Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **39** (2013), 1–16

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

COMPARISON AND OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

Ethiraju Thandapani, Renu Rama

Communicated by T. Gramchev

Abstract. In this paper we establish some comparison theorems for the oscillation of second order neutral differential equations of mixed type

$$(a(t)([x(t)+b(t)x(t-\sigma_1)+c(t)x(t+\sigma_2)]^{\alpha})')'+q(t)x^{\beta}(t-\tau_1)+p(t)x^{\gamma}(t+\tau_2)=0$$

where α, β and γ are ratios of odd positive integers, σ_1, σ_2, τ_1 and τ_2 are non negative integers. Our results are new even if $p(t)=0$ or $q(t)=0$. Examples are provided to illustrate the main results.

1. Introduction. In this paper, we shall study the oscillatory behavior of the second order nonlinear neutral differential equation of mixed type

$$\big(a(t)([x(t)+b(t)x(t-\sigma_1)+c(t)x(t+\sigma_2)]^{\alpha})'\big)' + q(t)x^{\beta}(t-\tau_1)+p(t)x^{\gamma}(t+\tau_2)=0, \tag{1.1}$$

where $t \geq t_0$. Throughout this paper, we assume without further mention that the following hypotheses hold:

2010 *Mathematics Subject Classification:* 34C15.
Key words: oscillation theorems, nonlinear, neutral differential equations.

(H_1) $a(t)$ is a positive and differentiable function for $t \geq t_0$ with $\int_{t_0}^{\infty} \frac{1}{a(t)}dt = \infty$;

(H_2) $b(t)$ *and* $c(t) \in C^2([t_0,\infty),[0,\infty))$, and there exist b and c such that $b(t) \leq b,$ *and* $c(t) \leq c$;

(H_3) $p(t)$ *and* $q(t)$ are nonnegative continuous real valued functions for all $t \geq t_0$;

(H_4) $\sigma_1,\ \sigma_2,\ \tau_1$ and τ_2 are nonnegative integers, and $\alpha,\ \beta$ and γ are the ratios of odd positive integers with $\alpha \in (0,\ \infty)$.

Set $z(t) = [x(t)+b(t)x(t-\sigma_1)+c(t)x(t+\sigma_2)]^{\alpha}$. By a solution of equation (1.1), we mean a function $x(t) \in C^1([T_x,\infty),\mathbb{R})$ defined for all $t \geq t_0 - \max(\sigma_1,\tau_1)$, which has the property $a(t)z'(t) \in C^1([T_x,\infty),\mathbb{R})$,and satisfying equation (1.1) for all $t \geq T_x \geq t_0$. A solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Neutral functional differential equations have numerous applications in electric networks. For instance, they are frequently used for the study of distribution networks containing lossless transmission lines which arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [13, 15].

In recent years, many results have been obtained on the oscillation of different classes of functional differential equations, we refer the reader to the papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 20, 22] and the references cited therein. However, there are few results regarding the oscillatory properties of neutral differential equations with mixed arguments. In [5, 14, 20], the authors established some oscillation criteria for the following mixed neutral equation

$$(1.2)\quad (x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2))'' = q_1(t)x(t-\sigma_1)+q_2(t)x(t+\sigma_2),\ t \geq t_0$$

with q_1 and q_2 are nonnegative real valued functions. Grace [11] obtained some oscillation theorems for the odd order neutral differential equation

$$(1.3)\quad (x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2))^{(n)} = q_1x(t-\sigma_1)+q_2x(t+\sigma_2),\ t \geq t_0$$

where $n \geq 1$ is odd. In [1, 12, 23] the authors obtained several sufficient conditions for the oscillation of solutions of higher-order neutral functional differential equation of the form

$$(1.4)\quad (x(t)+cx(t-h)+Cx(t+H))^{(n)} + qx(t-g)+Qx(t+G) = 0,\ t \geq t_0$$

where q and Q are nonnegative real constants.

