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AN ESTIMATION METHOD FOR THE RELIABILITY OF "CONSECUTIVE-k-OUT-OF-n SYSTEM"

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Abstract. This paper is concerned with consecutive-k-out-of- n system in which all the components have the same q lifetime probability, so, it's possible to estimate q from a sample by using the maximum likelihood principle. In the reliability formula of the consecutive-k-out-of-n system appears the term q^k. The goal in this work is to propose a direct estimation of q^k to avoid the accumulated errors owed to the power. More precisely, we establish a new method based on the Markov chains to calculate and estimate the reliability of the system.

1. Introduction. A consecutive-k-out-of- n system is formed by n linearly components. This system fails if and only if there's at least k failed consecutive components, $(1 \leq k \leq n)$. Until now, a great number of closed formulas, recursive and direct algorithms(cf. [1, 3, 6, 13]), limit formulas (cf. [2, 7, 8]) and

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bounds (cf. [5, 10, 11]) for the reliability of linear and circular "consecutive-k-out-of-n: F systems" are established by use of methods from combinatorics and probability theory.

In the most articles dealing with the reliability of such systems, it is supposed that the lifetime probability q of a component is known, and it will be estimated by choosing a sample in the expression of the reliability R:

$$R = (1 - q^k) \left[1 - \frac{q^k(1-q)}{1-q^k} \right]^{n-k}.$$

We can conclude this formula, (cf. [1, 6, 13]), by supposing that all components of the system are independent and identically distributed.

In this formula we have the term q^k, so, a little gaps in the evaluation of q yields a great one in the evaluation of q^k and then in the estimation of R. For this reason we propose a new method based on the Markov chains. This can help us to estimate directly R with the q^k estimator rather the q estimator.

We Consider that the structure function of this system is given by

$$S_n = \min_{1 \le i \le n-k} \max_{i \le j \le i+k-1} X_j$$

where X_j is the state of the component j , $j = 1, \dots, n$.
We suppose that the components are independent and have the same distribution.

2. Notations.

For $i = 1, \dots, n$:

$$X_i = \begin{cases} 0 & \text{if the component } i \text{ breaks down} \\ 1 & \text{if the component } i \text{ is working} \end{cases}$$

where $P(X_i = 0) = q$, $P(X_i = 1) = p = 1 - q$. Finally, let Z_k a sub-system with k components and $S_n = 1$ means that the system is working.

3. Results.

Let the variables $Z_n = \max_{n \le i \le n+k-1} X_i$.

Theorem 1. *The process $(Z_n)_{n \ge 1}$ is a Markov chain.*

Proof. For all $n \ge 1$ and $z_n \in \{0, 1\}$, we must proof that,

$$P(Z_{n+1} = z_{n+1}/Z_1 = z_1, Z_2 = z_2, \dots, Z_n = z_n) = P(Z_{n+1} = z_{n+1}/Z_n = z_n).$$

1) If

$$A_n = Z_1 = z_1, Z_2 = z_2, \ldots, Z_{n-1} = z_{n-1}, Z_n = 0,$$

then

$$\begin{aligned} P(Z_{n+1} = 1/A_n) &= \frac{1}{P(A_n)} \int_{[A_n]} 1_{[Z_{n+1}=1]} dp \\ &= \frac{1}{P(A_n)} \left(\int_{[A_n]\cap[X_n=0]} 1_{(Z_{n+1}=1)} dp + \int_{[A_n]\cap[X_n=1]} 1_{(Z_{n+1}=1)} dp \right) \end{aligned}$$

$$\begin{aligned} P(Z_{n+1} = 1/A_n) &= \frac{\int_{[A_n]\cap[X_n=0]} 1_{(X_{n+k}=1)} dp}{P(A_n)} \\ &= \frac{1}{P(A_n)} \int_{[A_n]} 1_{(X_{n+k}=1)} dp. \end{aligned}$$

