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## ON SELF-AVOIDING WALKS ON CERTAIN GRIDS AND THE CONNECTIVE CONSTANT

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**ABSTRACT.** We consider self-avoiding walks on the square grid graph. More precisely we investigate the number of walks of a fixed length on  $\mathbb{Z} \times \{-1, 0, 1\}$ . Using combinatorial arguments we derive the related generating function. We present the asymptotic estimates of the number of walks in consideration, as well as important connective constants.

**1. Introduction.** By a self-avoiding walk (*SAW* for convenience) we shall mean a non-intersecting path on a lattice. In other words, we consider it as a sequence of points  $(c_0, c_1, \dots, c_n)$ , where  $c_i = (x_i, y_i)$ , such that  $c_i \neq c_j$  for all different  $i$  and  $j$ , and  $|x_i - x_{i-1}| + |y_i - y_{i-1}| = 1$  for  $1 \leq i \leq n$ . Usually, the SAWs that we are going to consider start from the point  $(0, 0)$ . Otherwise, we shall mention explicitly the starting point. The interest in SAWs comes from their applications in physics and chemistry. Finding the number of SAWs with a fixed length on the lattice  $\mathbb{Z} \times \mathbb{Z}$  remains a difficult to approach problem in combinatorics.

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Let us denote with  $a_n$  the number of walks of length  $n$ , where the points  $(x_i, y_i)$  in the SAWs have the property  $y_i \in \{0, 1\}$ . Such walks are situated on  $\mathbb{Z} \times \{0, 1\}$ . Zeilberger [6] found a formula for  $a_n$  in terms of generating functions.

**Theorem 1.0.1** (Zeilberger [6]). *The following relation holds:*

$$a_n = 8f_n - \sigma_n,$$

where

$$\sigma_n = \begin{cases} n, & \text{if } n \text{ is even,} \\ 4, & \text{if } n \text{ is odd,} \end{cases}$$

and  $f_n$  is the  $n$ -th Fibonacci number.

Later Benjamin [2] and Nikolov [5] presented purely combinatorial proofs of the formula for  $a_n$ . A more complicated problem was to consider the same grid  $\mathbb{Z} \times \{0, 1\}$  with restrictions to the left and to the right. Jointly with Kalina Petrova in [3], we found an exact formula for the number of walks,  $w_{abn}$ , in  $\mathbb{Z} \times \{0, 1\}$ , where the points  $(x_i, y_i)$  in the SAWs have the property  $-a \leq x_i \leq b$ . As a consequence, we estimated this number asymptotically and derived the following result:

**Proposition 1.0.1** (Dangovski and Petrova [3]). *The following statements hold:*

$$\lim_{n \rightarrow \infty} \frac{w_{abn}}{q^n} = 0,$$

$$\lim_{n \rightarrow \infty} \frac{w_{a\infty n}}{q^n} = \frac{1}{\sqrt{5}} \left( 4 - \frac{2}{q^{2a+2}} \right),$$

$$\lim_{n \rightarrow \infty} \frac{w_{\infty\infty n}}{q^n} = \frac{8}{\sqrt{5}},$$

where  $a = \text{const}$ ,  $b = \text{const}$ , and  $q = \frac{1 + \sqrt{5}}{2}$ .

The problem of finding the number of SAWs on the lattice  $\mathbb{Z} \times \mathbb{Z}$  can be restated using the following definition.

**Definition 1.0.1.** *Let  $W_r$  denote the set of sequences  $C = (c_0, c_1, \dots, c_n)$  of pairwise different points  $c_i = (x_i, y_i)$  in the plane,  $n \in \mathbb{N}_0$ , such that  $x_i, y_i \in \mathbb{Z}$  and  $y_i \in [-r, r]$  for  $0 \leq i \leq n$ ,  $c_0 = (0, 0)$ , and  $|x_i - x_{i-1}| + |y_i - y_{i-1}| = 1$  for  $1 \leq i \leq n$ .*

Indeed, when  $r \rightarrow \infty$  a SAW in  $W_r$  is actually considered in the lattice  $\mathbb{Z} \times \mathbb{Z}$ . It is of interest to study  $W_r$  for small values of  $r$  (see also [6]) in order to obtain a connection with finding the asymptotic value of the number in the general case. In this paper we present the generating function for the number of SAWs of length  $n$  in  $W_1$ . We derive a classification of the walks by using combinatorial arguments. More precisely we find a connection between these SAWs and SAWs with more convenient properties and analyze the latter.