Clearly, equations(1.2), (1.3) and (1.4) with $n = 2$, and $\alpha = \beta = \gamma = 1$ are special cases of equation (1.1). Motivated by this observation in this paper we study the oscillatory behavior of equation (1.1) for different values of α, β, and γ.In section 2, we establish sufficient condition for the oscillation of all solution of equation (1.1) and in Section 3 we provide some examples to illustrate the main result.

In the sequel, when we write a functional inequality without specifying its domain of validity and we assume that it holds for all sufficiently large t.

2. Oscillation results. In this section, we shall establish some new oscillation criteria for the equation (1.1). Throughout this paper, we shall use the following notations, without further mention:

$$Q_1(t) = \min\{q(t), q(t-\sigma_1), q(t+\sigma_2)\}, \quad P_1(t) = \min\{p(t), p(t-\sigma_1), p(t+\sigma_2)\},$$

$$Q^*(t) = Q_1(t) + P_1(t), \text{ and } \quad R(t) = \int_{t_0}^{t} \frac{1}{a(s)} ds.$$

To prove our main results we need the following lemmas.

Lemma 2.1. *Let $A \geq 0$, $B \geq 0$ and $\delta \geq 1$. Then*

$$(A+B)^\delta \leq 2^{\delta-1}(A^\delta + B^\delta).$$

Lemma 2.2. *Assume that $A \geq 0$, $B \geq 0$ and $0 < \delta \leq 1$. Then*

$$(A+B)^\delta \leq A^\delta + B^\delta.$$

The proofs of Lemmas 2.1 and 2.2 may be found in [14, 20].

Theorem 2.3. *Assume that $\gamma = \beta \geq 1$, and*

$$u'(t) + \frac{Q^*(t) R^{\beta/\alpha}(t-\tau_1)}{4^{\beta-1}\left(1 + b^\beta + \dfrac{c^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} u^{\beta/\alpha}(t+\sigma_1-\tau_1) \leq 0 \tag{2.1}$$

has no positive solution for all sufficiently large $t \geq t_0$. Then every solution of equation (1.1) *is oscillatory.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$ and $x(t-\sigma_1) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for all $t \geq t_1$.

In view of equation (1.1), we obtain

$$(a(t)z'(t))' = -q(t)x^{\beta}(t-\tau_1) - p(t)x^{\beta}(t+\tau_2) \le 0, \; t \ge t_1. \tag{2.2}$$

Thus $a(t)z'(t)$ is nonincreasing. Then it is easy to conclude that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \ge t_1$. If there exists a $t_2 \ge t_1$ such that $z'(t_2) < 0$, then from (2.2), we see that

$$a(t)z'(t) \le a(t_2)z'(t_2) < 0, \quad t \ge t_2.$$

Integrating the last inequality from t_2 to t, we obtain

$$z(t) \le z(t_2) + a(t_2)z'(t_2) \int_{t_2}^{t} \frac{1}{a(s)} ds.$$

Letting $t \to \infty$, we obtain $z(t) \to -\infty$ due to (H_1), which is a contradiction. Thus, there exists $t_2 \ge t_1$ such that

$$z'(t) > 0 \tag{2.3}$$

for all $t \ge t_2$. From the equation (1.1), for sufficiently large t, we have

$$\begin{aligned} &(a(t)z'(t))' + q(t)x^{\beta}(t-\tau_1) + p(t)x^{\beta}(t+\tau_2) + b^{\beta}(a(t-\sigma_1)z'(t-\sigma_1))' + \\ &b^{\beta}q(t-\sigma_1)x^{\beta}(t-\tau_1-\sigma_1) + b^{\beta}p(t-\sigma_1)x^{\beta}(t+\tau_2-\sigma_1) + \\ &\frac{c^{\beta}}{2^{\beta-1}}(a(t+\sigma_2)z'(t+\sigma_2))' + \frac{c^{\beta}}{2^{\beta-1}}q(t+\sigma_2)x^{\beta}(t-\tau_1+\sigma_2) + \\ &\frac{c^{\beta}}{2^{\beta-1}}p(t+\sigma_2)x^{\beta}(t+\tau_2+\sigma_2) = 0. \end{aligned} \tag{2.4}$$

