in [8]. For three-dimensional Schrödinger operators without Stark effect, one of the results of J.F.Bony *et al.* showed that there exist infinitely many resonances in a vicinity of each Landau level (see [2]).

By using the arguments in [9, Chapter 7], we obtain that $P(B,\omega)$ is unitarily equivalent to

$$P_1(B,\omega) := \sqrt{B^2+\omega^2}(D_y^2+y^2) + \omega^2 x^2$$

$$+V^w\left((B^2+\omega^2)^{-\frac{1}{4}}D_y + \left(1+\frac{\omega^2}{B^2}\right)^{-\frac{1}{2}}x,\ (B^2+\omega^2)^{-\frac{1}{2}}D_x + B^{-\frac{1}{2}}\left(1+\frac{\omega^2}{B^2}\right)^{-\frac{3}{4}}y\right).$$

Here we use the Weyl quantization (see [13, 16]).

Let θ be real. Consider the unitary operator $U_\theta : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $u(x,y) \mapsto u(e^\theta x, e^{-\theta} y)$. One has

$$P_{1,\theta}(B,\omega) := U_\theta P_1(B,\omega) U_\theta^{-1} = \sqrt{B^2+\omega^2}(e^{2\theta} D_y^2 + e^{-2\theta} y^2) + e^{2\theta}\omega^2 x^2$$
$$+ V^w \left(e^\theta \left((B^2+\omega^2)^{-\frac{1}{4}} D_y + \left(1+\frac{\omega^2}{B^2}\right)^{-\frac{1}{2}} x \right), \right.$$
$$\left. e^{-\theta} \left((B^2+\omega^2)^{-\frac{1}{2}} D_x + B^{-\frac{1}{2}} \left(1+\frac{\omega^2}{B^2}\right)^{-\frac{3}{4}} y \right) \right).$$

We set

$$P_{0,\theta}(B,\omega) := \sqrt{B^2+\omega^2}(e^{2\theta} D_y^2 + e^{-2\theta} y^2) + e^{2\theta}\omega^2 x^2, \tag{1.2}$$

and

$$V_\theta^w(B,\omega) = V^w \left(e^\theta \left((B^2+\omega^2)^{-\frac{1}{4}} D_y + \left(1+\frac{\omega^2}{B^2}\right)^{-\frac{1}{2}} x \right), \right.$$
$$\left. e^{-\theta} \left((B^2+\omega^2)^{-\frac{1}{2}} D_x + B^{-\frac{1}{2}} \left(1+\frac{\omega^2}{B^2}\right)^{-\frac{3}{4}} y \right) \right),$$

then we have

$$P_{1,\theta}(B,\omega) = P_{0,\theta}(B,\omega) + V_\theta^w(B,\omega).$$

By using the analytic extension of the potential V (see $(\mathbf{H_1})$), we can extend the formula of $P_{1,\theta}(B,\omega)$ from the real axis to a small complex neighbourhood of 0 with respect to θ. In this paper we define the resonances of $P(B,\omega)$ as the eigenvalues of the non-selfadjoint operator $P_{1,\theta}(B,\omega)$ for $\theta \in \mathbb{C}$, $|\theta|$ small and $\operatorname{Im}\theta < 0$. Moreover the eigenvalues and their multiplicities are independent of θ (see [4, 15]).

From now on, we fix $\theta \in \mathbb{C}$ satisfying $\operatorname{Im}\theta < 0$ and $|\theta|$ small enough. Note that the essential spectra of $P_{0,\theta}(B,\omega)$ and $P_{1,\theta}(B,\omega)$ are coincident and given by $\bigcup_{n\in\mathbb{N}} \{(2n+1)\sqrt{B^2+\omega^2} + e^{2\theta}\lambda,\ \lambda \geq 0\}$ (see Lemma 3.1). Then the resonances are distributed outside these semi-lines. As is mentioned above, we are interested in localizing the resonances near Landau levels. To do this, we follow the strategy used by X. P. Wang in [27, 28] and the recent progress in the analysis of two-dimensional Schrödinger operators with magnetic fields (see [6, 7, 8]).

We fix $n \in \mathbb{N}$. Then we set $\mu_n := (2n+1)\sqrt{B^2+\omega^2}$ and $h := \dfrac{1}{\sqrt{B^2+\omega^2}}$. Let $E \in \mathbb{R} \setminus \{0\}$. In Section 3 we prove that z is an eigenvalue of $P_{1,\theta}(B,\omega) - \mu_n$ near E if and only if 0 is an eigenvalue of an h-pseudodifferential operator (called the effective Hamiltonian). Here the effective Hamiltonian is given by

$$E_{-+}(z) = z - A_\theta(h) + h^2 G_\theta(z;h), \tag{1.3}$$

where $G_\theta(z;h)$ is holomorphic for z in some large, bounded set T_n (see (3.14)) and $A_\theta(h)$ does not depend on z (see Theorem 3.5). Moreover $A_\theta(h)$ is also an h-pseudodifferential operator with symbol $a(e^\theta x, e^{-\theta}\xi; h)$ which admits a complete expansion in powers of h (see identity (3.25)):

$$a(x,\xi;h) - a_0(x,\xi) \sim \sum_{j\geq 1} h^j a_j(x,\xi),$$

where

$$a_0(x,\xi) = \omega^2 x^2 + V(x,\xi) \ \text{ and } \ a_1(x,\xi) = \frac{(2n+1)}{4}\Delta V(x,\xi). \tag{1.4}$$

For the h-pseudodifferential operators, we refer the readers to [9, 11].

Therefore, the localization of resonances of $P(B,\omega)$ can be deduced from studying the spectrum of $A_\theta(h)$. In fact, the crucial steps to prove the existence of resonances are the following:

• Prove the exponential decay of the eigenfunctions of $A_\theta(h)$ associated to the eigenvalues near E (see Theorem 4.4).

• Establish a resolvent estimate in the non-selfadjoint case (see Proposition 4.6).

For these two points, we need a non-trapping condition (see $(\mathbf{H_3})$).

For the width of resonances, as in [20] we use the WKB method to construct an approximate solution of the problem $E_{-+}(z)u = \mathcal{O}(h^\infty)$. Thus by studying a suitable Grushin problem, we obtain the expansion of each resonance in powers of h with real coefficients. This means that the width of resonances is at least of size $\mathcal{O}(h^\infty)$.

The rest of paper is organized as follows. In Section 2 we give our assumptions and results. The essential spectrum of $P_{1,\theta}(B,\omega)$ is computed in Subsection 3.1. Next the Grushin problem is constructed in Subsection 3.2 to establish a reduction to the effective Hamiltonian. In Section 4, we study the spectral property of the leading term of the effective Hamiltonian. The existence of resonances of $P_{1,\theta}(B,\omega)$ is proved in Subsection 5.1 and the width of resonances is showed in Subsection 5.2.

2. Assumptions and results. In this section, we will present our hypotheses and our main result. We recall the operator

$$P(B,\omega) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 + V(x,y),\ (x,y) \in \mathbb{R}^2,$$

where the potential V satisfies the following hypothesis:

($\mathbf{H_1}$) There exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\delta > 0$ such that V admits an analytic extension on the domain

$$\mathcal{A} = \{(z_1,z_2) \in \mathbb{C}^2;\ |\mathrm{Im}\,(z_1,z_2)| \le \alpha_1 |\mathrm{Re}\,(z_1,z_2)| + \alpha_2\},$$

and for all $(z_1,z_2) \in \mathcal{A}$

$$|V(z_1,z_2)| \le \alpha_3 \langle \mathrm{Re}\,(z_1,z_2)\rangle^{-\delta}.$$

Here we denote $\langle (t,s)\rangle = (1+t^2+s^2)^{\frac{1}{2}}$, $(t,s) \in \mathbb{R}^2$.

We recall that the total electric potential $a_0(x,y) = \omega^2 x^2 + V(x,y)$ (see (1.4)). We introduce the following assumption:

($\mathbf{H_2}$) Let $E \in \mathbb{R} \setminus \{0\}$. Suppose that a_0 has a local non-degenerate maximum (or minimum) point E at (x_0,y_0), i.e., the definite Hessian $a_0''(x_0,y_0) < 0$ (or $a_0''(x_0,y_0) > 0$).

By the translation, we can always assume that $(x_0,y_0) = (0,0)$. Set

$$\Omega_E = \{(x,y) \in \mathbb{R}^2;\ a_0(x,y) = E\}.$$

($\mathbf{H_3}$) (On the non-trapping condition) Assume that $\Omega_E = \{(0,0)\} \cup \Gamma$, where Γ is a connected curve and $(0,0)$ is an isolated point, and that the classical Hamiltonian $a_0(x,\xi)$ is non-trapping on Γ:

$$\begin{aligned}(2.1)\qquad \{a_0(x,\xi), G_0(x,\xi)\} &= \partial_\xi a_0 \partial_x G_0 - \partial_x a_0 \partial_\xi G_0 \\ &= \xi \partial_\xi a_0(x,\xi) - x\partial_x a_0(x,\xi) \neq 0,\ \forall (x,\xi) \in \Gamma,\end{aligned}$$

where $G_0(x,\xi) = x.\xi$, $\forall (x,\xi) \in \mathbb{R}^2$.

Our main result is the following:

Theorem 2.1. *Assume that the assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. For each* n, *we define*

$$U_n = \Big\{ z \in \mathbb{C};\ \operatorname{Re} z \in \Big](2n+1)B + E - \frac{C_0}{B}, (2n+1)B + E + \frac{C_0}{B}\Big[, \quad \operatorname{Im} z \in \Big] - \frac{1}{C_0 B}, 0\Big] \Big\}, \tag{2.2}$$

where $C_0 > 1$ *can be arbitrarily large outside a discrete set in* $\mathbb{R}$. *Then for* B *large enough, the resonances of* $P(B,\omega)$ *in* U_n *exist and are all given by complete expansions in powers of* B^{-1}:

$$E_{n,j}(B,\omega) \sim (2n+1)B + E + \frac{1}{2}\left(\pm(2j+1)(\lambda\mu)^{\frac{1}{2}} + (2n+1)\frac{\lambda+\mu}{2}\right)B^{-1} + \sum_{k\geq 2} c^{(k)}_{\pm;n,j} B^{-k}, \tag{2.3}$$

where $c^{(k)}_{\pm;n,j} \in \mathbb{R}$, λ *and* μ *are eigenvalues of the Hessian* $a_0''(0,0)$, *and the sign* $+(-)$ *corresponds to a local minimum, maximum respectively.*

Moreover, the resonances of $P(B,\omega)$ *in* U_n *are all algebraically simple and the width of resonances is of order* $\mathcal{O}(B^{-\infty})$.

Remark 2.2. Notice that in the half-plane $\{z \in \mathbb{C};\ \operatorname{Re} z < \sqrt{B^2+\omega^2}\}$, the poles of the meromorphic extension of the resolvent from $\{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$ to $\{z \in \mathbb{C};\ \operatorname{Im} z < 0\}$ are all given by the discrete eigenvalues of $P(B,\omega)$. Then the resonances in this half-plane are identically equal to the set of discrete eigenvalues of $P(B,\omega)$. Therefore, let $E < 0$, the width of $E_{0,k}(B,\omega)$ is equal to 0.

We want to give an example to illustrate our main result.