**Definition 1.0.2.** Let  $c_X(n)$  denote the number of self-avoiding walks of length  $n$  starting at the origin on a given lattice  $X$ . We call  $\mu_X = \lim_{n \rightarrow \infty} c_X(n)^{\frac{1}{n}}$  the connective constant for the lattice  $X$ .

Cutting a SAW in two parts we obtain two separated SAWs, but when contracting two walks we do not always maintain the self-avoiding property. Hence  $c_X(n + m) \leq c_X(n) \cdot c_X(m)$  and Fekete's lemma shows that the connective constant exists and is finite. Duminil-Copin and Smirnov [4] derive  $\mu_H = \sqrt{2 + \sqrt{2}}$ , where  $H$  is the hexagonal lattice (see also [1]). It is shown in [3] that  $\mu_{\mathbb{Z} \times \{0,1\}} = \frac{1 + \sqrt{5}}{2}$ . Here we present  $\mu_{\mathbb{Z} \times \{-1,0,1\}}$  along with other asymptotic estimates.

Let  $s_n$  denote the number of SAWs of length  $n$  in a given set  $S$  (the notations of the number and the set are obviously connected). We shall consider the ordinary generating function for the sequence  $s_n$ ,

$$G_S(t) = G(s_n; t) = \sum_{n=0}^{\infty} s_n t^n.$$

In the proofs we use the notions as follows: *u-move* for an upward move, *d-move* for a downward move, *l-move* for a move to the left and *r-move* for a move to the right. A sequence of the letters  $u, d, l$  and  $r$  is a SAW, which starts from  $(0, 0)$  (we may consider the start as a fixed point that is specified if needed) and follows the directions in the string. If a direction  $x \in \{u, d, l, r\}$  is repeated  $k$  times we write  $x^k$ . We consider connected SAWs as one (the beginning of a walk in a given set is the end of a walk in the previous set). We say that two SAWs are avoidable if they do not share same points. Unions of the form  $X_{(i)} \cup Y_{(j)}$ , where  $X$  and  $Y$  are avoidable SAWs, are treated as one with a beginning – the one of  $X$  at  $y$ -value  $i$  (if connected  $i$  is not necessary) and end – the one of  $Y$  ( $Y$  starts at  $y$ -value  $j$ ).

The author has used computer calculations to suggest some of the formulas, presented in this paper. The classifications of the sets, further in the project, have been obtained via exhaustive search.

**2. Self-avoiding walks on  $\mathbb{Z} \times \{-1, 0, 1\}$ .** In order to find the generating function for the number of SAWs on  $\mathbb{Z} \times \{-1, 0, 1\}$  of a given length we note a connection with simpler walks. Actually, we consider special sub-walks and we remind that not all of them start from the origin.

**2.1. Generating function for  $T$ -walks.** To simplify a walk we need to add some restrictions. In the walks of the following type we set a restriction for the  $x$ -coordinates and add the special property that the first and the last point of the walk have the same  $x$ -coordinate.

**Definition 2.1.1.** Let  $T$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ , such that  $n \in \mathbb{N}_0$ ,  $c_0 = (0, -1)$ ,  $c_n = (0, 1)$ , and  $x_i \geq 0$ ,  $y_i \in \{-1, 0, 1\}$  for  $0 \leq i \leq n$ .

To find the generating function for  $T$  we need some basic results.

**Definition 2.1.2.** Let  $T'$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ ,  $n \in \mathbb{N}_0$ , where  $c_0 = (0, -1)$ ,  $y_n = -1$  and  $x_i \geq x_{i-1}$ ,  $y_i \in \{-1, 0\}$  for  $1 \leq i \leq n$ .

**Definition 2.1.3.** Let  $T''$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ ,  $n \in \mathbb{N}_0$ , where  $y_0 = 1$ ,  $c_n = (0, 1)$ , and  $x_i \leq x_{i-1}$ ,  $y_i \in \{0, 1\}$  for  $1 \leq i \leq n$ .

**Definition 2.1.4.** Let  $T_1$  denote the set of all pairs  $(t', t'')$ , where  $t' \in T'$  and  $t'' \in T''$ , such that  $t'$  and  $t''$  are avoidable.