It is clear that the two events, $(X_{n+k} = 1)$ and A_n are independent. So

$$P(Z_{n+1} = 1/A_n) = \frac{P\left(Z_n = 0, X_{n+k} = 1\right)}{P\left(Z_n = 0\right)},$$

we have used the fact that $(Z_n = 0)$ and $(X_{n+k} = 1)$ are independent events. We can write:

$$\begin{aligned} (Z_n = 0, X_{n+k} = 1) &= \{Z_n = 0\} \cap \{X_{n+k} = 1\} \\ &= \{Z_n = 0\} \cap \{Z_{n+1} = 1\} \\ &= \left(Z_n = 0, Z_{n+1} = 1\right). \end{aligned}$$

Then,

$$P(Z_{n+1} = 1/A_n) = P\left(Z_{n+1} = 1/Z_n = 0\right).$$

In the same way we obtain:

$$P(Z_{n+1} = 0/A_n) = P(Z_{n+1} = 0/Z_n = 0).$$

Now, put $B_n = Z_1 = z_1,\ Z_2 = z_2, \ldots, Z_{n-1} = z_{n-1},\ Z_n = 1$. $C_{n-1} = X_1 = x_1$, $X_2 = x_2, \ldots, X_{n-1} = x_{n-1}$. $D_{n,n-k+1} = X_n = 1,\ X_{n+1} = 0, \ldots, X_{n+k-1} = 0$.

$$
\begin{aligned}
P(Z_{n+1} = 0/B_n) &= \frac{1}{P(B_n)} \int\limits_{[B_n]} 1_{[Z_{n+1}=1]} dp \\
&= \frac{1}{P(B_n)} \left(\int\limits_{[B_n]\cap[X_n=0]} 1_{[Z_{n+1}=1]} dp + \int_{[B_n]\cap[X_n=1]} 1_{[Z_{n+1}=1]} dp \right) \\
&= \frac{1}{P(B_n)} \int\limits_{[B_n]\cap[X_n=1]} 1_{[\max(X_{n+1},\ldots,X_{n+k})=0]} dp \\
&= \frac{1}{P(B_n)} \int\limits_{[B_n]\cap[X_n=1]} 1_{[(X_{n+1}=0,\ldots,X_{n+k}=0)]} dp \\
&= \frac{1}{P(B_n)} \int\limits_{[B_n]\cap[X_n=1]\cap[X_{n+1}=0,\ldots,X_{n+k-1}=0]} 1_{[X_{n+k}=0]} dp \\
&= \frac{P(B_n, D_{n;n-k+1}).P(X_{n+k}=0)}{P(B_n)} \\
&= \frac{P(C_{n-1}, D_{n;n-k+1}).P(X_{n+k}=0)}{P(C_{n-1}, Z_n = 1)}
\end{aligned}
$$

It is clear that C_{n-1} and $D_{n,n-k+1}$ are two independent events, and so are the two events C_{n-1} and $(Z_n = 1)$. Then,

$$
P(Z_{n+1} = 0/B_n) = \frac{P\left(D_{n,n-k+1}\right)}{P\left(Z_n = 1\right)}.
$$

We can write $(Z_n = 1, Z_{n+1} = 0)$ as:

$$
\begin{aligned}
(Z_n = 1, \max(X_{n+1}, \ldots, X_{n+k}) = 0) &= (Z_n = 1, X_{n+1} = 0, \ldots, X_{n+k} = 0) \\
&= (X_n = 0, X_{n+1} = 0, \ldots, X_{n+k} = 0).
\end{aligned}
$$

So,

$$P(Z_{n+1} = 0/B_n) = P\left(Z_{n+1} = 0/Z_n = 1\right).$$

Finally:

$$\begin{aligned} P(Z_{n+1} = 1/B_n) &= 1 - P(Z_{n+1} = 0/B_n) \\ &= 1 - P\left(Z_{n+1} = 0/Z_n = 1\right) \\ &= P\left(Z_{n+1} = 1/Z_n = 1\right). \end{aligned}$$

We conclude that $(Z_n)_{n\geq 1}$ is a Markov chain.

Now, To compute the reliability of the consecutive-k-out-of-n system, we must compute the transition probabilities from the state i (at the time n) to the state j (at the time $n+1$) where i and $j \in \{0,1\}$. For this end, let $\Pi(l,m) = P(Z_{n+1} = j\ /Z_n = i)$ where $i, j, l, m \in \{0,1\}$.