Consider $V(x,y) = -\frac{c_1}{x^4+1} - \frac{c_2}{y^2+1}$, $c_1, c_2 > 0$. Then $a_0(x,y) = \omega^2 x^2 - \frac{c_1}{x^4+1} - \frac{c_2}{y^2+1}$ and one has

$$\begin{cases} \partial_x a_0(x,y) = 0 \\ \partial_y a_0(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 2\omega^2 x + \dfrac{4c_1 x^3}{(x^4+1)^2} = 0 \\ \dfrac{2c_2 y}{(y^2+1)^2} = 0. \end{cases} \tag{2.4}$$

The system (2.4) has only one solution $(x,y) = (0,0)$ and $a_0(0,0) = -c_1 - c_2$. It is easy to compute the Hessian at $(0,0)$ of a_0:

$$a_0''(0,0) = \begin{pmatrix} 2\omega^2 & 0 \\ 0 & 2c_2 \end{pmatrix} > 0.$$

It shows that a_0 has a local minimum point at $(0,0)$. On the other hand, $a_0^{-1}(-c_1-c_2)=\{(0,0)\}$. Therefore we do not need to verify the non-trapping condition as in this case $\Gamma=\emptyset$. Our main result shows that there exist resonances of $P(B,\omega)$ near $(2n+1)B-c_1-c_2$ for B large enough and for all $n\in\mathbb{N}$.

3. Reduction to the semiclassical effective Hamiltonian. In this section, we reduce the study of resonances of $P(B,\omega)$ to the spectral study of an h-pseudodifferential operator.

3.1. Spectral properties of $\boldsymbol{P_{1,\theta}(B,\omega)}$. In this subsection, we compute the essential spectrum of $P_{1,\theta}(B,\omega)$ and we give some resolvent estimates.

Lemma 3.1. *Let θ be in a small complex neighbourhood of* 0*. Then,*

$$\sigma_{\mathrm{ess}}(P_{1,\theta}(B,\omega))=\sigma_{\mathrm{ess}}(P_{0,\theta}(B,\omega))=\bigcup_{n\in\mathbb{N}}\{\mu_n+e^{2\theta}\lambda,\ \lambda\geq 0\},\tag{3.1}$$

where $\mu_n=(2n+1)\sqrt{B^2+\omega^2}$ and $P_{0,\theta}(B,\omega)$ given by (1.2)*.*

Proof. Recall that $P_{0,\theta}(B,\omega)=\sqrt{B^2+\omega^2}(e^{2\theta}D_y^2+e^{-2\theta}y^2)+e^{2\theta}\omega^2x^2$. Then when $\omega=0$, one has $P_{0,\theta}(B,0)=\sqrt{B^2+\omega^2}(e^{2\theta}D_y^2+e^{-2\theta}y^2)$. We are going to determine the spectrum of $P_{0,\theta}(B,0):L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$.

Denote by $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ the restriction of $P_{0,\theta}(B,0)$ on $L^2(\mathbb{R}_y)$. Let us consider

$$\mathfrak{A}=\{\psi_n(y);\ \psi_n(y)=H_n(y)e^{-\frac{y^2}{2}}\ \text{where } H_n(\cdot) \text{ is the } n\text{-th Hermite polynomial, } n\in\mathbb{N}\}$$

the set of normalized eigenfunctions of one-dimensional harmonic operator, i.e.,

$$(D_y^2+y^2)(\psi_n(y))=(2n+1)\psi_n(y).$$

For $\theta\in\mathbb{C}$ near 0, we set $\psi_{n,\theta}(y)=e^{-\frac{\theta}{2}}\psi_n(e^{-\theta}y)$, $n\in\mathbb{N}$. We put $\mathfrak{A}_\theta=\{\psi_{n,\theta};\ n\in\mathbb{N}\}$. Then one has $\mathfrak{A}_\theta\subset L^2(\mathbb{R})$ and $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}(\psi_{n,\theta}(y))=\mu_n\psi_{n,\theta}(y)$, $n\in\mathbb{N}$. It shows that $\bigcup_{n\in\mathbb{N}}\{\mu_n\}\subset\sigma\Big(P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}\Big)$. On the other hand, since $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ is elliptic for $|\theta|$ small, then its spectrum is discrete. In addition, $\mathfrak{A}_\theta$ is a dense set in $L^2(\mathbb{R})$ (see [15, Chapter 16]). Then if $\lambda\notin\bigcup_{n\in\mathbb{N}}\{\mu_n\}$ is an eigenvalue of $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ and the corresponding eigenfunction f, one has $f \in \mathfrak{A}_\theta^\perp = \{0\}$. Therefore,

$$\sigma\Big(P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}\Big) = \bigcup_{n\in\mathbb{N}} \{\mu_n\}. \tag{3.2}$$

In fact, we can write $P_{0,\theta}(B,0) = \mathrm{Id}_{L^2(\mathbb{R}_x)} \otimes P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ and the multiplication operator $e^{2\theta}\omega^2x^2 = e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)} \otimes \mathrm{Id}_{L^2(\mathbb{R}_y)}$. Here $e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}$ is the natural restriction of the multiplication operator $e^{2\theta}\omega^2x^2$ on $L^2(\mathbb{R}_x)$. Then it is easy to verify that the operator $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ and the multiplication operator $e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}$ satisfy [26, Theorem XIII.35]. It enables us to obtain:

$$\sigma(P_{0,\theta}(B,\omega)) = \sigma\Big(P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}\Big) + \sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big). \tag{3.3}$$

Moreover, $\sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big) = \{e^{2\theta}\lambda;\ \lambda \geq 0\}$. Combining this with (3.2) and (3.3), one obtains

$$\sigma(P_{0,\theta}(B,\omega)) = \bigcup_{n\in\mathbb{N}} \{\mu_n + e^{2\theta}\lambda; \lambda \geq 0\}, \tag{3.4}$$

and then the discrete spectrum of $P_{0,\theta}(B,\omega)$ is empty.

Now we prove that the essential spectrum of $P_{1,\theta}(B,\omega)$ is equal to that of $P_{0,\theta}(B,\omega)$. Firstly we can show as in [1] that $V(P_0(B,\omega)-z)^{-1}$ is a compact operator for $z \notin \sigma(P_0(B,\omega))$. By the unitary equivalence, $T(\theta) := V_\theta^w(B,\omega)(P_{0,\theta}(B,\omega)-z)^{-1}$ is also a compact operator for θ real and $z \notin \sigma(P_{0,\theta}(B,\omega))$. Further, since $T(\theta)$ is an analytic bounded operator-valued function in θ near 0, it is compact for all θ near 0 (see [26, page 126, Lemma 5]). From this we can apply [17, page 244, Theorem 5.35] and achieve

$$\sigma_{\mathrm{ess}}(P_{1,\theta}(B,\omega)) = \sigma_{\mathrm{ess}}(P_{0,\theta}(B,\omega)) = \bigcup_{n\in\mathbb{N}} \{\mu_n + e^{2\theta}\lambda; \lambda \geq 0\}, \tag{3.5}$$

where $\mu_n = (2n+1)\sqrt{B^2+\omega^2}$. $\square$

For each fixed $n \in \mathbb{N}$, we denote by ψ_n the normalized eigenfunction of the harmonic oscillator corresponding to the eigenvalue $(2n+1)$ (i.e., $(D_y^2 +$

$y^2)\psi_n(y) = (2n+1)\psi_n(y)$ and $\|\psi_n\|_{L^2(\mathbb{R})} = 1$). Put $\psi_{n,\theta}(y) = e^{-\frac{\theta}{2}}\psi_n(e^{-\theta}y)$. Since $\psi_n(y)$ is of the form $H_n(y)e^{-y^2/2}$, where H_n is the n-th Hermite polynomial, we have $\overline{\psi_{n,\theta}} = \psi_{n,\bar\theta}$ and $\langle\psi_{n,\theta}, \psi_{n,\bar\theta}\rangle = \|\psi_n\|^2 = 1$. We next consider the following operators

$$R_- : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2), \qquad R_+ : L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$$
$$u(x) \mapsto u(x)\psi_{n,\theta}(y) \qquad u(x,y) \mapsto \langle u, \psi_{n,\bar\theta}\rangle_{L^2(\mathbb{R}_y)}$$

and

$$\Pi_n : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$
$$u(x,y) \mapsto \langle u, \psi_{n,\bar\theta}\rangle_{L^2(\mathbb{R}_y)}\psi_{n,\theta}(y),$$

here the scalar product $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}_y)}$ denotes the integration in the y variable. The natural restriction of Π_n on $L^2(\mathbb{R}_y)$ we also denote by Π_n. From the definition, one has

$$R_+R_-u(x) = R_+(u(x)\psi_{n,\theta}(y)) = u(x)$$
$$R_-R_+v(x,y) = \langle u, \psi_{n,\bar\theta}\rangle_{L^2(\mathbb{R}_y)}\psi_{n,\theta}(y) = \Pi_n v(x,y).$$

Lemma 3.2 *Let θ be in a small complex neighbourhood of* 0. *Then we have*

$$\sigma(\Pi_n P_{0,\theta}(B,\omega)\Pi_n) = \{\mu_n + e^{2\theta}\lambda;\ \lambda \geq 0\}, \tag{3.6}$$

and

$$\sigma((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)) = \bigcup_{k\in\mathbb{N}\setminus\{n\}} \{\mu_k + e^{2\theta}\lambda;\ \lambda \geq 0\}. \tag{3.7}$$

Here $\Pi_n P_{0,\theta}(B,\omega)\Pi_n : \Pi_n L^2(\mathbb{R}^2) \to \Pi_n L^2(\mathbb{R}^2)$ *and* $(1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n) : (1-\Pi_n)L^2(\mathbb{R}^2) \to (1-\Pi_n)L^2(\mathbb{R}^2)$.

Proof. First we demonstrate (3.6).

We observe that the Hilbert space $\Pi_n L^2(\mathbb{R}_y)$ is generated by $\psi_{n,\theta}$. Then it can be readily verified that

$$\sigma\Big(\Pi_n P_{0,\theta}(B,0)\Pi_n\Big|_{\Pi_n L^2(\mathbb{R}_y)}\Big) = \{\mu_n\}. \tag{3.8}$$

We recall that $\sigma\Big(e^{2\theta}\omega^2 x^2\Big|_{L^2(\mathbb{R}_x)}\Big) = \{e^{2\theta}\lambda;\ \lambda \geq 0\}$. Then as in (3.3), it follows from [26, Theorem XIII.35] that:

$$\sigma(\Pi_n P_{0,\theta}(B,\omega)\Pi_n) = \sigma\Big(\Pi_n P_{0,\theta}(B,0)\Pi_n\Big|_{\Pi_n L^2(\mathbb{R}_y)}\Big) + \sigma\Big(e^{2\theta}\omega^2 x^2\Big|_{L^2(\mathbb{R}_x)}\Big) \tag{3.9}$$

$$= \{\mu_n + e^{2\theta}\lambda;\ \lambda \geq 0\}. \tag{3.10}$$

Secondly, we prove (3.7) in the same way as above.

By applying [15, Proposition 6.9], one has

$$\sigma\Big((1-\Pi_n)P_{0,\theta}(B,0)(1-\Pi_n)\Big|_{L^2(\mathbb{R}_y)}\Big) = \bigcup_{k\in\mathbb{N}\setminus\{n\}} \{\mu_k\}. \tag{3.11}$$

Then we use again [26, Theorem XIII.35] and derive

$$\sigma((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)) = \sigma\Big((1-\Pi_n)P_{0,\theta}(B,0)(1-\Pi_n)\Big|_{L^2(\mathbb{R}_y)}\Big) + \sigma\Big(e^{2\theta}\omega^2 x^2\Big|_{L^2(\mathbb{R}_x)}\Big) \tag{3.12}$$

$$= \bigcup_{k\in\mathbb{N}\setminus\{n\}} \{\mu_k + e^{2\theta}\lambda;\ \lambda \geq 0\}. \tag{3.13}$$

□

From now on, we fix $n \in \mathbb{N}$. We put

$$T_n = \Big\{z \in \mathbb{C};\ |\mathrm{Re}\, z| \leq 2\beta\sqrt{B^2+\omega^2}, |\mathrm{Im}\, z| \leq 2|\mathrm{Im}\,\theta|(1-\beta)\sqrt{B^2+\omega^2}\Big\}, \tag{3.14}$$

where $0 < \beta < 1$.