By using Lemma 2.1, the equation (2.4) becomes

$$\begin{aligned} &(a(t)z'(t))' + b^{\beta}(a(t-\sigma_1)z'(t-\sigma_1))' + \frac{c^{\beta}}{2^{\beta-1}}(a(t+\sigma_2)z'(t+\sigma_2))' \\ &\qquad + \frac{Q_1(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\tau_1) + \frac{P_1(t)}{4^{\beta-1}}z^{\beta/\alpha}(t+\tau_2) \le 0. \end{aligned} \tag{2.5}$$

Since $z(t)$ is increasing, we have $z(t+\tau_2) \ge z(t-\tau_1)$, and therefore from (2.5), we obtain

$$\begin{aligned} &(a(t)z'(t))' + b^{\beta}(a(t-\sigma_1)z'(t-\sigma_1))' \\ &\qquad + \frac{c^{\beta}}{2^{\beta-1}}(a(t+\sigma_2)z'(t+\sigma_2))' + \frac{Q^*(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\tau_1) \le 0. \end{aligned} \tag{2.6}$$

It follows from (2.2) that

$$z(t) = z(t_2) + \int_{t_2}^{t} \frac{a(s)z'(s)}{a(s)} ds \ge a(t)z'(t)R(t). \tag{2.7}$$

Set $y(t) = a(t)z'(t)$. Then $y(t) > 0$ and nonincreasing. From (2.6) and (2.7), we obtain

$$\left(y(t) + b^\beta y(t-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} y(t+\sigma_2)\right)' + \frac{Q^*(t)}{4^{\beta-1}} R^{\beta/\alpha}(t-\tau_1) y^{\beta/\alpha}(t-\tau_1) \le 0. \tag{2.8}$$

Define $u(t)$ by

$$u(t) = y(t) + b^\beta y(t-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} y(t+\sigma_2).$$

Then $u(t) > 0$. Since $y(t)$ is nonincreasing, we have

$$u(t) \le \left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right) y(t-\sigma_1).$$

Substituting the above inequality in (2.8), we see that $u(t)$ is a positive solution of the inequality (2.1), which is a contradiction. This completes the proof. □

Corollary 2.4. *Assume that* $\alpha = \beta = \gamma$ *and* $\sigma_1 - \tau_1 < 0$ *hold. If*

$$\liminf_{t\to\infty} \int_{t+\sigma_1-\tau_1}^{t} Q^*(s)R(s-\tau_1)ds > \frac{4^{\beta-1}}{e}\left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right), \tag{2.9}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.3and [13, Theorem 2.3.1]. □

Corollary 2.5. *Assume that* $\gamma = \beta < \alpha$ *and* $\sigma_1 - \tau_1 < 0$ *hold. If*

$$\int_{t_0}^{\infty} Q^*(s)R^{\beta/\alpha}(s-\tau_1)ds = \infty, \tag{2.10}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.3 and [21]. □

Corollary 2.6. *Assume that* $\beta = \gamma > \alpha$ *and* $\tau_1 - \sigma_1 > 0$ *hold. If there exists* $\lambda > \dfrac{1}{\tau_1 - \sigma_1} \log(\beta/\alpha)$ *such that*

$$\liminf_{t\to\infty} \left[Q^*(t) R^{\beta/\alpha}(t-\tau_1) \exp(-e^{\lambda t}) \right] > 0, \tag{2.11}$$

then every solution of equation (1.1) *is oscillatory.*

P r o o f. The proof follows from Theorem 2.3 and [21]. □

Theorem 2.7. *Assume that* $\gamma = \beta \geq 1$ *and* $\sigma_1 - \tau_1 > 0$ *hold. If*

$$w'(t) - \frac{Q^*(t+\sigma_1)}{4^{\beta-1}\left(1 + b^\beta + \dfrac{c^\beta}{2^{\beta-1}}\right)} \left(\int_{t_1}^{t+\sigma_1} \frac{1}{a(s-\sigma_1)} ds \right) w^{\beta/\alpha}(t+\sigma_1-\tau_1) \geq 0 \tag{2.12}$$

has no positive solution for sufficiently large $t_1 \geq t_0$, *then every solution of equation* (1.1) *is oscillatory.*