Computation of the transition probabilities. We have:

$$\begin{aligned} \Pi(0,0) &= \frac{P(Z_{n+1} = 0, Z_n = 0)}{P(Z_n = 0)} \\ &= \frac{P(Z_{n+1} = 0, Z_n = 0, X_n = 0)}{P(Z_n = 0)} + \frac{P(Z_{n+1} = 0, Z_n = 0, X_n = 1)}{P(Z_n = 0)} \\ &= \frac{P(Z_{n+1} = 0/Z_n = 0, X_n = 0)}{P(X_n = 0)} \times \frac{P(Z_n = 0/X_n = 0)P(X_n = 0)}{P(X_n = 0)} \\ &= q, \end{aligned}$$

and

$$\begin{aligned} \Pi(0,1) &= P(Z_{n+1} = 1/Z_n = 0) \\ &= 1 - P(Z_{n+1} = 0/Z_n = 0) \\ &= 1 - q. \end{aligned}$$

In the other hand, we have:

$$\begin{aligned}
\Pi(1,0) &= P(Z_{n+1}=0/Z_n=1)\\
&= \frac{P(Z_{n+1}=0, Z_n=1)}{P(Z_n=1)}\\
&= \frac{P(Z_{n+1}=0, Z_n=1, X_n=0)}{P(Z_n=1)} + \frac{P(Z_{n+1}=0, Z_n=1, X_n=1)}{P(Z_n=1)}\\
&= \frac{P(Z_{n+1}=0/Z_n=1, X_n=1)}{P(Z_n=1)} \times \frac{P(Z_n=1/X_n=1)P(X_n=1)}{P(Z_n=1)}\\
&= \frac{q^k(1-q)}{1-q^k}.
\end{aligned}$$

and finally:

$$\begin{aligned}
\Pi(1,1) &= P(Z_{n+1}=1/Z_n=1)\\
&= 1-P(Z_{n+1}=0/Z_n=1)\\
&= 1-\Pi(1,0)\\
&= 1-\frac{q^k(1-q)}{1-q^k}.
\end{aligned}$$

Now, we can establish a computing formula of the system reliability.

Theorem 2. *The reliability of the "consecutive k-out-of-n" system is given by*

$$R = (1-q^k)\left[1-\frac{q^k(1-q)}{1-q^k}\right]^{n-k}.$$

Proof. With $S_n = Z_1.Z_2\ldots Z_{n-k}.Z_{n-k+1}$ We have $R = P(S_n = 1)$. Then,

$$\begin{aligned}
P(S_n=1) &= P(Z_1.Z_2\ldots Z_{n-k}.Z_{n-k+1}=1)\\
&= P(Z_1=1, Z_2=1\ldots, Z_{n-k+1}=1)\\
&= P(Z_{n-k+1}=1/Z_1=1,\ldots,Z_{n-k}=1)\\
&\quad \times P(Z_{n-k}=1/Z_1=1,\ldots,Z_{n-k-1}=1)\\
&\quad \times\cdots\times P(Z_2=1/Z_1=1)\times P(Z_1=1).
\end{aligned}$$

Since, $(Z_i)_{i\geq 1}$ is a Markov chain, we conclude that:

$$\begin{aligned} P(S_n = 1) &= P(Z_{n-k+1} = 1/Z_{n-k} = 1) \\ &\quad \times P(Z_{n-k} = 1/Z_{n-k-1} = 1) \\ &\quad \times \cdots \times P(Z_2 = 1/Z_1 = 1).P(Z_1 = 1) \end{aligned}$$

and

$$\begin{aligned} R &= P(S_n = 1) \\ &= \Pi(1,1) \times \cdots \times \Pi(1,1).P(Z_1 = 1) \\ &= [\Pi(1,1)]^{n-k}[1 - P(Z_1 = 0)] \\ &= \left[1 - \frac{q^k(1-q)}{1-q^k}\right]^{n-k}\left(1 - q^k\right). \end{aligned}$$

Likelihood Estimator of *R*. We estimate R with two methods. We set $R_1 = R$ in the first method and $R_2 = R$ in the second. $\widehat{R_1}$ and $\widehat{R_2}$ denote the estimators of R_1 and R_2 respectively.