Proposition 3.3. *Let $\theta \in \mathbb{C}$ with $|\theta|$ small and $\mathrm{Im}\,\theta < 0$. Then for $z \in T_n$ the operator $R_{0,\theta}(B,\omega,z) := \Big((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n) - \mu_n - z\Big)^{-1}(1-\Pi_n)$ exists and the following estimate holds:*

There exists $C_1 > 0$ independent of B such that

$$\|R_{0,\theta}(B,\omega,z)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_1}{\sqrt{B^2+\omega^2}},\ \textit{uniformly in } z \in T_n. \tag{3.15}$$

Moreover, for B large enough, the operator $R_{1,\theta}(B,\omega,z) = \Big((1-\Pi_n)P_{1,\theta}(B,\omega)(1-\Pi_n) - \mu_n - z\Big)^{-1}(1-\Pi_n)$ exists for all $z \in T_n$ and the following estimate holds:

There exists $C_2 > 0$ independent of B such that

$$\|R_{1,\theta}(B,\omega,z)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_3}{\sqrt{B^2+\omega^2}}, \text{ uniformly in } z \in T_n. \tag{3.16}$$

P r o o f. From (3.7) and the definition of T_n, one has $\sigma((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)-\mu_n)\cap T_n=\emptyset$. It implies the existence of $R_{0,\theta}(B,\omega,z)$, $z\in T_n$. For $z\in T_n$, we put

$$C(z)=\Big((1-\Pi_n)(e^{2\theta}D_y^2+e^{-2\theta}y^2+e^{2\theta}\omega^2x^2)(1-\Pi_n)-(2n+1)-\frac{z}{\sqrt{B^2+\omega^2}}\Big)^{-1}(1-\Pi_n).$$

Since $\frac{1}{\sqrt{B^2+\omega^2}}T_n = \left\{\frac{1}{\sqrt{B^2+\omega^2}}z;\ z\in T_n\right\}$ is a compact set independent of both B and ω, then there exists $z_0 \in T_n$ such that $\|C(z_0)\| = \sup_{z\in T_n}\|C(z)\|$. Remark that $\|C(z_0)\|$ does not depend on both B and ω. On the other hand, by a change of variables $x\mapsto (B^2+\omega^2)^{\frac14}x$, we have $R_{0,\theta}(B,\omega,z)$ is unitarily equivalent to $\frac{1}{\sqrt{B^2+\omega^2}}C(z)$. Then

$$\|R_{0,\theta}(B,\omega,z)\| \leq \frac{1}{\sqrt{B^2+\omega^2}}\|C(z_0)\|,$$

uniformly in $z\in T_n$. Note that $\|C(z_0)\|$ is finite for $\operatorname{Im}\theta<0$. And this proves (3.15).

As a consequence of (3.15), we prove (3.16). Indeed, for $z\in T_n$,

$$\begin{aligned}
&(1-\Pi_n)P_{1,\theta}(B,\omega)(1-\Pi_n)-\mu_n-z\\
&=(1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)-\mu_n-z+(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\\
&=\Big((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)-\mu_n-z\Big)\\
&\qquad\times\Big(1+R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\Big).
\end{aligned}$$

Since V is bounded together with all its derivatives, there exists $C>0$ such that $\|R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\| \leq \frac{C}{\sqrt{B^2+\omega^2}}$ uniformly in $z\in T_n$. Then for B large enough, $\|R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\| \leq \frac12$ and then

$R_{1,\theta}(B,\omega,z)$ exists. Moreover, for B sufficiently large, there exists $C_2 > 0$ such that

(3.17)
$$\|R_{1,\theta}(B,\omega,z)\| = \left\| \Big(1 + R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\Big)^{-1} R_{0,\theta}(B,\omega,z)\right\| \leq \frac{C_2}{\sqrt{B^2+\omega^2}},$$

uniformly in $z \in T_n$. □

Remark 3.4. Let Q be equal to $R_{0,\theta}(B,\omega,z)$ or $R_{1,\theta}(B,\omega,z)$. Let K be a compact set in $\mathbb{R}$. Using the theory of h-pseudodifferential operators of operator-valued symbols, we can view Q as an h-pseudodifferential operator in the x-variable whose symbol $q(x,\xi,\theta;h)$ is bounded operator in the y-variable. In particular, the proof of Proposition 3.3 shows that $q(x,\xi,\theta;h)$ is well-defined on $\mathbb{R}^2_{x,\xi}$ (resp. $K\times\mathbb{R}$) for $\operatorname{Im}\theta < 0$ (resp. $\operatorname{Im}\theta = 0$).

3.2. Grushin problem. From now on, we use $h = \dfrac{1}{\sqrt{B^2+\omega^2}}$. To indicate that the operators depend on h, we replace the indices (B,ω) by h. For example, we write $P_{1,\theta}(h)$ (resp. $V_\theta^w(h)$) instead of $P_{1,\theta}(B,\omega)$ (resp. $V_\theta^w(B,\omega)$).

We now study the Grushin problem for $P_{1,\theta}(h) - \mu_n$: Set

$$\mathcal{P}(z) = \begin{pmatrix} P_{1,\theta}(h) - \mu_n - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}\times L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)\times L^2(\mathbb{R}), \tag{3.18}$$

where $\mathcal{D} \subset L^2(\mathbb{R}^2)$ is the domain of $P_{1,\theta}(h)$.

Fix $\theta \in \mathbb{C}$ with $|\theta|$ small enough and $\operatorname{Im}\theta < 0$.

Theorem 3.5. *For B large enough, the operator $\mathcal{P}(z)$ is invertible uniformly for $z \in T_n$. Moreover, the inverse of $\mathcal{P}(z)$ is holomorphic in $z \in T_n$ and given by*

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix},$$

where

$$E_{-+}(z) = z - e^{2\theta}\omega^2 x^2 - R_+V_\theta^w(h)R_- - R_+b(z)[V_\theta^w(h),\Pi_n]R_- \tag{3.19}$$

with $b(z) = (I + [\Pi_n, V_\theta^w(h)]R_{1,\theta}(h,z))^{-1}$.

Here the operators $E(z)$, $E_-(z)$ and $E_+(z)$ are given by

$$E(z) = R_{1,\theta}(h,z)b(z)\ ;\ E_-(z) = R_+b(z)$$
$$E_+(z) = R_- - R_{1,\theta}(h,z)b(z)[V_\theta^w(h),\Pi_n]R_-. \tag{3.20}$$

In addition, z is an eigenvalue of $P_{1,\theta}(h)-\mu_n$ if and only if 0 is an eigenvalue of $E_{-+}(z)$. Here the notation $[\cdot,\cdot]$ is the commutator which is defined by $[A,C] = AC - CA$.

Proof. The proof follows from a simple modification of [27, Theorem 2.2] (see also [8, Section 6]). So we omit the details. □

Now we are interested in studying the operator $E_{-+}(z)$. In fact, for $z \in T_n$ (T_n is defined in (3.14)) and h sufficiently small, we prove that $E_{-+}(z) - (z - e^{2\theta}\omega^2x^2)$ is an h-pseudodifferential operator with bounded symbol.

By applying the Beal's characterization of pseudodifferential operators (see [9]), one easily sees that Π_n is a pseudodifferential operator with bounded symbol $\pi_n(y,\eta)$. Then making use of the pseudodifferential calculus (see [9, Chapter 7]), one obtains that the symbol of the commutator $[V_\theta^w(h),\Pi_n]$ has the asymptotics $\sum\limits_{j\geq 1} b_j h^{\frac{j}{2}}$ in $S^0(\mathbb{R}^4)$. It follows that

$$[V_\theta^w(h),\Pi_n] = \mathcal{O}(h^{\frac{1}{2}}) \tag{3.21}$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$ – the space of bounded operators from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.

Now for $z \in T_n$ and h small enough,

$$\begin{aligned} E_{-+}(z) &= z - e^{2\theta}\omega^2x^2 - R_+V_\theta^w(h)R_- - R_+b(z)[V_\theta^w(h),\Pi_n]R_- \\ &= z - e^{2\theta}\omega^2x^2 - R_+V_\theta^w(h)R_- \\ &\quad - R_+\sum_{j\geq 0}\big([V_\theta^w(h),\Pi_n]R_{1,\theta}(h,z)\big)^j[V_\theta^w(h),\Pi_n]R_-. \end{aligned} \tag{3.22}$$

Here we used (3.16), (3.21) and the Neumann series. From this and the fact that $R_+[V_\theta^w(h),\Pi_n]R_- = 0$, one obtains

$$\begin{aligned} E_{-+}(z) &= z - e^{2\theta}\omega^2x^2 - R_+V_\theta^w(h)R_- \\ &\quad - R_+\sum_{j\geq 1}\big([V_\theta^w(h),\Pi_n]R_{1,\theta}(h,z)\big)^j[V_\theta^w(h),\Pi_n]R_- \\ &= z - e^{2\theta}\omega^2x^2 - R_+V_\theta^w(h)R_- + h^2G_\theta(z;h), \end{aligned} \tag{3.23}$$

where $G_\theta(z;h)$ is holomorphic for $z \in T_n$. We set

$$A_\theta(h) = e^{2\theta}\omega^2 x^2 + R_+ V_\theta^w(h) R_-. \tag{3.24}$$

As in [7], we can prove that $R_+ V_\theta^w(h) R_-$ is an h-pseudodifferential operator with bounded symbol which belongs to $S^0((x^2+\xi^2)^{-\frac{\delta}{2}})$, where δ is given in Assumption $(\mathbf{H_1})$. Here $S^0((x^2+\xi^2)^{-\frac{\delta}{2}})$ is the class of symbols with order function $(x^2+\xi^2)^{-\frac{\delta}{2}}$ (see [9, Chapter 7]). Therefore,

$$A_\theta(h) = a^w(e^\theta x, e^{-\theta} h D_x; h), \tag{3.25}$$

where $a(\cdot,\cdot;h)$ is holomorphic in some conic neighbourhood of $\mathbb{R}^2$ and we have the following complete expansion in powers of h:

$$a(x,\xi;h) - a_0(x,\xi) \sim \sum_{j\geq 1} h^j a_j(x,\xi) \tag{3.26}$$

with

$$a_0(x,\xi) = \omega^2 x^2 + V(x,\xi) \ \text{ and } \ a_1(x,\xi) = \frac{(2n+1)}{4}\Delta V(x,\xi). \tag{3.27}$$

By using the arguments in [27], we can prove that $E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2)$ is an h-pseudodifferential operator with bounded symbol. Moreover the symbol also admits a complete expansion in powers of h.

Fix $\theta \in \mathbb{C}$ with $|\theta|$ small enough and $\operatorname{Im}\theta < 0$. We have obtained the following:

Proposition 3.6. *For $z \in T_n$ and h sufficiently small, the operator $E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2)$ is an h-pseudodifferential operator with bounded symbol. Moreover, the symbol admits a complete expansion in powers of h in $S^0(\mathbb{R}^2)$:*

$$E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2) = a_0^w(e^\theta x, e^{-\theta} h D_x) + a_1^w(e^\theta x, e^{-\theta} h D_x) h + \mathcal{O}(h^2), \tag{3.28}$$

where a_0, a_1 are given in (3.27).

Remark 3.7. It follows from the theory of h-pseudodifferential operators of operator-valued symbols, formula (3.23) and Remark 3.4 that the symbol corresponding to $E_{-+}(z)$ is well-defined for x in a compact set and $\operatorname{Im}\theta = 0$.

Thanks to Theorem 3.5, our purpose is now to study the spectrum of the effective Hamiltonian $E_{-+}(z)$.