**Definition 2.1.5.** Let  $T_1''$  denote the set of all pairs of SAWs  $s = (s_0, s_1, \dots, s_n)$ ,  $s_i = (x_{s_i}, y_{s_i})$ ,  $r = (r_0, r_1, \dots, r_m)$ ,  $r_i = (x_{r_i}, y_{r_i})$ , where  $n, m \in \mathbb{N}$ , such that  $y_{s_i} = 1$ ,  $s_0 = (0, 1)$ ,  $x_{s_i} = x_{s_{i-1}} + 1$ , for  $1 \leq i \leq n$ ,  $r_0 = (0, -1)$ ,  $y_{r_n} = -1$ ,  $x_{r_i} \geq x_{r_{i-1}}$ ,  $y_{r_i} \in \{-1, 0\}$ , for  $1 \leq i \leq m$ , and  $x_{s_n} = x_{r_m}$ .

We remind that the coefficient of  $t^n$  in  $G_S(t)$ , which we denote with  $[t^n]G_S(t)$  is actually the number of SAWs with length  $n$  in the given set of SAWs  $S$ , which generates the formal power series  $G_S(t)$ .

**Proposition 2.1.1.** The following identity holds:

$$G_{T_1''}(t) = \frac{1 - t^2 + t^4}{1 - 2t^2 + t^4 - t^6}.$$

**Proof.** One can see that a SAW in  $T_1''$  is a chain of mini-walks  $L$  of type  $ur^i_{(-1)} \cup r^i_{(1)}$  or of type  $r^i u_{(-1)} \cup r^i_{(1)}$ ,  $i \geq 1$ . Since the type of the next step of

the walk is determined from the previous one, we conclude that

$$(2.1) \quad G_L(t) = G_{ur^i_{(-1)} \cup r^i_{(1)}}(t) = \sum_{j=1}^{\infty} t^{2j+1} = \frac{t^3}{1-t^2}.$$

A walk in  $T_1''$  can start with  $r \cup r_{(1)}$  or with  $u$ . In the first case we need an even number of walks of type  $L$  and in the second – an odd one in order to set the  $y$ -value of the last point  $-1$ . The walk may finish with two straight lines of type  $r^i \cup r_{(1)}^i$ . Hence,

$$(2.2) \quad T_1'' = \begin{cases} L^{2i} (r^j \cup r_{(1)}^j), & \text{where } i \geq 0, j \geq 0, \\ uL^{2i+1} (r^j \cup r_{(1)}^j), & \text{where } i \geq 0, j \geq 0. \end{cases}$$

Thus,

$$\begin{aligned} G_{T_1''}(t) &= G_{L^{2i}}(t)G_{r^j \cup r_{(1)}^j}(t) + tG_{L^{2i+1}}(t)G_{r^j \cup r_{(1)}^j}(t) \\ &= \sum_{j=0}^{\infty} (G_L(t))^{2j} \frac{1}{1-t^2} + t \frac{t^3}{1-t^2} \sum_{j=0}^{\infty} (G_L(t))^{2j} \frac{1}{1-t^2} \\ &= \frac{1}{1 - (\frac{t^3}{1-t^2})^2} \frac{1}{1-t^2} + \frac{t^4}{(1-t^2)^2} \frac{1}{1 - (\frac{t^3}{1-t^2})^2} \end{aligned}$$

and after simplification the proof is completed.  $\square$

**Proposition 2.1.2.** *The following identity holds:*

$$G_{T_1}(t) = \frac{(t-1)(t+1)(2t^4 - t^2 + 2)}{(2t^2 - 1)(t^4 + 1)}.$$

**Proof.** By considering all of the possible cases, we note the following classification of the set:

$$(2.3) \quad T_1 = \begin{cases} (r^i \cup dr^i u_{(1)}) T_1', & \text{where } i \geq 1, \\ T_1'' T_2', \\ \emptyset. \end{cases}$$

In the classification  $T_1'$  is the set of SAWs in  $T_1$  with beginning  $(r \cup r_{(1)})$  and  $T_2'$  is the set of walks in  $T_1'$  that do not start with  $T_1''$ . We should note that