P r o o f. Proceeding as in the proof of Theorem 2.3, we obtain (2.2)–(2.6) for all $t \geq t_2 \geq t_1$. Integrating (2.6) from t to ∞ yields

$$a(t)z'(t) + b^\beta a(t-\sigma_1)z'(t-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} a(t+\sigma_2)z'(t+\sigma_2) \geq \int_t^\infty \frac{Q^*(s)}{4^{\beta-1}} z^{\beta/\alpha}(s-\tau_1) ds. \tag{2.13}$$

Since $a(t)z'(t) > 0$ and nonincreasing, we have

$$a(t)z'(t) + b^\beta a(t-\sigma_1)z'(t-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} a(t+\sigma_2)z'(t+\sigma_2) \leq \left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right) a(t-\sigma_1)z'(t-\sigma_1). \tag{2.14}$$

In view of (2.13) and (2.14), we have

$$z'(t-\sigma_1) \geq \frac{1}{\left(1 + b^\beta + \dfrac{c^\beta}{2^{\beta-1}}\right) a(t-\sigma_1)} \int_t^\infty \frac{Q^*(s)}{4^{\beta-1}} z^{\beta/\alpha}(s-\tau_1) ds. \tag{2.15}$$

Integrating (2.15) from t_2 to t, we see that

$$z(t-\sigma_1) \geq \int_{t_2}^{t} \frac{1}{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right) a(s-\sigma_1)} \left(\int_{s}^{\infty} \frac{Q^*(v)}{4^{\beta-1}} z^{\beta/\alpha}(v-\tau_1)dv \right) ds$$
$$\geq \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)} \int_{t_2}^{t} Q^*(s) z^{\beta/\alpha}(s-\tau_1) \left(\int_{t_2}^{s} \frac{1}{a(v-\sigma_1)} dv \right) ds.$$

Thus

$$z(t) \geq \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)} \int_{t_2}^{t+\sigma_1} Q^*(s) z^{\beta/\alpha}(s-\tau_1) \left(\int_{t_2}^{s} \frac{1}{a(v-\sigma_1)} dv \right) ds.$$

Let

$$w(t) = \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)} \int_{t_2}^{t+\sigma_1} Q^*(s) z^{\beta/\alpha}(s-\tau_1) \left(\int_{t_2}^{s} \frac{1}{a(v-\sigma_1)} dv \right) ds > 0.$$

Then

$$w'(t) = \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)} Q^*(t+\sigma_1) z^{\beta/\alpha}(t+\sigma_1-\tau_1) \int_{t_2}^{t+\sigma_1} \frac{1}{a(v-\sigma_1)} dv.$$

Thus $z(t) > w(t)$, and

$$w'(t) \geq \frac{Q^*(t+\sigma_1)}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)} \left(\int_{t_2}^{t+\sigma_1} \frac{1}{a(v-\sigma_1)} dv \right) w^{\beta/\alpha}(t+\sigma_1-\tau_1).$$

Hence we find $w(t)$ is a positive solution of (2.12). This contradiction completes the proof. □

From Theorem 2.7 and Theorem 2.3.4 of [13], we obtain the following corollary.