4. Spectral properties of the leading term of $E_{-+}(z)$. In this section, we investigate the spectrum of $A_\theta(h)$ near E . We recall that $A_\theta(h) = e^{2\theta}\omega^2 x^2 + R_+ V_\theta^w(h) R_-$ which satisfies (3.25). For $\theta \in \mathbb{C}$ with $|\theta|$ small enough and $\operatorname{Im}\theta < 0$, we have the following:

Lemma 4.1. *The essential spectrum of $A_\theta(h)$ is equal to the set $\{e^{2\theta}\lambda;\ \lambda \geq 0\}$.*

P r o o f. We regard $R_+ V_\theta^w(h) R_-$ as the perturbation of the multiplication operator $e^{2\theta}\omega^2 x^2$. Recall that the decay of V at infinity implies that the symbol of $R_+ V_\theta^w(h) R_-$ belongs to $S^0((x^2+\xi^2)^{-\frac{\delta}{2}})$ (see Assumption ($\mathbf{H_1}$)). Therefore it follows that $R_+ V_\theta^w(h) R_-(e^{2\theta}\omega^2 x^2 - z)^{-1}$ is a compact operator, for $z \notin \sigma(e^{2\theta}\omega^2 x^2)$ (see [22, page 62]). Thus, we apply [17, Theorem 52.35] to obtain the essential spectrum of $A_\theta(h)$ is equal to that of $e^{2\theta}\omega^2 x^2$. In addition, the essential spectrum of the multiplication operator $e^{2\theta}\omega^2 x^2$ is nothing but $\{e^{2\theta}\lambda;\ \lambda \geq 0\}$. The lemma is proved. □

Without loss of generality, we may next assume that the total electric potential $a_0(x,\xi)$ has a local minimum at $(0,0)$ (otherwise we study $-A_\theta(h)$). Moreover, the real part of θ can be ignored by a unitary transformation, it is then sufficient to consider $\operatorname{Re}\theta = 0$.

To study the spectrum of $A_\theta(h)$ near E, it is very important to know some properties of the principal symbol. We will see below that $a_0(e^\theta x, e^{-\theta}\xi) - E$ is elliptic outside $(0,0)$.

So far, we want to show the exponential decay of eigenfunctions of $A_\theta(h)$ corresponding to eigenvalues near E. Then it is essential to study an operator of the form $e^{\frac{f(x)}{h}} A_\theta(h) e^{-\frac{f(x)}{h}}$, of which the principal symbol is $a_0(e^\theta x, e^{-\theta}(\xi + i f'(x))$. So by choosing a suitable function f, we show below that $a_0(e^\theta x, e^{-\theta}(\xi + i f'(x))) - E$ has the same properties as $a_0(e^\theta x, e^{-\theta}\xi) - E$.

From now on we fix $\theta = i\gamma$ with $\gamma < 0$. Let $\beta > 0$, we set $B(\beta) = \{(x,\xi) \in \mathbb{R}^2;\ |x| + |\xi| < \beta\}$.

Lemma 4.2. *For $\beta > 0$ sufficiently small and $|\gamma|$ small enough, there exists a smooth function $f(x)$ such that*

$$f(x) > 0 \ \text{ for } \ x \in \mathbb{R} \setminus \{0\}, \tag{4.1}$$

$$f(x) = c_1 x^2 \ \text{ for } \ x \text{ near } 0, \tag{4.2}$$

where c_1 is a small positive constant, and the following lower bounds hold:

There exists $C > 0$ large enough such that

$$\operatorname{Re}\Big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + i f'(x))) - E\Big) \geq \frac{1}{C}(x^2 + \xi^2) \ \text{ for } \ (x,\xi) \in B(\beta), \tag{4.3}$$

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \frac{|\gamma|}{C} \ \textit{for} \ (x,\xi) \in \mathbb{R}^2 \setminus B(\beta). \tag{4.4}$$

P r o o f. Put $\Gamma_1 := \{x \in \mathbb{R};\ \exists \xi \in \mathbb{R}, (x,\xi) \in \Gamma\}$. Remember that $\Gamma = \{(x,\xi) \in \mathbb{R}^2;\ \omega^2 x^2 + V(x,\xi) = E\}$ and $\lim\limits_{|(x,\xi)|\to\infty} V(x,\xi) = 0$, then Γ_1 is a bounded set.

For $\beta, |\gamma|$ small enough chosen later on, we construct a real smooth function f depending on these constants:

$$f(x) = c_1 x^2 \chi_1(x) + c_2\chi_2(x) + c_3\chi_3(x),$$

where $\chi_1 \in C_0^\infty(\mathbb{R};[0,1])$, $\chi_3, \chi_2 \in C^\infty(\mathbb{R};[0,1])$ satisfy:

• $\chi_1 = 1$ on $\{x \in \mathbb{R};\ \exists \xi \in \mathbb{R} \text{ s.t. } (x,\xi) \in B(\beta)\}$, the support of χ_1 lies in some small neighbourhood of 0 and $\chi_1 + \chi_2 = 1$.

• The support of χ_3 lies outside a neighbourhood of $\Gamma_1 \cup \{0\}$ and $\chi_3(x) = 1$ for $|x|$ large,

and positive constants $c_1, c_2, c_3 > 0$ small enough (to be chosen later on). Remark that c_1, c_2 depend on γ, c_3 is independent of γ.

Then it is easy to see that (4.1), (4.2) are verified. By using a symplectic change of coordinates if necessary, one can assume that Hessian of a_0 at $(0,0)$ is given by

$$a_0''(0,0) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

We start by proving (4.3). Notice that the constant C may change from line to line in what follows. For $\beta > 0$ small enough, by applying Taylor formula of order three to a_0 at $(0,0)$, one obtains:

$$a_0(e^{i\gamma}x, e^{-i\gamma}\xi) = E + \frac{1}{2}(\lambda e^{i2\gamma}x^2 + \mu e^{-i2\gamma}\xi^2) + \mathcal{O}((x,\xi)^3) \tag{4.5}$$

for $(x,\xi) \in B(\beta)$. We replace ξ by $\xi + if'(x)$ in (4.5). Since $f'(x) = 2c_1 x$ for $x \in \{y \in \mathbb{R};\ \exists \eta \in \mathbb{R} \text{ s.t. } (y,\eta) \in B(\beta)\}$, we can choose c_1 small enough such that there exists $C > 0$ large:

$$\mathrm{Re}\Big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big) \geq \frac{1}{C}(x^2 + \xi^2),$$

for $(x,\xi) \in B(\beta)$. Thus the lower bound estimate (4.3) is proved.

Now we demonstrate the estimate (4.4). The proof is divided into two cases according to the sign of E.

Case $E < 0$. First remark that Ω_E is a compact set in this case. Since $(0,0)$ is an isolated point, then for β small enough, $\Gamma = \Gamma \cap \mathbb{R}^2 \setminus B(\beta) = \Omega_E \cap \mathbb{R}^2 \setminus B(\beta)$. It implies that Γ is also a compact set. We choose a neighbourhood of Γ as follows:

The non-trapping condition on Γ (see Assumption ($\mathbf{H}_3$)) implies that, for each $(x_0,\xi_0) \in \Gamma$, there exists $\varepsilon(x_0,\xi_0) > 0$ such that

$$(4.6) \qquad |x\partial_x a_0(x,\xi) - \xi\partial_\xi a_0(x,\xi)| \geq \frac{1}{C(x_0,\xi_0)} > 0,$$

for all $(x,\xi) \in D\big((x_0,\xi_0),\varepsilon(x_0,\xi_0)\big)$. Here $C(x_0,\xi_0)$ is a large constant depending on (x_0,ξ_0) and $D\big((x_0,\xi_0),\varepsilon(x_0,\xi_0)\big) = \{(x,\xi) \in \mathbb{R}^2;\ |x-x_0|^2 + |\xi-\xi_0|^2 < \varepsilon(x_0,\xi_0)^2\}$. The compactness of Γ gives: there exists a finite number of such discs such that

$$\Gamma \subset \bigcup_{j=1}^{k} D\big((x_j,\xi_j),\varepsilon(x_j,\xi_j)\big) =: \mathcal{V}(\Gamma).$$

a) For $(x,\xi) \in \mathcal{V}(\Gamma)$:

$$(4.7) \qquad |x\partial_x a_0(x,\xi) - \xi\partial_\xi a_0(x,\xi)| \geq \frac{1}{\max\limits_{1\leq j\leq k} C(x_j,\xi_j)} > 0,$$

where $C(x_j,\xi_j), j = 1,\dots,k$, are given in (4.6).

Since $\mathrm{Im}\,(e^{i\gamma}x, e^{-i\gamma}\xi) = \sin(\gamma)(x,-\xi)$ and $\mathrm{Re}\,(e^{i\gamma}x, e^{-i\gamma}\xi) = \cos(\gamma)(x,\xi)$, then

$$\begin{aligned}&\mathrm{Im}\,(e^{i\gamma}x, e^{-i\gamma}\xi).\nabla a_0(\mathrm{Re}\,(e^{i\gamma}x, e^{-i\gamma}\xi))\\ &\qquad\qquad = \sin(\gamma)\Big(x\partial_x a_0(\cos(\gamma)(x,\xi)) - \xi\partial_\xi a_0(\cos(\gamma)(x,\xi))\Big).\end{aligned}$$

Combining this with (4.7) we have, for $|\gamma|$ small enough,

$$(4.8) \qquad \Big|\mathrm{Im}\,(e^{i\gamma}x, e^{-i\gamma}\xi).\nabla a_0(\mathrm{Re}\,(e^{i\gamma}x, e^{-i\gamma}\xi))\Big| \geq \frac{|\gamma|}{C},\ \forall (x,\xi) \in \mathcal{V}(\Gamma),$$

where C is a large constant.

Notice that (x,ξ) near Γ corresponds to x near Γ_1. Then we can choose c_1, c_2 small depending on γ such that $|f'(x)| \leq c'|\gamma|$ for x near Γ_1. Here the constant c' is small enough. Thus the inequality (4.8) remains true when we replace $(e^{i\gamma}x, e^{-i\gamma}\xi)$ by $w := (e^{i\gamma}x, e^{-i\gamma}(\xi+if'(x)))$, i.e., there exists C sufficiently large such that

$$|\mathrm{Im}\, w.\nabla a_0(\mathrm{Re}\, w)| \geq \frac{|\gamma|}{C}. \tag{4.9}$$

Remark that the Hessian of a_0 is given by

$$a_0''(x,\xi) = \begin{pmatrix} 2\omega^2 + \partial^2_{xx}V(x,\xi) & \partial^2_{x\xi}V(x,\xi) \\ \partial^2_{\xi x}V(x,\xi) & \partial^2_{\xi\xi}V(x,\xi) \end{pmatrix}$$

and $\partial^\alpha_{x\xi}a_0(x,\xi) = \partial^\alpha_{x\xi}V(x,\xi)$ for all $\alpha \in \mathbb{N}^2$, $|\alpha| \geq 3$. Then we apply the Taylor formula of order two to $a_0(w)$ at $\mathrm{Re}\, w$:

$$a_0(w) = a_0(\mathrm{Re}\, w) + i\,\mathrm{Im}\, w\nabla a_0(\mathrm{Re}\, w) + \omega^2 x^2 \sin^2(\gamma) + r(w), \tag{4.10}$$

where $|r(w)| \leq C\sin^2(|\gamma|)$. Combine this with (4.9), we have for C large enough,

$$|a_0(w) - E| \geq |\mathrm{Im}\,(a_0(w) - E)| \geq \frac{|\gamma|}{C},\ \forall (x,\xi) \in \mathcal{V}(\Gamma). \tag{4.11}$$

b) For $(x,\xi) \in \mathbb{R}^2 \setminus B(\beta)$ and $(x,\xi) \notin \mathcal{V}(\Gamma)$: From now on, we set $\tilde{\mathcal{V}}(\Gamma) := \mathbb{R}^2 \setminus (B(\beta) \cup \mathcal{V}(\Gamma))$.