$T'_1$  is a chain of repeated mini-walks  $T_m$  of the form  $r^{i+1} \cup r dr^i u_{(1)}$ ,  $i \geq 1$ , or  $(r \cup r_{(1)}) T''_1 (r^{i+1} \cup r dr^i u_{(1)})$ ,  $i \geq 1$ , that can end in  $(r \cup r_{(1)}) T''_1$ . The classification of  $T'_2$  is the same, but a walk in it should begin with  $r^{i+1} \cup r dr^i u_{(1)}$ ,  $i \geq 1$ . In other words:

$$(2.4) \quad G_{T'_1}(t) = G_{T_m^j}(t) \left( G_\emptyset(t) + G_{r \cup r_{(1)}}(t) G_{T''_1}(t) \right),$$

$$G_{T'_2}(t) = G_{r^{i+1} \cup r dr^i u_{(1)}}(t) G_{T_m^j}(t) \left( G_\emptyset(t) + G_{r \cup r_{(1)}}(t) G_{T''_1}(t) \right),$$

$$G_{T_m}(t) = G_{r^{i+1} \cup r dr^i u_{(1)}}(t) + G_{r \cup r_{(1)}}(t) G_{T''_1}(t) G_{r^{i+1} \cup r dr^i u_{(1)}}(t),$$

where  $j \geq 0$ ,  $i \geq 1$ .

Thus, from (2.4) we derive that

$$G_{T_m}(t) = \left( G_\emptyset(t) + G_{r \cup r_{(1)}}(t) G_{T''_1}(t) \right) G_{r^{i+1} \cup r dr^i u_{(1)}}(t)$$

$$= \left( 1 + t^2 \frac{1 - t^2 + t^4}{1 - 2t^2 + t^4 - t^6} \right) \sum_{j=3}^{\infty} t^{2j}$$

$$= \frac{t^6}{-t^6 + t^4 - 2t^2 + 1},$$

$$G_{T'_1}(t) = \frac{1}{1 - G_{T_m}(t)} \left( 1 + t^2 \frac{1 - t^2 + t^4}{1 - 2t^2 + t^4 - t^6} \right)$$

$$= \frac{(t - 1)(t + 1)}{(2t^2 - 1)(t^4 + 1)},$$

$$G_{T'_2}(t) = G_{r^{i+1} \cup r dr^i u_{(1)}}(t) G_{T_m^j}(t) \left( G_\emptyset(t) + G_{r \cup r_{(1)}}(t) G_{T''_1}(t) \right) + G_\emptyset(t)$$

$$= \frac{t^6}{1 - t^2} \frac{(t - 1)(t + 1)}{(2t^2 - 1)(t^4 + 1)} + 1$$

$$= \frac{t^6 - t^4 + 2t^2 - 1}{(2t^2 - 1)(t^4 + 1)}.$$

From (2.3) we have that  $G_{T_1}(t) = G_{r^i \cup dr^i u_{(1)}}(t) G_{T'_1}(t) + G_{T''_1}(t) G_{T'_2}(t) + G_\emptyset(t)$ . Hence,

$$G_{T_1}(t) = \sum_{j=2}^{\infty} t^{2j} \frac{(t - 1)(t + 1)}{(2t^2 - 1)(t^4 + 1)} + \frac{1 - t^2 + t^4}{1 - 2t^2 + t^4 - t^6} \frac{t^6 - t^4 + 2t^2 - 1}{(2t^2 - 1)(t^4 + 1)} + 1,$$





where  $\phi_T$  is the generating function for the duplicates we obtain when calculating the generating function for  $T_1$ . One can see that the walks  $ru^2l$ ,  $r^{i+1}ul^iul$  and  $rr^iul^{i+1}$  (for  $i \geq 1$ ) are counted twice, so

$$\phi_T = G_{ru^2l}(t) + G_{r^{i+1}ul^iul}(t) + G_{rr^iul^{i+1}}(t) = \frac{t^4 + t^6}{1 - t^2}.$$

Hence, we obtain

$$G_T(t) = \left( t^4 + \frac{2t^6}{1 - t^2} \right) \frac{(t - 1)(t + 1)(2t^4 - t^2 + 2)}{(2t^2 - 1)(t^4 + 1)} + \frac{t^2 + t^4}{1 - t^2} - \frac{t^4 + t^6}{1 - t^2},$$

from which the result follows.  $\square$

**2.2. Generating function for  $R$ -walks.** Here we present another type of walks that have a restriction. Finding their generating function would help us consider the general problem.

**Definition 2.2.1.** Let  $R$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ ,  $n \in \mathbb{N}_0$ , where  $c_0 = (0, -1)$ ,  $x_i \geq 0$ , and  $y_i \in \{-1, 0, 1\}$  for  $0 \leq i \leq n$ .

We need to divide these walks in simpler ones.