First method: Consider a system with n components $X_1, X_2, \ldots, X_n$ such that:

$$P(X_i = 1) = p \qquad \text{and} \qquad P(X_i = 0) = q.$$

Then,

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{i=n}(X_i = x_i) = p^{\sum_{i=1}^{n} x_i}(1-p)^{\sum_{i=1}^{n}(1-x_i)}.$$

where $x_i \in \{0, 1\}$.

The probabilities p and q are estimated by the maximum likelihood method:

$$\begin{aligned} L(x_1, \ldots, x_n, p) &= P(X_1 = x_1, \ldots, X_n = x_n) \\ &= p^{\sum_{i=1}^{n} x_i}(1-p)^{\sum_{i=1}^{n}(1-x_i)} \end{aligned}$$

$$\frac{\partial \ln L\ (x_1, \ldots, x_n, p)}{\partial p} = \frac{\partial \ln L\ (x_1, \ldots, x_n, p)}{\partial p}.$$

Then,

$$\frac{\partial \ln L(x_1,\ldots,x_n,p)}{\partial p} = \sum_{i=1}^{n} x_i \frac{1}{p} - \sum_{i=1}^{n} (1-x_i)\frac{1}{1-p} = 0.$$

Then,

$$\widehat{p} = \frac{\sum\limits_{i=1}^{n} x_i}{n} \qquad \text{and} \qquad \widehat{q} = \frac{\sum_{i=1}^{n}(1-x_i)}{n}.$$

Consequently:

$$\widehat{R_1} = \left(1-\widehat{q}^{\,k}\right)\left[1-\frac{\widehat{q}^{\,k}\left(1-\widehat{q}\right)}{1-\widehat{q}^{\,k}}\right]^{n-k}.$$

Second method: We consider

$$Z_i = \max_{1\le j\le i+k-1} X_j, \quad i=1,2,\ldots,n-k+1.$$

and we put:

$$A_{n-k+1} = Z_1 = z_1,\ldots,Z_{n-k+1} = z_{n-k+1}.$$

Then

$$\begin{aligned} P(A_{n-k+1}) &= P\left(Z_{n-k+1} = z_{n-k+1}/A_{n-k}\right) \\ &= P\left(Z_{n-k} = z_{n-k}/A_{n-k-1}\right)\times\cdots\times P\left(Z_2/Z_1\right)\times P\left(Z_1\right). \end{aligned}$$

We have already proved that $(Z_n)_{n\ge1}$ is a Markov chain, therefore, we deduce that:

$$P(A_{n-k+1}) = P(Z_{n-k+1} = z_{n-k+1}/Z_{n-k} = z_{n-k}).$$

$$P(Z_{n-k} = z_{n-k}/Z_{n-k-1} = z_{n-k-1})\times\cdots\times P(Z_2 = z_2/Z_1 = z_1).P(Z_1 = z_1).$$

We set:

$$\begin{aligned} \Pi(0,0) &= P\left(Z_{n+1} = 0/Z_n = 0\right) = \alpha, \\ \Pi(1,1) &= P\left(Z_{n+1} = 0/Z_n = 1\right) = \beta, \end{aligned}$$

so,

$$\Pi(0,1) = 1-\alpha \ \text{ and } \Pi(1,0) = 1-\beta.$$

The precedent expression can be written as:

$$P(Z_1 = z_1, Z_2 = z_2, \ldots, Z_{n-k+1} = z_{n-k+1}) \\ = P\left(Z_1 = z_1\right).\alpha^{n_{0,0}}\left(1-\alpha\right)^{n_{01}}.\beta^{n_{1,1}}\left(1-\beta\right)^{n_{1,0}},$$

where, for $i = 1, \ldots, n-k+1$:

$$\begin{aligned} n_{0,0} &= \text{Card}\left\{i : Z_i = 0 \text{ and } Z_{i+1} = 0\right\} \\ n_{0,1} &= \text{Card}\left\{i : Z_i = 0 \text{ and } Z_{i+1} = 1\right\} \\ n_{1,1} &= \text{Card}\left\{i : Z_i = 1 \text{ and } Z_{i+1} = 1\right\} \\ n_{1,0} &= \text{Card}\left\{i : Z_i = 1 \text{ and } Z_{i+1} = 0\right\}. \end{aligned}$$