Choose $R > 0$ sufficiently large such that: $\omega^2 R^2 > \sup_{\mathbb{R}^2} |V|$ and $\sup_{\{(x,\xi)\in\mathbb{R}^2;\ |\xi|\geq R\}} |V(x,\xi)| < -\frac{E}{2}$. Then

$$\omega^2 x^2 + V(x,\xi) - E \geq -\frac{E}{2} > 0 \tag{4.12}$$

for all $(x,\xi) \in \{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \geq R \ \text{ or } \ |\xi| \geq R\}$. In fact for $|x| \geq R$ we use $\omega^2x^2 + V(x,\xi) - E \geq \omega^2 R^2 - \sup_{\mathbb{R}^2}|V| - E > -E$ and for $|\xi| \geq R$ we use $\omega^2x^2 + V(x,\xi) - E \geq -\sup_{\{(x,\xi)\in\mathbb{R}^2;\ |\xi|\geq R\}} |V(x,\xi)| - E > -\frac{E}{2}$. Then for $|\gamma|$ small and c_3 small (c_3 is independent of γ),

$$\mathrm{Re}\Big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big) \geq -\frac{E}{4} > 0 \tag{4.13}$$

on the set $\{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \geq R \text{ or } |\xi| \geq R\}$.

Since $\Gamma \cap \tilde{\mathcal{V}}(\Gamma) = \emptyset$ and $a_0(x,\xi) - E = 0$ if and only if $(x,\xi) \in \Gamma \cup \{(0,0)\}$, one has $a_0(x,\xi) - E \neq 0$ on $\tilde{\mathcal{V}}(\Gamma)$. Thus, on the compact set $\{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \leq R, |\xi| \leq R\}$,

$$|a_0(x,\xi) - E| \geq \text{const} > 0.$$

By a perturbation argument, for $|\gamma|$ and c_1, c_2, c_3 small enough, one obtains

$$\left| a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E \right| \geq \text{const} > 0, \tag{4.14}$$

for $(x,\xi) \in \{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \leq R, |\xi| \leq R\}$.

Therefore, from (4.13) and (4.14), one has

$$\left| a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E \right| \geq \text{const} > 0, \tag{4.15}$$

for all $(x,\xi) \in \tilde{\mathcal{V}}(\Gamma)$.

It follows from (4.11), (4.15) that, for $|\gamma|$ and c_1, c_2, c_3 small enough, there exists C large enough such that

$$\left| a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E \right| \geq \frac{|\gamma|}{C}, \tag{4.16}$$

for all $(x,\xi) \in \mathbb{R}^2 \setminus B(\beta)$.

Case $E > 0$. Note that Ω_E is no longer a compact set. In fact, there are two asymptotes of Γ: $x = \pm\frac{\sqrt{E}}{\omega}$ since V vanishes at infinity. Let $\varepsilon > 0$ small, we can choose $R > 0$ large enough such that $\Gamma \cap \mathbb{R}^2 \setminus D((0,0),R) \subset \{(x,\xi) \in \mathbb{R}^2 \setminus D((0,0),R);\ |\omega^2 x^2 - E| < \varepsilon\}$. Here $R >> \beta$.

First, for $(x,\xi) \in D((0,0),R) \setminus B(\beta)$, by using the same arguments as in case $E < 0$, i.e., the non-trapping condition on Γ and the compactness of $\overline{D((0,0),R)}$, we obtain (4.4).

Next we divide the set $\mathbb{R}^2 \setminus D((0,0),R)$ into two sets

$$\mathbb{R}^2 \setminus D((0,0),R) = R(\varepsilon) \cup R(\varepsilon)^c,$$

where

$$R(\varepsilon) := \{(x,\xi) \in \mathbb{R}^2 \setminus D((0,0),R);\ |\omega^2 x^2 - E| < \varepsilon\}.$$

- In $R(\varepsilon)$, when $\varepsilon \to 0$ one has

$$V(x,\xi),\ x\partial_x V(x,\xi),\ \xi\partial_\xi V(x,\xi) = o(1),$$

and

$$\omega^2 x^2 - E = o(1).$$

Therefore, for ε sufficiently small and $(x,\xi) \in R(\varepsilon)$

$$\Big|\{x.\xi, a_0(x,\xi)\}\Big| = \Big|x(2\omega^2 x + \partial_x V(x,\xi)) - \xi\partial_\xi V(x,\xi)\Big| \geq |2E + o(1)| \geq E. \tag{4.17}$$

Apply again the perturbation argument, one gets

$$\Big|\mathrm{Im}\,(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))).\nabla a_0(\mathrm{Re}\,(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))))\Big| \geq \frac{E|\gamma|}{2} > 0. \tag{4.18}$$

The same arguments as in (4.11), one obtains

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \frac{E|\gamma|}{2} > 0. \tag{4.19}$$

• In $R(\varepsilon)^c$, one has $|\omega^2 x^2 - E| \geq \varepsilon$. Since $\lim\limits_{|(x,\xi)|\to\infty} V(x,\xi) = 0$, we choose R large enough such that

$$|\omega^2 x^2 + V(x,\xi) - E| \geq \frac{\varepsilon}{2},$$

for all $(x,\xi) \in R(\varepsilon)^c$. Now we apply again the perturbation argument to obtain, for $|\gamma|$ and c_1, c_2, c_3 small enough,

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \frac{\varepsilon}{4} > 0. \tag{4.20}$$

The proof of the lemma is thus complete. □

Since x and ξ play the same role, the following lemma can be proved by using the same arguments as above:

Lemma 4.3. *For β sufficiently small and $|\gamma|$ small enough, there exists a smooth function $g(\xi)$ such that*

$$g(\xi) > 0 \ \text{ for } \ \xi \in \mathbb{R}\setminus\{0\} \ \text{ and } \ g(\xi) = c_2\xi^2 \ \text{ for } \ \xi \text{ near } 0, \tag{4.21}$$

where c_2 is a small positive constant, and the following lower bounds hold:

There exists $C > 0$ sufficiently large such that

$$\mathrm{Re}\Big(a_0(e^{i\gamma}(x + ig'(\xi)), e^{-i\gamma}\xi) - E\Big) \geq \frac{1}{C}(x^2 + \xi^2) \text{ for } (x,\xi) \in B(\beta), \tag{4.22}$$

(4.23) $$\left|a_0(e^{i\gamma}(x+ig'(\xi)),e^{-i\gamma}\xi)-E\right|\geq\frac{|\gamma|}{C}\ \textit{for}\ (x,\xi)\in\mathbb{R}^2\setminus B(\beta).$$

Recall that $\gamma=\operatorname{Im}\theta<0$.

By relying on Lemma 4.2 and Lemma 4.3, we prove the exponential decay of eigenfunctions corresponding to eigenvalues of $A_{i\gamma}(h)$ near E:

Theorem 4.4. *Let f be constructed in Lemma 4.2. Let $C_0>0$ be a large and fixed constant, we define a neighbourhood of E,*

$$D=\{z\in\mathbb{C};\ |z-E|<C_0h\}.$$

Suppose that $\lambda(h)\in D$ is an eigenvalue of $A_{i\gamma}(h)$ and $u(h)$ is a normalized eigenfunction associated to $\lambda(h)$, then there exists $C>0$ such that

(4.24) $$\|e^{f(x)/h}u(h)\|\leq C,$$

Proof. Let $\tilde{a}_{i\gamma}$ be the symbol of $A_{i\gamma}(h)-e^{2i\gamma}\omega^2x^2$. We recall that $(x,\xi)\mapsto\tilde{a}_{i\gamma}(x,\xi;h)$ is holomorphic in some conic neighbourhood of $\mathbb{R}^2$. Let us put $A_f(h)=e^{\frac{f}{h}}A_{i\gamma}(h)e^{-\frac{f}{h}}$. Then by using the contour integration in the ξ variable, one has for $u\in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\Big(A_f(h)-e^{2i\gamma}\omega^2x^2\Big)u(x)&=\frac{1}{2\pi h}\iint e^{(i(x-y)\xi+f(x)-f(y))/h}\tilde{a}_{i\gamma}\Big(\frac{x+y}{2},\xi;h\Big)u(y)dyd\xi\\&=\frac{1}{2\pi h}\iint e^{i(x-y)(\xi-if'(x,y))/h}\tilde{a}_{i\gamma}\Big(\frac{x+y}{2},\xi;h\Big)u(y)dyd\xi\\&=\frac{1}{2\pi h}\iint e^{i(x-y)\xi/h}\tilde{a}_{i\gamma}\Big(\frac{x+y}{2},\xi+if'(x,y);h\Big)dyd\xi,\end{aligned}$$

where $f'(x,y)$ is determined by $f(x)-f(y)=(x-y)f'(x,y)$.

Using the analyticity of $\tilde{a}_{i\gamma}$, it follows from Cauchy formula that $A_f(h)-e^{2i\gamma}\omega^2x^2$ is also an h-pseudodifferential operator with bounded symbol. Moreover, the symbol of $A_f(h)-e^{2i\gamma}\omega^2x^2$ can also be expanded in powers of h in $S^0(\mathbb{R}^2)$ (the set of bounded symbols) with the principal symbol is $V(e^{i\gamma}x,e^{-i\gamma}(\xi+if'(x)))$. Then the principal symbol of $A_f(h)$ is $a_0(e^{i\gamma}x,e^{-i\gamma}(\xi+if'(x)))$.

Let us put $u_f(h):=e^{\frac{f}{h}}u(h)$ which belongs to $L^2(\mathbb{R})$ since f is bounded. Let $B(\beta)$ be as in Lemma 4.2, we choose a partition of unity $\chi_1+\chi_2=1$, supp$\chi_1\subset B(\beta)$, $\chi_1=1$ near $(0,0)$. The idea of the proof is to estimate separately $\chi_1^w(x,hD_x)u_f(h)$ and $\chi_2^w(x,hD_x)u_f(h)$.

Firstly we want to evaluate $\chi_2^w(x, hD_x)u_f(h)$. From (4.4), the symbol $\chi_2(x,\xi)\big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - \lambda(h)\big)^{-1}$ exists. Moreover since $(A_f(h) - \lambda(h))u_f(h) = 0$, one has

$$\Big(\chi_2(x,\xi)\big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi+if'(x)))-\lambda(h)\big)^{-1}\Big)^w(x, hD_x)(A_f(h)-\lambda(h))u_f(h) = 0.$$

By applying the h-pseudodifferential calculus in the right hand side of this equality, one obtains

$$\Big(\chi_2^w(x, hD_x) + \mathcal{O}(h)\Big)u_f(h) = 0. \tag{4.25}$$

Secondly, we estimate $u_f^1(h) := \chi_1^w(x, hD_x)u_f(h)$. In fact, it results from the compactness of the support of χ_1 and $\big(A_f(h) - \lambda(h)\big)u_f(h) = 0$ that

$$\operatorname{Re}\big\langle (A_f(h) - \lambda(h))u_f^1(h), u_f^1(h)\big\rangle = \operatorname{Re}\big\langle [A_f(h), \chi_1^w(x, hD_x)]u_f(h), u_f^1(h)\big\rangle \tag{4.26}$$

$$= \mathcal{O}(h)\Big\langle u_f(h), u_f^1(h)\Big\rangle. \tag{4.27}$$

In addition, by using (4.3) and the Gårding inequality, one obtains

$$\operatorname{Re}\big\langle (A_f(h) - \lambda(h))u_f^1(h), u_f^1(h)\big\rangle$$

$$= e^{-i\gamma}(hD_x + if'(x))) - E\big)u_f^1(h), u_f^1(h)\Big\rangle + \mathcal{O}(h)\|u_f^1(h)\|^2$$

$$\geq \frac{1}{C}\Big\langle (h^2D_x^2 + x^2 - C_1h)u_f^1(h), u_f^1(h)\Big\rangle, \tag{4.28}$$

for some large constants C, C_1.