**Definition 2.2.2.** Let  $R'$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ ,  $n \in \mathbb{N}_0$ , where  $c_0 = (0, -1)$ ,  $x_i \geq x_{i-1} \geq 0$ , and  $y_i \in \{-1, 0, 1\}$  for  $1 \leq i \leq n$ .

**Definition 2.2.3.** Let  $R''$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ ,  $n \in \mathbb{N}_0$ , where  $c_0 = (0, -1)$ ,  $x_i \geq 0$  and  $y_i \in \{-1, 0, 1\}$  for  $0 \leq i \leq n$ , such that there exists  $j < n$  with the property that  $x_r \geq x_{r-1}$ , for  $1 \leq r \leq j$ ,  $x_{j+1} < x_j$  and  $x_r < x_j$ , for  $j + 2 \leq r \leq n$ .

**Definition 2.2.4.** Let  $R'''$  denote the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ ,  $n \in \mathbb{N}_0$ , where  $c_0 = (0, -1)$ ,  $x_i \geq 0$  and  $y_i \in \{-1, 0, 1\}$  for  $0 \leq i \leq n$ , such that  $\exists j < k \leq n$  with the property  $x_r \geq x_{r-1}$ , for  $1 \leq r \leq j$ ,  $x_{j+1} < x_j$  and  $x_k > x_j$ .

From the definitions we have that  $R = R' + R'' + R'''$ . So, we need to find the generating functions of these three sets in order to find  $G_R(t)$ .

**Proposition 2.2.1.** The following identity holds:

$$G_{R'}(t) = -\frac{t^3 + t^2 + 1}{t^4 + t^3 - t^2 + 2t - 1}.$$

Proof. One can see that a walk in  $R'$  is a chain of mini-SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ , such that  $x_i \geq x_{i-1}$  for  $1 \leq i \leq n$ , where  $y_i \in \{-1, 0\}$  for  $1 \leq i \leq n - 1$  and  $y_0 = y_1 = -1, y_n = 1$ , or  $y_i \in \{0, 1\}$ , for  $1 \leq i \leq n - 1$  and  $y_0 = y_1 = 1, y_n = -1$ , in the set  $BL$ . Since the type of the next  $BL$ -walk is determined from the previous one, we have the following classification:

$$(2.7) \quad BL = \begin{cases} rT'ru^2, \\ rT' rur^i u, & \text{where } i \geq 1, \\ ru^2, \\ rur^i u, & \text{where } i \geq 1. \end{cases}$$

A walk in  $T'$  is a cluster of mini-walks  $L'$  of type  $r^i u$  or  $ur^i$  ( $i \geq 1$ ). Since the type of the next mini-walk is determined from the previous one, we have that

$$(2.8) \quad G_{L'}(t) = G_{r^i u}(t) = \sum_{j=2}^{\infty} t^j = \frac{t^2}{1-t}.$$

A  $T'$ -SAW can start with  $r$  or with  $u$ . In the first case we would need an even number of walks of type  $L'$  and in the second – odd, in order to set the  $y$ -level of the last point  $-1$ . The walk may finish with a straight line of type  $r^i$ . Hence,

$$(2.9) \quad T' = \begin{cases} (L')^{2i} r^j, & \text{where } i \geq 0, j \geq 0, \\ u(L')^{2i+1} r^j, & \text{where } i \geq 0, j \geq 0. \end{cases}$$

Thus,

$$\begin{aligned} G_{T'}(t) &= G_{(L')^{2i}}(t)G_{r^j}(t) + tG_{(L')^{2i+1}}(t)G_{r^j}(t) \\ &= \sum_{j=0}^{\infty} (G_{L'}(t))^{2j} \frac{1}{1-t} + t \frac{t^2}{1-t} \sum_{j=0}^{\infty} (G_{L'}(t))^{2j} \frac{1}{1-t} \\ &= \frac{1}{1 - (\frac{t^2}{1-t})^2} \frac{1}{1-t} + \frac{t^3}{(1-t)^2} \frac{1}{1 - (\frac{t^2}{1-t})^2}. \end{aligned}$$