The estimators likelihood of the parameters α and β are:

$$\widehat{\alpha} = \frac{n_{0,0}}{n_{0,0} + n_{0,1}} \quad \text{and} \quad \widehat{\beta} = \frac{n_{1,1}}{n_{1,0} + n_{1,1}}.$$

Since,

$$\begin{aligned} P(S_n = 1) &= P(Z_{n-k+1} = 1/Z_{n-k} = 1) \times P(Z_{n-k} = 1/Z_{n-k-1} = 1) \\ &\quad \times \cdots \times P(Z_2 = 1/Z_1 = 1).P(Z_1 = 1). \end{aligned}$$

Then the reliability of the system is:

$$\begin{aligned} R_2 &= P\left(S_n = 1\right) \\ &= \Pi\left(1,1\right) \times \cdots \times \Pi\left(1,1\right).P\left(Z_1 = 1\right) \\ &= \left[\Pi\left(1,1\right)\right]^{n-k}.\left[1 - P\left(Z_1 = 0\right)\right] \end{aligned}$$

It is not difficult to derive:

$$\Pi\left(1,0\right) = \frac{q^k\left(1-q\right)}{1-q^k} \text{ and } 1-\beta = \frac{q^k\left(1-q\right)}{1-q^k}.$$

After elementary calculus we obtain:

$$q^k = \frac{1-\beta}{2-\alpha-\beta}.$$

Then,

$$R_2 = \beta^{n-k}\left(\frac{1-\alpha}{2-\alpha-\beta}\right).$$

Finally, the estimator of R_2 is:

$$\widehat{R_2} = \widehat{\beta}^{n-k}\left(\frac{1-\widehat{\alpha}}{2-\widehat{\alpha}-\widehat{\beta}}\right)$$

4. Numerical applications We propose some numerical examples.

Example 1. Consider the system with 15 components with states given by: 111001000011000. After some computation, we obtain:

n	k	$\widehat{q}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{R_1}$	$\widehat{R_2}$
15	2	0.6	0.6	0.625	0.0232	0.0011
15	3	0.6	0.5	0.8181	0.1931	0.0659
15	4	0.6	0	0.9	0.4429	0.2852
15	5	0.6	0	1	0.6544	1
15	6	0.6	0	1	0.7979	1

Example 2. Consider the system with 20 components with states given by: 00011011100000100001. After some computations, we obtain:

n	k	$\widehat{q}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{R_1}$	$\widehat{R_2}$
20	2	0.65	0.6666	0.7777	0.0028	0.0065
20	3	0.65	0.5	0.8181	0.0647	0.0241
20	4	0.65	0.3333	0.8461	0.2317	0.0561
20	5	0.65	0	0.9333	0.4365	0.3330
20	6	0.65	0	1	0.6163	1

Example 3. Consider the system with 30 components with states disposed as bellow : 100000011001000111100011001111. After some computations,

we obtain the numerical values:

n	k	$\widehat{q}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{R_1}$	$\widehat{R_2}$
30	2	0.5333	0.5454	0.7058	0.0001	$3.5E-05$
30	3	0.5333	0.5	0.8571	0.0275	0.012
30	4	0.5333	0.6666	0.9565	0.1907	0.2784
30	5	0.5333	0.5	0.9583	0.4460	0.3185
30	6	0.5333	0	0.9565	0.6681	0.3297
30	7	0.5333	0	1	0.8157	1

5. Conclusion. In these numerical examples, we notice the difference between the estimators $\widehat{R_1}$ and $\widehat{R_2}$. We believe that the second method is better than the first to estimate the reliability (R) of the consecutive k-out-of-n system. Since, in the first method, to estimate R it's necessary to estimate q then to calculate q^k. In the second, we use directly $\widehat{\beta}$ (*not depending to* $\widehat{q}$ *and* $\widehat{q}^k$) to estimate the reliability of the system. Then this method permits to avoid the accumulated errors owed to the power.

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