Since $h^2D_x^2$ is a positive operator, then

$$\Big\langle (h^2D_x^2 + x^2 - C_1h)u_f^1(h), u_f^1(h)\Big\rangle \geq \Big\langle (x^2 - C_1h)u_f^1(h), u_f^1(h)\Big\rangle. \tag{4.29}$$

Let M be a large constant. We decompose the scalar product in (4.29) into two parts according to $|x| > Mh$ and $|x| \leq Mh$. Then one has

$$\Big\langle (x^2 - C_1h)u_f^1(h), u_f^1(h)\Big\rangle = \Big\langle (x^2 - C_1h)u_f^1(h), u_f^1(h)\Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|>Mh^{1/2}\})} + \Big\langle (x^2 - C_1h)u_f^1(h), u_f^1(h)\Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})}. \tag{4.30}$$

By using $x^2 \geq 0$, one obtains

$$\begin{aligned}(4.31)\quad \Big\langle (x^2 - C_1 h) u_f^1(h), u_f^1(h) \Big\rangle &\geq \Big\langle (M^2 - C_1) h u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|>Mh^{1/2}\})} \\ &\quad - C_1 h \Big\langle u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})} \\ &= \Big\langle (M^2 - C_1) h u_f^1(h), u_f^1(h) \Big\rangle \\ &\quad - M^2 h \Big\langle u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})}.\end{aligned}$$

For $|x| \leq Mh^{1/2}$, one has $\dfrac{f(x)}{h} = \dfrac{c_1 x^2}{h} \leq c_1 M^2$. Combining this with the fact that $\|u(h)\| = 1$, one derives: There exists $C(M) > 0$ such that

$$(4.32)\qquad \Big\langle u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})} < C(M).$$

Then, combining (4.29), (4.31) and (4.32) one has

$$(4.33)\quad \Big\langle (h^2 D_x^2 + x^2 - C_1 h) u_f^1(h), u_f^1(h) \Big\rangle \geq (M^2 - C_1) h \|u_f^1(h)\|^2 - M^2 C(M) h.$$

From (4.26), (4.28) and (4.33), there exists $C_2 > 0$ such that

$$(4.34)\qquad (M^2 - C_1) h \|u_f^1(h)\|^2 - M^2 C(M) h \leq C_2 h \|u_f^1(h)\| \|u_f(h)\|,$$

which implies that

$$\|u_f^1(h)\|^2 \leq \frac{M^2 C(M)}{(M^2 - C_1)} + \frac{C_2}{(M^2 - C_1)} \|u_f(h)\|^2.$$

From (4.25), (4.34) and the fact that $\chi_1^w(x, hD_x) + \chi_2^w(x, hD_x) = 1$ we have

$$\begin{aligned}\|u_f(h)\|^2 &= \|(\chi_1^w(x, hD_x) + \chi_2^w(x, hD_x)) u_f(h)\|^2 \\ &\leq \frac{M^2 C(M)}{(M^2 - C_1)} + \frac{C_2}{(M^2 - C_1)} \|u_f(h)\|^2 + C_3 h \|u_f(h)\|^2,\end{aligned}$$

for some $C_3 > 0$. The proof follows by choosing M sufficiently large, h small enough. □

Since x and ξ play the same role, we also obtain the following:

Theorem 4.5. *Let g and D be as in Lemma 4.3 and Theorem 4.4. Suppose that $\lambda(h) \in D$ is an eigenvalue of $A_{i\gamma}(h)$ and $u(h)$ is a normalized eigenfunction associated to $\lambda(h)$, then there exists $C > 0$ such that*

$$\|e^{g(hD_x)/h}u(h)\| \leq C. \tag{4.35}$$

Thanks to the Theorems 4.4 and 4.5, we need only study the symbol of the operator $A_{i\gamma}(h)$ near $(0,0)$. We also have assumed near $(0,0)$ that $a_0(x,\xi) = E + \frac{1}{2}(\lambda x^2 + \mu\xi^2) + \mathcal{O}((x,\xi)^3)$ (i.e., the matrix $a_0''(0,0)$ is diagonal). Putting

$$A^0_{i\gamma}(h) = \frac{1}{2}(\lambda e^{2i\gamma}x^2 + \mu e^{-2i\gamma}h^2D_x^2) + \frac{2n+1}{4}\Delta V(0,0)h. \tag{4.36}$$

Since $\Delta V(0,0) = \lambda+\mu-\omega^2$, then it is clear that the spectrum of $A^0_{i\gamma}(h), \sigma(A^0_{i\gamma}(h)) = \{he_k;\ k \in \mathbb{N}\}$ where

$$e_k = \frac{(2k+1)\sqrt{\lambda\mu}}{2} + \frac{(2n+1)(\lambda+\mu-2\omega^2)}{4},\ k \in \mathbb{N}. \tag{4.37}$$

Notice here that in the work of X. P. Wang (see [28]) the total electric potential is $W(x,y) = \omega x + V(x,y)$ and then $\Delta W(x,y) = \Delta V(x,y)$. But in our case, the total electric potential is $a_0(x,y) = \omega^2x^2 + V(x,y)$. Then $\Delta a_0(x,y) = 2\omega^2 + \Delta V(x,y)$. This explains why there is $-2\omega^2$ in the formula of e_k.

Let C_0 be a large fixed constant such that $C_0 \neq e_k$, $k \in \mathbb{N}$. Henceforth we denote the neighbourhood of E,

$$D = \{z \in \mathbb{C};\ |z - E| \leq C_0h\}. \tag{4.38}$$

For $\beta > 0$ small and $j \in \mathbb{N}$, let $D_j = \{z \in D;\ |z - E - he_j| \leq \beta h\}$. Then we prove that the spectrum of $A_{i\gamma}(h)$ in $D \setminus \bigcup_{j\in\mathbb{N}} D_j$ is empty.

Proposition 4.6. *Let z be in $D \setminus \bigcup_{j\in\mathbb{N}} D_j$. For h small enough, the following resolvent estimate holds:*

$$\|(A_{i\gamma}(h) - z)^{-1}\| \leq Ch^{-1},$$

for some $C > 0$.

Proof. Choosing $\{\chi_1, \chi_2\}$ a partition of unity on $\mathbb{R}^2$, $\chi_1 = 1$ near $(0,0)$ and $\chi_1 = 0$ outside an β-neighbourhood of $(0,0)$.

From the proof of the inequality (4.4), it is easy to see that for $z \in D$, one has $|a_0(e^{i\gamma}x, e^{-i\gamma}\xi) - z| \geq \dfrac{|\gamma|}{C}$ for $(x,\xi) \in \mathrm{supp}\chi_2$. Denote by $B_1(z)$ the h-pseudodifferential operator with symbol $\chi_2(x,\xi)(a_0(e^{i\gamma}x, e^{-i\gamma}\xi) - z)^{-1}$. Then one has

$$B_1(z)(A_{i\gamma}(h) - z) = \chi_2^w(x, hD_x) + \mathcal{O}(h). \tag{4.39}$$

For $z \in D \setminus \bigcup_{j\in\mathbb{N}} D_j$, then $z - E \in \rho(A^0_{i\gamma}(h))$ which shows the existence of $B_2(z) := (A^0_{i\gamma}(h) + E - z)^{-1}$. Our purpose is to study $\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z)$. It can be readily verified that $x^j(hD_x)^k B_2(z)$ is unitarily equivalent to

$$h^{\frac{k+j}{2}-1}x^j D_x^k\Big(\frac{1}{2}(\lambda e^{2i\gamma}x^2 + \mu e^{-2i\gamma}D_x^2) + \frac{2n+1}{4}\Delta V(0,0) + \frac{E-z}{h}\Big)^{-1}.$$

Then,

$$\|x^j(hD_x)^k B_2(z)\| \leq Ch^{\frac{k+j}{2}-1},\ 0 \leq k+j \leq 2. \tag{4.40}$$

We have

$$\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z) = \chi_1^w(x, hD_x)\Big(1 + B_2(z)(A_{i\gamma}(h) - A^0_{i\gamma}(h) - E)\Big)$$

$$= \chi_1^w(x, hD_x)\Big(1 + B_2(z)(A_{i\gamma}(h) - A^0_{i\gamma}(h) - E)\Big)\Big(\chi_3^w(x, hD_x) + 1 - \chi_3^w(x, hD_x)\Big),$$

where $\chi_3 = 1$ near $\mathrm{supp}\chi_1$ and $\chi_3 = 0$ outside 2β-neighbourhood of $(0,0)$. From (4.40) and $\mathrm{supp}\chi_1 \cap \mathrm{supp}\chi_3 = \emptyset$, we obtain by inductive arguments that

$$\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A^0_{i\gamma}(h) - E)(1 - \chi_3^w(x, hD_x)) = \mathcal{O}(h^\infty). \tag{4.41}$$

One has the symbol of $A_{i\gamma}(h) - A^0_{i\gamma}(h) - E$ on the support of χ_3 is $\mathcal{O}((x,\xi))h + \mathcal{O}(h^2)$. Thus we use again (4.40) to obtain

$$\|\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A^0_{i\gamma}(h) - E)\chi_3^w(x, hD_x)\| \leq C\beta + Ch, \tag{4.42}$$

for some C large.

These arguments give

$$\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z) = \chi_1^w(x, hD_x) + \mathcal{O}(\beta) + \mathcal{O}(h). \tag{4.43}$$

From (4.39) and (4.43), there exist $C_1, C_2, C_3, C_4 > 0$ such that

$$(1 - C\beta - C_1 h)\|u\| \le \|\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z)u\| + \|B_1(z)(A_{i\gamma}(h) - z)u\|$$
$$\le C_2 h^{-1}\|(A_{i\gamma}(h) - z)u\| + C_3\|(A_{i\gamma}(h) - z)u\|$$
$$\le C_4 h^{-1}\|(A_{i\gamma}(h) - z)u\|, \ \forall u \in C_0^\infty(\mathbb{R}).$$

We choose β and h small such that $(1 - C\beta - C_1 h) > \frac{1}{2}$. It implies that $A_{i\gamma}(h)$ does not have any eigenvalue in $D \setminus \bigcup_{j\in\mathbb{N}} D_j$. On the other hand Lemma 4.1 asserts that $\sigma_{\text{ess}}(A_{i\gamma}(h)) \cap D \setminus \bigcup_{j\in\mathbb{N}} D_j = \emptyset$. Thus $D \setminus \bigcup_{j\in\mathbb{N}} D_j \subset \rho(A_{i\gamma}(h))$ and $\|(A_{i\gamma}(h) - z)^{-1}\| \le Ch^{-1}$ for all $z \in D \setminus \bigcup_{j\in\mathbb{N}} D_j$. □

From Proposition 4.6, the spectrum of $A_{i\gamma}(h)$ near E is contained in $\bigcup_{j\in\mathbb{N}} D_j$. Further, one obtains the following:

Theorem 4.7. *Let D be the set defined in* (4.38)*. Then for h sufficiently small, there is one to one correspondence between the eigenvalues of $A_{i\gamma}(h)$ in D and the set $\{e_j;\ e_j < C_0,\ j \in \mathbb{N}\}$. Moreover, we can rearrange the eigenvalues of $A_{i\gamma}(h)$ in D such that the jth eigenvalue is*

$$E_j(h) = E + he_j + \mathcal{O}(h^{\frac{3}{2}}). \tag{4.44}$$

Proof. According to Proposition 4.6, it suffices to show that for each j such that $e_j < C_0$, in $D_j = \{z \in D;\ |z - E - he_j| \le \beta h\}$ there exists uniquely an eigenvalue of $A_{i\gamma}(h)$. Denote $\Gamma_j = \{z \in \mathbb{C};\ |z - E - he_j| = 2\beta h\} \subset D \setminus \bigcup_{j\in\mathbb{N}} D_j$. For $z \in \Gamma_j$, one has $\|(z - A_{i\gamma}(h))^{-1}\| \le Ch^{-1}$. Let us define for each $j \in \mathbb{N}$

$$\Xi_j(h) = \frac{1}{2\pi i}\int_{\Gamma_j} (z - A_{i\gamma}(h))^{-1} dz. \tag{4.45}$$

In the same notations as in Proposition 4.6, one has

$$B_1(z)(A_{i\gamma}(h) - z) = \chi_2^w(x, hD_x) + \mathcal{O}(h)$$

and

$$\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z) = \chi_1^w(x, hD_x)(1 + B_2(z)(A_{i\gamma}(h) - A_{i\gamma}^0(h) - E)).$$