From the last observation we obtain

$$(2.10) \quad G_{T'}(t) = \frac{-1 + t - t^3}{-1 + 2t - t^2 + t^4}.$$

Now using (2.7) and (2.10) we continue with the following calculations:

$$G_{BL}(t) = \left( t^3 + \frac{t^4}{1-t} \right) \left( 1 + t \frac{-1+t-t^3}{-1+2t-t^2+t^4} \right)$$

to obtain

$$(2.11) \quad G_{BL}(t) = \frac{t^3}{-t^4 + t^2 - 2t + 1}.$$

After exhaustive search, we present the classification of  $R'$ :

$$(2.12) \quad R' = \begin{cases} R_1, \\ BLE(BL)^i, & \text{where } i \geq 0, \\ BLE(BL)^i t R_1, & \text{where } i \geq 0, \end{cases}$$

where  $BLE$  are 'extended'  $BL$ -walks, i.e.,  $BLE = \{u^2\} + \{ur^i u\} + BL$ , for  $i \geq 1$  and  $R_1$  is the set of SAWs  $(c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ , such that  $x_i \geq x_{i-1}$ , for  $1 \leq i \leq n$ ,  $x_0 = -1$  and  $y_i \in \{-1, 0\}$  for  $0 \leq i \leq n$  (Fig. 2 contains an example of a SAW in  $R'$ ). Now we can easily see that

$$(2.13) \quad G_{BLE}(t) = \frac{t^2}{-t^4 + t^2 - 2t + 1}.$$



Fig. 2

A walk in  $R_1$  is again a chain of  $L'$ -walks, but here we are interested in the type of the first  $L'$ -walk. The classification of the set is the following one:

$$(2.14) \quad R_1 = \begin{cases} u, \\ r^i, & \text{where } i \geq 0, \\ r^i u (L')^j r^k, & \text{where } i \geq 1, j \geq 0, k \geq 0, \\ ur^i (L')^j, & \text{where } i \geq 1, j \geq 0, \\ ur^i (L')^j t, & \text{where } i \geq 1, j \geq 0. \end{cases}$$

Hence, from (2.8) and (2.14) we obtain that

$$(2.15) \quad G_{R_1}(t) = \frac{-t - 1}{t^2 + t - 1}.$$

Now, from (2.12), (2.11), (2.13), (2.15), we proceed with

$$G_{R'}(t) = G_{BLE}(t) \frac{1}{1 - G_{BL}(t)} (1 + tG_{R_1}(t)) + G_{R_1}(t)$$

to complete the proof.  $\square$

**Remark 2.2.1.** One can notice that the set  $T'$  is very similar to  $T''_1$  (the only difference is the straight line in the second set). Therefore, obtaining the generating function for  $T'$  is analogous to obtaining the function for  $T''_1$ .

**Proposition 2.2.2.** *The following identity holds:*

$$G_{R''}(t) = \frac{t^3(1 + 3t + 3t^2 + 3t^5 + 5t^6 + 3t^7 + t^8)}{(1 + t + t^2)(2t^2 - 1)(1 + t^4)(t^4 + t^3 - t^2 + 2t - 1)}.$$

*Proof.* One can see that the set  $R''$  is similar to  $R'$ , but we can once change the  $x$ -direction of the points. This way we obtain a combination of two type of walks – one with a straight  $x$ -direction and one with a straight and backward  $x$ -directions, restricted to the left and to the right. Let us consider a SAW  $C = (c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i) \in R''$  with the existing  $j < n$  such that  $x_r \geq x_{r-1}$ , for  $1 \leq r \leq j$ ,  $x_{j+1} < x_j$  and  $x_r < x_j$ , for  $j + 2 \leq r \leq n$ . We take  $m := \min_{j < i \leq n} \{x_i\}$  and remove all the points  $c_i$  with  $x_i \geq m$  from  $C$ . The result is a new SAW  $C'$ . Let  $BT$  denote the set of SAWs  $C - C'$  where  $C \in R'$ . By considering all the possible cases we note the classification of  $BT$ :

$$(2.16) \quad BT = \begin{cases} T - \{u^2\}, \\ rTld, \\ r^i T l^i d r^{i-1}, & \text{where } i \geq 2, \\ r^i T l d l^{i-1}, & \text{where } i \geq 2, \\ r^i T l d l^{i-1} u r^j, & \text{where } i \geq 3, 1 \leq j \leq i - 2, \\ r^i u l^i, & \text{where } i \geq 1, \\ r^i u l^i u r^j, & \text{where } i \geq 1, 1 \leq j \leq i. \end{cases}$$





where  $\phi_{R''}$  is the set of walks that we have counted in  $R'$  and  $R''$ .