Then,

$$(B_1(z) + \chi_1^w(x, hD_x)B_2(z))(A_{i\gamma}(h) - z)$$
$$= 1 + \mathcal{O}(h) + \chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A_{i\gamma}^0(h) - E)$$
$$= 1 + \mathcal{O}(h) + \chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A_{i\gamma}^0(h) - E)\chi_3^w(x, hD_x),$$

which shows that

(4.46)
$$(A_{i\gamma}(h) - z)^{-1} = B_1(z) + \chi_1^w(x, hD_x)B_2(z)$$
$$- \Big(\mathcal{O}(h) + \chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A_{i\gamma}^0(h) - E)\chi_3^w(x, hD_x)\Big)(A_{i\gamma}(h) - z)^{-1}.$$

We notice here that $B_1(z)$ is holomorphic in $z \in D$. Then, by inserting (4.46) into (4.45), one obtains

$$\Xi_j(h) = \chi_1^w(x, hD_x)\Pi^0(h) + \int_{\Gamma_j} (\mathcal{O}(1) + \mathcal{O}(\beta)h^{-1})dz$$
$$= \chi_1^w(x, hD_x)\Pi^0(h) + B_3(h, \beta), \tag{4.47}$$

where $\Pi^0(h)$ is the spectral projector associated to e_j of $A_{i\gamma}^0(h)$ and $\|B_3(h, \beta)\| \to 0$ as $\beta, h \to 0$. It implies that, for $j \in \mathbb{N}$ such that $e_j < C_0$, $\mathrm{rank}\Xi_j(h) = \mathrm{rank}\chi_1\Pi^0(h) = 1$. According to [26, Theorem XII.6] there exists uniquely a simple eigenvalue of $A_{i\gamma}(h)$ inside Γ_j denoted by $E_j(h)$.

Now we prove the estimate (4.44). Let $\phi_{j,i\gamma}(x; h)$ be the unique eigenvalue associated to he_j of $A_{i\gamma}^0(h)$. Since $\phi_{j,i\gamma}(x; h)$ is of the form $p_j(\frac{e^{i\gamma}x}{\sqrt{h}})h^{-\frac{1}{4}}e^{-c\frac{e^{i2\gamma}x^2}{2h}}$ where p_j is a polynomial, by a scaling argument, one gets

$$\|x^j(hD_x)^k(\chi_1^w(x, hD_x)\phi_{j,i\gamma}(x; h))\| = \mathcal{O}(h^{\frac{k+j}{2}}),\ 0 \le k + j \le 3.$$

It follows from this that

$$(A_{i\gamma}(h) - E - A_{i\gamma}^0(h))\chi_1^w(x, hD_x)\phi_{j,i\gamma}(x; h) = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.48}$$

A direct computation gives

$$(A_{i\gamma}(h) - E - he_j)\chi_1^w\phi_{j,i\gamma}(x; h)$$
$$= [A_{i\gamma}(h), \chi_1^w]\phi_{j,i\gamma}(x; h) + \chi_1^w(A_{i\gamma}(h) - E - A_{i\gamma}^0(h))(\phi_{j,i\gamma}(x; h))$$
$$= [A_{i\gamma}(h), \chi_1^w]\phi_{j,i\gamma}(x; h) + \chi_1^w(A_{i\gamma}(h) - E - A_{i\gamma}^0(h))(\chi_3^w + 1 - \chi_3^w)(\phi_{j,i\gamma}(x; h)).$$

Combining this with (4.41) and (4.48), one has

$$(A_{i\gamma}(h) - E - he_j)\chi_1^w \phi_{j,i\gamma}(x;h) = [A_{i\gamma}(h), \chi_1^w]\phi_{j,i\gamma}(x;h) + \mathcal{O}(h^{\frac{3}{2}}). \tag{4.49}$$

Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ such that $\tilde{\chi} = 1$ near $(0,0)$ and $\chi_1\tilde{\chi} = \tilde{\chi}$. It follows from the h-pseudodifferential calculus that $[A_{i\gamma}(h), \chi_1^w]\tilde{\chi}^w = \mathcal{O}(h^\infty)$. Combining this with the exponential decay of $\phi_{j,i\gamma}(x;h)$, one obtains

$$\begin{aligned}[A_{i\gamma}(h), \chi_1^w]\phi_{j,i\gamma}(x;h) &= [A_{i\gamma}(h), \chi_1^w]\tilde{\chi}^w \phi_{j,i\gamma}(x;h) + [A_{i\gamma}(h), \chi_1^w](1-\tilde{\chi}^w)\phi_{j,i\gamma}(x;h) \\ &= \mathcal{O}(h^\infty). \qquad (4.50)\end{aligned}$$

From (4.49) and (4.50), one obtains

$$(A_{i\gamma}(h) - E - he_j)\chi_1^w(x, hD_x)\phi_{j,i\gamma}(x;h) = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.51}$$

Let $u_{j,i\gamma}(h)$ be a normalized eigenfunction of $A_{i\gamma}(h)$ associated to $E_j(h)$. We denote by $A_{i\gamma}(h)^*$ the adjoint of $A_{i\gamma}(h)$. Now we take $u_{j,i\gamma}(h)^*$ an eigenfunction of $A_{i\gamma}(h)^*$ such that $A_{i\gamma}(h)^* u_{j,i\gamma}(h)^* = \bar{E}_j(h) u_{j,i\gamma}(h)^*$ and $\langle u_{j,i\gamma}(h), u_{j,i\gamma}(h)^*\rangle = 1$.

It follows from (4.47), (4.51) that

$$\Big\langle (A_{i\gamma}(h) - E - he_j)(u_{j,i\gamma}(h) - B_3(h,\beta)u_{j,i\gamma}(h)), u_{j,i\gamma}(h)^* \Big\rangle = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.52}$$

Remark that $u_{j,i\gamma}(h)$ (resp. $u_{j,i\gamma}(h)^*$) is the eigenfunction of $A_{i\gamma}(h)$ (resp. $A_{i\gamma}(h)^*$). Then (4.52) follows that

$$(E_j(h) - E - he_j)(1 - \langle B_3(h,\beta)u_{j,i\gamma}(h), u_{j,i\gamma}(h)^*\rangle) = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.53}$$

Recall that $\|B_3(h,\beta)\| \to 0$ as $h, \beta \to 0$. Then for h, β sufficiently small, it results from (4.53) that $E_j(h) = E + he_j + \mathcal{O}(h^{\frac{3}{2}})$. □

By using the fact that $E_j(h)$ is a simple eigenvalue and repeating the same arguments as in [20] (see also [14]), we obtain:

Theorem 4.8. *The eigenvalue $E_j(h)$ of $A_{i\gamma}(h)$ can be expanded asymptotically in powers of h, i.e.,*

$$E_j(h) \sim \sum_{k\geq 0} \lambda_{j,k} h^k,$$

where $\lambda_{j,0} = E, \lambda_{j,1} = e_j$.

5. Proof of the main theorem. In this section, we prove the existence of resonances and show the width of resonances is $\mathcal{O}(h^\infty)$.

5.1. The existence of resonances. In this subsection, we prove the existence of resonances of $P(B,\omega)$ in each set U_n, $n \in \mathbb{N}$. For fix $j \in \mathbb{N}$, let us recall some notations used in the proof of Theorem 4.7. We consider $u_{j,i\gamma}(h)$ be a normalized eigenfunction of $A_{i\gamma}(h)$ associated to $E_j(h)$. We denote by $A_{i\gamma}(h)^*$ the adjoint of $A_{i\gamma}(h)$. Take $u_{j,i\gamma}(h)^*$ an eigenfunction of $A_{i\gamma}(h)^*$ such that $A_{i\gamma}(h)^* u_{j,i\gamma}(h)^* = \bar{E}_j(h) u_{j,i\gamma}(h)^*$, $\langle u_{j,i\gamma}(h), u_{j,i\gamma}(h)^*\rangle = 1$. Let $\tilde{\Pi}_j(h)$ be the spectral projection associated to $E_j(h)$ of $A_{i\gamma}(h)$ defined by $\tilde{\Pi}_j(h)(u) = \langle u, u_{j,i\gamma}(h)^*\rangle u_{j,i\gamma}(h)$.

Lemma 5.1. *For ε small enough, we put $\Omega_j := \{z \in \mathbb{C};\ |z - E_j(h)| < \varepsilon h\}$. Let $z \in \Omega_j$, then for h sufficiently small, one has*

$$\tilde{R}(z) = ((1-\tilde{\Pi}_j(h))A_{i\gamma}(h)(1-\tilde{\Pi}_j(h)) - z)^{-1}(1-\tilde{\Pi}_j(h)) \tag{5.1}$$

exists. Moreover, $\tilde{R}(z)$ is holomorphic in $z \in \Omega_j$ and $\|\tilde{R}(z)\| \le Ch^{-1}$.

Proof. By Theorem 4.7, there is only a simple eigenvalue $E_j(h)$ of $A_{i\gamma}(h)$ in Ω_j. This shows that

$$\sigma\Big((1-\tilde{\Pi}_j(h))A_{i\gamma}(h)(1-\tilde{\Pi}_j(h))\Big) = \sigma(A_{i\gamma}(h)) \setminus \{E_j(h)\}.$$

Then,

$$\Omega_j \subset \rho\Big((1-\tilde{\Pi}_j(h))A_{i\gamma}(h)(1-\tilde{\Pi}_j(h))\Big),$$

which gives the existence of $\tilde{R}(z)$. The estimate of $\tilde{R}(z)$ can be followed immediately by imitating the proof of Proposition 4.6. Then we omit the details. □

Theorem 5.2. *For each $n \in \mathbb{N}$ fixed above, and let h be small enough, there exists only one resonance $E_{n,j}(h)$ of $P(B,\omega)$ in $\{z \in \mathbb{C};\ |z-(2n+1)h^{-1} - E_j(h)| < \varepsilon h\}$,*

$$E_{n,j}(h) = (2n+1)h^{-1} + E + he_j + \mathcal{O}(h^2) \tag{5.2}$$

which is algebraically simple. In particular, for all j such that $e_j < C_0$ (C_0 is defined in (2.2)), $E_{n,j}(h)$ is a resonance of $P(B,\omega)$ in U_n. Remark that U_n is defined by (2.2).

Proof. Let

$$R^1_- : \mathbb{C} \to L^2(\mathbb{R}),\ \lambda \mapsto \lambda u_{j,i\gamma}(h)$$

$$R_+^1 : L^2(\mathbb{R}) \to \mathbb{C},\ v \mapsto \langle v, u_{j,i\gamma}(h)^* \rangle.$$

Then $R_-^1 R_+^1 = \tilde{\Pi}_j(h)$ and $R_+^1 R_-^1 = 1$. Let us consider the following Grushin problem for $E_{-+}(z)$:

$$\mathcal{P}_1(z) = \begin{pmatrix} E_{-+}(z) & R_-^1 \\ R_+^1 & 0 \end{pmatrix} : L^2(\mathbb{R}) \times \mathbb{C} \to L^2(\mathbb{R}) \times \mathbb{C}.$$

We treat this problem in the same way as Theorem 3.5. In the same notations as in Lemma 5.1, we put

$$\tilde{\mathcal{E}}_1(z) = \begin{pmatrix} -\tilde{R}(z) & R_-^1 \\ R_+^1 & E_j(h) - z \end{pmatrix}.$$

By a simple computation, we also get $\mathcal{P}_1(z)\tilde{\mathcal{E}}_1(z) = I + \mathcal{O}(h)$ and $\tilde{\mathcal{E}}_1(z)\mathcal{P}_1(z) = I + \mathcal{O}(h)$ uniformly in $z \in \Omega_j$. So $\mathcal{P}_1(z)$ is invertible, whose inverse is

$$\begin{aligned} &\mathcal{E}_1(z) \\ &= \tilde{\mathcal{E}}_1(z) \begin{pmatrix} (1 - h^2 G_{i\gamma}(z;h)\tilde{R}(z))^{-1} & -(1 - h^2 G_{i\gamma}(z;h)\tilde{R}(z))^{-1} h^2 G_{i\gamma}(z;h) u_{j,i\gamma}(h) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e(z) & e_+(z) \\ e_-(z) & e_{-+}(z) \end{pmatrix}, \end{aligned}$$

where $h^2 G_{i\gamma}(z;h) = (E_{-+}(z) - z + A_{i\gamma}(h)) = \mathcal{O}(h^2)$ (see (3.23)). The right lower corner element of $\mathcal{E}_1(z)$ is

$$\begin{aligned} e_{-+}(z) &= E_j(h) - z - \left\langle (1 - h^2 G_{i\gamma}(z;h)\tilde{R}(z))^{-1} h^2 G_{i\gamma}(z;h) u_{j,i\gamma}(h), u_{j,i\gamma}(h)^* \right\rangle \\ &= E_j(h) - z + \mathcal{O}(h^2). \end{aligned}$$

We have $e_{-+}(z) : \mathbb{C} \to \mathbb{C}$ and $0 \in \sigma(E_{-+}(z))$ if and only if $z \in \sigma(e_{-+}(z))$. Combining this with (3.18), $z \in \sigma(e_{-+}(z))$ if and only if $(2n+1)h^{-1} + z \in \sigma(P_{1,i\gamma}(h))$.