The set  $\phi_{R'''}$  has the following classification:

$$(2.23) \quad \phi_{R'''} = \begin{cases} r^i ul^i ur^i, & \text{where } i \geq 1, \\ T'r^{i+1}ul^i ur^i, & \text{where } i \geq 1, \\ BLE(BL)^i r^{i+1}ul^i ur^i, & \text{where } i \geq 1, \\ BLE(BL)^i rT'r^{i+1}ul^i ur^i, & \text{where } i \geq 1. \end{cases}$$

Now, from (2.22), (2.23), (2.10), (2.20), (2.21), Proposition 2.2.1 and Proposition 2.2.2 we have that

$$\begin{aligned} G_{R'''}(t) &= G_{SE}(t) \frac{1}{1 - G_S(t)} (1 + t(G_{R'}(t) + G_{R''}(t))) \\ &\quad - \frac{t^5}{1 - t^3} - G_{T'}(t) \frac{t^6}{1 - t^3} - G_{BLE}(t) \frac{1}{1 - G_{BL}(t)} \frac{t^6}{1 - t^3} \\ &\quad - G_{BLE}(t) \frac{1}{1 - G_{BL}(t)} G_{T'}(t) \frac{t^7}{1 - t^3} \end{aligned}$$

and after replacing with the known generating functions and simplifying, we obtain the result.  $\square$

From these observations, we derive the main result in this subsection.

**Proposition 2.2.4.** *The following identity holds:*

$$G_R(t) = \frac{1 + t + t^3 + 2t^4 + t^5 - 4t^6 - 2t^7 + t^8 + 2t^9 - t^{10} - t^{11}}{(-1 + 2t^2 - t^4 + 2t^6)(-1 + t + 2t^3 + t^4 + 2t^5 + 2t^6)}.$$

*Proof.* We have already mentioned that  $R = R' + R'' + R'''$ . Here we shall present the argumentation about that. First, we shall note that:

- $R' \cap R'' = \emptyset$ , because if  $r' \in R'$ ,  $r'' \in R''$ , we have that  $BT \notin r'$ ,  $BT \in r''$ ;
- $R' \cap R''' = \emptyset$ , because if  $r' \in R'$ ,  $r''' \in R'''$ , we have that  $S \notin r'$ ,  $S \in r'''$ ;
- $R'' \cap R''' = \emptyset$ , because if  $r'' \in R''$ ,  $r''' \in R'''$ , we have that  $S \notin r''$ ,  $S \in r'''$ .

Now, if  $c \in R$  and the  $x$ -coordinates of  $c$  are increasing, then  $c \in R'$ , otherwise  $c \in R'' + R'''$ . Hence,

$$G_R(t) = G_{R'}(t) + G_{R''}(t) + G_{R'''}(t)$$

and from Proposition 2.2.1, Proposition 2.2.2 and Proposition 2.2.3 the result follows.  $\square$

**2.3. Generating functions of  $W_1$ -walks and asymptotic estimates.**

To find  $G_{W_1}(t)$  and the number of self-avoiding walks in  $\mathbb{Z} \times \{-1, 0, 1\}$  respectively we are going to make a connection between the set  $W_1$  and the sets  $T$  and  $R$ .

**Theorem 2.3.1.** *The following identity holds:*

$$G_{W_1}(t) = \frac{N(t)}{D(t)},$$

where

$$\begin{aligned} N(t) = & 1 + 3t + 2t^2 - 3t^3 - 10t^4 - 2t^5 + 14t^6 + 21t^7 - 11t^8 - 35t^9 - 10t^{10} \\ & + 31t^{11} + 32t^{12} - 16t^{13} - 38t^{14} + 3t^{15} + 24t^{16} + 26t^{17} - 4t^{18} + 4t^{20} \\ & + 4t^{21} + 4t^{22} \end{aligned}$$

and

$$D(t) = (1 - 2t^2 + t^4 - 2t^6)^2(1 - t - 3t^3 - 2t^5 + t^7 + 2t^8 + 2t^9).$$

*Proof.* We consider  $W_X$  such that  $W_X \in W_1$  and for every  $w_i \in W_X$  we have  $w_i = XY_i$ , where  $X$  is a fixed sub-SAW. In other words,  $W_X$  is the set of the SAWs in  $W_1$  that start with  $X$ . We have that  $W_1 = W_u + W_d + W_l + W_r$ , but  $G_{W_u}(t) = G_{W_d}(t)$  and  $G_{W_l}(t) = G_{W_r}(t)$  by symmetry. Hence (Fig. 5 gives an example of a SAW in  $W_1$ ),