By applying the Rouché theorem, $e_{-+}(z) = 0$ has a unique simple solution in Ω_j, $z = E_j(h) + \mathcal{O}(h^2)$. Therefore $E_{n,j}(h) := (2n+1)h^{-1} + E_j(h) + \mathcal{O}(h^2)$ is an unique resonance of $P(B,\omega)$ in $\{z \in \mathbb{C};\ |z - (2n+1)h^{-1} - E_j(h)| < \varepsilon h\}$. □

5.2. The width of resonances. In this subsection, we use the same notations as in the preceding sections. Let $j \in \mathbb{N}$ and $\Omega_j := \{z \in \mathbb{C};\ |z - E_j(h)| < \varepsilon h\}$ be fixed as above (here $\varepsilon > 0$ small enough). We want to construct an approximate eigenvalue $\tilde{z}_j(h) \in \Omega_j$ and an approximate eigenfunction $\tilde{u}_{j,\theta}(h)$ such that

$$E_{-+}(\tilde{z}_j(h))\tilde{u}_{j,\theta}(h) = \mathcal{O}(h^\infty), \tag{5.3}$$

where we recall that

$$E_{-+}(z) = z - A_\theta(h) + h^2 G_\theta(z;h)$$

and $\theta \in \mathbb{C}$ with $|\theta|$ small enough, $\operatorname{Im}\theta < 0$ (see (3.23)).

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ near $[-R, R]$ ($R > 0$). It follows from Remark 3.7 that $\tilde{G}_\theta(z;h) := \chi(x)G_\theta(z;h)$ is well-defined and holomorphic in $\theta \in \mathbb{C}$, $|\theta|$ small enough. In addition, $\tilde{G}_\theta(z;h)$ is self-adjoint if z real and $\operatorname{Im}\theta = 0$.

Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\tilde{\chi} \subset (-R, R)$. Using the h-pseudodifferential calculus, one has

$$\|\tilde{\chi}(x)[\tilde{G}_\theta(z;h) - G_\theta(z;h)]\| = \mathcal{O}(h^\infty). \tag{5.4}$$

Then we are led to construct an approximate solution for the following problem

$$\Big(A_0(h) - h^2\tilde{G}_0(z;h)\Big)u(x) = zu(x),\ z \in \Omega_j. \tag{5.5}$$

In fact, the same problem was studied in [20, 21, 28] by using the WKB method, so we only want to recall main steps. First of all, with the help of Theorem 4.4, one can construct an approximate solution near 0 to

$$(A_0(h) - z^{(j)}(h))u^{(j)}(x;h) = \mathcal{O}(e^{\frac{-d(x)}{h}}h^\infty) \tag{5.6}$$

in the form

$$z^{(j)}(h) \sim \sum_{l\geq 0}\lambda_l^{(j)}h^l$$

and

$$u^{(j)}(x;h) \sim e^{\frac{-d(x)}{h}}\sum_{l\geq 0}u_l^{(j)}(x)h^l,$$

where $d(x)$ is some phase function holomorphic in x near 0 and $\operatorname{Re}(d(x)) > 0$ for $x \in \mathbb{R}\setminus\{0\}$, $\lambda_0^{(j)} = E$, $\lambda_1^{(j)} = e_j$, $\lambda_l^{(j)} \in \mathbb{R}$ for all $l, j \in \mathbb{N}$. Here $z^{(j)}(h) - E_j(h) = \mathcal{O}(h^\infty)$ and $\|u_0^{(j)}\| > \text{const} > 0$.

After that, one solves the following problem by using inductive arguments in $k \in \mathbb{N}$:

$$\left(A_0(h) - h^2\tilde{G}_0(t_{k-1}^{(j)}(h); h) - t_k^{(j)}(h)\right)v_k^{(j)}(x; h) = \mathcal{O}\left(e^{-\frac{d(x)}{h}}h^{\infty}\right), \tag{5.7}$$

where $v_0^{(j)}(x; h) = u^{(j)}(x; h)$, $t_0^{(j)}(h) = z^{(j)}(h)$. The solution of (5.7) is of the form

$$t_k^{(j)}(h) \sim \sum_{l\geq 0} \lambda_{l,k}^{(j)} h^l \text{ and } v_k^{(j)}(x; h) \sim e^{-\frac{d(x)}{h}} \sum_{l\geq 0} v_{l,k}^{(j)}(x)h^l,$$

where $t_{k+1}^{(j)}(h) - t_k^{(j)}(h) = \mathcal{O}(h^{k+2})$ and $v_{k+1}^{(j)}(x; h) - v_k^{(j)}(x; h) = \mathcal{O}(h^{k+2})$, $\lambda_{l,k}^{(j)}$ are real and $v_{l,k}^{(j)}$ are holomorphic near 0. Here $\lambda_{0,k}^{(j)} = E$ and $\lambda_{1,k}^{(j)} = e_j$. Taking the diagonal series, we get an approximate solution of (5.5):

$$\tilde{z}_j(h) \sim \sum_{l\geq 0} \lambda_{l,l}^{(j)} h^l,$$

and

$$\tilde{u}_j(x; h) \sim e^{-\frac{d(x)}{h}} \sum_{l\geq 0} v_{l,l}^{(j)}(x)h^l,$$

where $\lambda_{0,0}^{(j)} = E$ and $\lambda_{1,1}^{(j)} = e_j$.

Let $\chi_1 \in C_0^{\infty}(\mathbb{R})$, $\chi_1 = 1$ near 0. Let us choose $\tilde{u}_{j,\theta}(x; h) = \chi_1(x)\tilde{u}_j(e^{\theta}x; h)$. From the analyticity of $\tilde{G}_\theta(z; h)$ with respect to θ near 0 and (5.4), we obtain an approximate eigenvalue $\tilde{z}_j(h)$ and an approximate eigenfunction $\tilde{u}_{j,\theta}(x; h)$ of (5.3).

Thus, we have proved the following theorem:

Theorem 5.3. *Let* $\Omega_j := \{z \in \mathbb{C};\ |z - E_j(h)| < \varepsilon h\}$, *for some small constant* $\varepsilon > 0$, $j \in \mathbb{N}$. *There exist* $\tilde{z}_j(h) \in \Omega_j$ *and* $\tilde{u}_{j,\theta}(\cdot; h) \in L^2(\mathbb{R})$ *verifying* $\|\tilde{u}_{j,\theta}(\cdot; h)\| > \text{const} > 0$, *such that*

$$E_{-+}(\tilde{z}_j(h))\tilde{u}_{j,\theta}(x; h) = \mathcal{O}(h^{\infty}), \tag{5.8}$$

where

$$\tilde{z}_j(h) \sim \sum_{l\geq 0} \lambda_{l,l}^{(j)} h^l,$$

$\lambda_{0,0}^{(j)} = E$, $\lambda_{1,1}^{(j)} = e_j$ *and* $\lambda_{l,l}^{(j)} \in \mathbb{R}$, $\forall j, l \in \mathbb{N}$.

Now we carry out again Subsection 5.1 in which $\theta = i\gamma$, the eigenfunction $u_{j,i\gamma}(h)$ of $A_{i\gamma}(h)$ is replaced by $\tilde{u}_{j,i\gamma}(\cdot;h)$ given in Theorem 5.3, $u_{j,i\gamma}(h)^*$ is replaced by $\dfrac{1}{\langle \tilde{u}_{j,i\gamma}(\cdot;h), \tilde{u}_{j,-i\gamma}(\cdot;h)\rangle}\tilde{u}_{j,-i\gamma}(\cdot;h)$ and $E_j(h)$ is replaced by $\tilde{z}_j(h)$. We then obtain

$$E_{n,j}(h) \sim (2n+1)h^{-1} + \sum_{l\geq 0} \lambda_{l,l}^{(j)} h^l. \tag{5.9}$$

5.3. End of the proof of Theorem 2.1. We end the proof of our main result in this subsection by proving the following

Proposition 5.4. *The resonance $E_{n,j}(B,\omega)$ has an asymptotic expansion in powers of B^{-1} as $B \to \infty$:*

$$\begin{aligned} E_{n,j}(B,\omega) &\sim (2n+1)B + E \\ &+ \frac{1}{2}\Big((2j+1)\sqrt{\lambda\mu} + \frac{(2n+1)(\lambda+\mu)}{2}\Big)B^{-1} + \sum_{k\geq 2} c_{n,j}^{(k)} B^{-k}, \end{aligned} \tag{5.10}$$

where $c_{n,j}^{(k)} \in \mathbb{R}$ and λ, μ are two eigenvalues of $a_0''(0,0)$. In particular, the width of resonance $E_{n,j}(B,\omega)$ is of order $\mathcal{O}(B^{-\infty})$.

Proof. For B large enough, one has $|\frac{\omega}{B}| < 1$. Thus, for all $N \in \mathbb{N}$,

$$\sqrt{B^2+\omega^2} = B\sqrt{1+\frac{\omega^2}{B^2}} = B\Big(1 + \frac{1}{2}\frac{\omega^2}{B^2} + \sum_{k\geq 2} a_k B^{-2k} + \mathcal{O}(B^{-2(N+1)})\Big), \tag{5.11}$$

$$\frac{1}{\sqrt{B^2+\omega^2}} = \frac{1}{B}\frac{1}{\sqrt{1+\frac{\omega^2}{B^2}}} = \frac{1}{B}\Big(1 - \frac{1}{2}\frac{\omega^2}{B^2} + \sum_{k\geq 2} b_k B^{-2k} + \mathcal{O}(B^{-2(N+1)})\Big), \tag{5.12}$$

where $a_k, b_k \in \mathbb{R}$.

Replacing h by $\dfrac{1}{\sqrt{B^2+\omega^2}}$ in (5.9) and taking into account (5.11), (5.12), we obtain, for all $N \in \mathbb{N}$,

$$E_{n,j}(B,\omega) = (2n+1)B + \sum_{k=0}^{N} c_{n,j}^{(k)} B^{-k} + \mathcal{O}(B^{-N-1}), \tag{5.13}$$

where $c^{(0)}_{n,j} = E$, $c^{(1)}_{n,j} = e_j + \frac{2n+1}{2}\omega^2 = \frac{1}{2}\left((2j+1)\sqrt{\lambda\mu} + \frac{(2n+1)(\lambda+\mu)}{2}\right)$, $c^{(k)}_{n,j} \in \mathbb{R}$. In particular, the imaginary part of $E_{n,j}(B,\omega)$ is of order $\mathcal{O}(B^{-\infty})$. This ends the proof of Proposition 5.4. $\square$

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