$$(2.24) \quad G_{W_1}(t) = 2(G_{W_u}(t) + G_{W_r}(t)).$$

Now,  $W_u = W_{ur} + W_{ul} + u$ , but  $G_{W_{ur}}(t) = G_{W_{ul}}(t)$  by symmetry and it follows that

$$(2.25) \quad G_{W_u}(t) = 2G_{W_{ur}}(t) + t.$$

We note the classification of  $W_{ur}$ :

$$(2.26) \quad W_{ur} = \begin{cases} urR, \\ urTl, \\ urTl^2R. \end{cases}$$



We have that  $W_r = r^i + W_{r^j d} + W_{r^j u}$ , but  $G_{W_{r^j d}}(t) = G_{W_{r^j u}}(t)$  by symmetry, so

$$(2.27) \quad G_{W_r}(t) = \frac{1}{1-t} + 2G_{W_{r^j d}}(t).$$

Using exhaustive search, we can note the classification of  $W_{r^j d}$ :

$$(2.28) \quad W_{r^j d} = \begin{cases} r^i d, & \text{where } i \geq 1, \\ r^i drR, & \text{where } i \geq 1, \\ r^i drTl^j, & \text{where } i \geq 1, j \leq i+1, \\ r^i drTl^{i+2}R, & \text{where } i \geq 1, \\ r^i drTl^{i+2}Tr^j, & \text{where } i \geq 1, j \leq i, \\ r^i dl^j, & \text{where } i \geq 1, j \leq i, \\ r^i dl^{i+1}R, & \text{where } i \geq 1, \\ r^i dl^{i+1}Tr^j, & \text{where } i \geq 1, j \leq i+1, \\ r^i dl^{i+1}Tr^{i+2}R, & \text{where } i \geq 1. \end{cases}$$

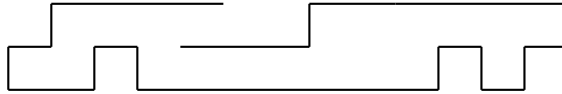


Fig. 5

Now we can calculate the generating functions of  $W_{ur}$  and  $W_{r^j d}$ , in order to obtain the result. From (2.24), (2.25), (2.26), (2.27) and (2.28) we derive

$$\begin{aligned} G_{W_1}(t) &= 4(t^2 G_R(t) + t^3 G_T(t) + t^4 G_T(t) G_R(t)) + 2t + 2\frac{t}{1-t} \\ &+ 4\left(\frac{t^2}{1-t} + \frac{t^3}{1-t} G_R(t) + \left(\frac{1}{(1-t^2)(1-t)} - 1\right) t^3 G_T(t)\right) \\ &+ 4\left(\frac{t^6}{1-t^2} G_T(t) G_R(t) + \frac{t^7}{(1-t^3)(1-t^2)} (G_T(t))^2 + \frac{t^3}{(1-t^2)(1-t)}\right) \\ &+ 4\left(\frac{t^4}{1-t^2} G_R(t) + \frac{t^5}{(1-t^2)(1-t^3)} G_T(t) + \frac{t^6}{1-t^3} G_T(t)\right) \\ &+ 4\frac{t^7}{1-t^3} G_T(t) G_R(t). \end{aligned}$$

We replace with the formulas in Proposition 2.1.3 and Proposition 2.2.4 to obtain the result.  $\square$

The last proposition implies the asymptotic of the number of walks in  $\mathbb{Z} \times \{-1, 0, 1\}$ .

**Definition 2.3.1.** Let  $w_{(1)n}$  denote the number of sequences  $C = (c_0, c_1, \dots, c_n)$ ,  $c_i = (x_i, y_i)$ , of pairwise different points in the plane such that  $x_i \in \mathbb{Z}$  and  $y_i \in \{-1, 0, 1\}$  for  $0 \leq i \leq n$ ,  $c_0 = (0, 0)$  and  $|x_i - x_{i-1}| + |y_i - y_{i-1}| = 1$  for  $1 \leq i \leq n$ .

**Corollary 2.3.1.** The following relations hold:

$$\mu_{\mathbb{Z} \times \{-1, 0, 1\}} = \lim_{n \rightarrow \infty} \frac{w_{(1)n}}{w_{(1)n-1}} = 1/v_{\min} \approx 1.9146267907190664,$$

where  $v_{\min}$  is the minimal modulus zero of  $D(t)$ .

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