Corollary 2.8. *Assume that* $\gamma = \beta = \alpha$ *and* $\sigma_1 - \tau_1 > 0$ *and*

$$\liminf_{t\to\infty} \int_{t}^{t+\sigma_1-\tau_1} Q^*(s+\sigma_1)\left(\int_{t_1}^{s+\sigma_1} \frac{1}{a(v-\sigma_1)}dv\right) ds$$

$$> \frac{4^{\beta-1}}{e}\left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right) \tag{2.16}$$

for all sufficiently large $t_1 \geq t_0$. *Then every solution of equation* (1.1) *is oscillatory.*

Next we present oscillation criteria for equation (1.1) when $0 < \beta < 1$.

Theorem 2.9. *Assume that* $0 < \gamma = \beta < 1$, *and*

$$u'(t) + Q^*(t)R^{\beta/\alpha}(t-\tau_1)u^{\beta/\alpha}(t+\sigma_1-\tau_1) \leq 0 \tag{2.17}$$

has no positive solution for all sufficiently large $t \geq t_0$. *Then every solution of equation* (1.1) *is oscillatory.*

P r o o f. The proof is similar to that of Theorem 2.3, except we use Lemma 2.2 instead of Lemma 2.1, and hence the details are omitted. □

Similar to Corollaries 2.4 to 2.6, we have the following Corollaries 2.10 to 2.12.

Corollary 2.10. *Assume that* $\alpha = \beta = \gamma$ *and* $\sigma_1 - \tau_1 < 0$ *hold. If*

$$\liminf_{t\to\infty} \int_{t+\sigma_1-\tau_1}^{t} Q^*(s)R(s-\tau_1)ds > \frac{1}{e}, \tag{2.18}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 2.11. *Assume that* $1 > \gamma = \beta > \alpha$, *and* $\sigma_1 - \tau_1 < 0$ *hold. If*

$$\int_{t_0}^{\infty} Q^*(s)R^{\beta/\alpha}(s-\tau_1)ds = \infty, \tag{2.19}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 2.12. *Assume that* $1 > \gamma = \beta > \alpha$ *and* $\sigma_1 - \tau_1 < 0$ *holds. If there exists a* $\lambda > \dfrac{1}{\tau_1 - \sigma_1}\log(\beta/\alpha)$ *such that*

$$\liminf_{t\to\infty}\left[Q^*(t)R^{\beta/\alpha}(t-\tau_1)\exp(-e^{\lambda t})\right] > 0, \tag{2.20}$$

then every solution of equation (1.1) *is oscillatory.*

Theorem 2.13. *Assume that* $0 < \gamma = \beta < 1$ *and* $\sigma_1 - \tau_1 > 0$ *hold. If*

$$w'(t) - \frac{Q^*(t+\sigma_1)}{(1+b^\beta+c^\beta)}\left(\int_{t_1}^{t+\sigma_1}\frac{1}{a(s-\sigma_1)}ds\right)w^{\beta/\alpha}(t+\sigma_1-\tau_1) \ge 0 \tag{2.21}$$

has no positive solution for sufficiently large $t \ge t_0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.7 and hence the details are omitted. □

Similar to Corollary 2.8, we obtain the following corollary.

Corollary 2.14. *Assume that* $0 < \beta = \gamma = \alpha < 1$ *and* $\sigma_1 - \tau_1 > 0$ *hold. If*

$$\liminf_{t\to\infty}\int_t^{t+\sigma_1-\tau_1} Q^*(s+\sigma_1)\left(\int_{t_1}^{s+\sigma_1}\frac{1}{a(v-\sigma_1)}dv\right)ds > \frac{(1+b^\beta+c^\beta)}{e},$$

for all sufficiently large $t_1 \ge t_0$, *then every solution of equation* (1.1) *is oscillatory.*

Next we discuss the oscillation of equation (1.1) when $a(t) \equiv 1$.

Theorem 2.15. *Assume that* $a(t) \equiv 1$, $0 < \beta < 1$, $\gamma > 1$ *with* $\frac{\gamma}{\alpha} > 1 > \frac{\beta}{\alpha}$, $b \le 1$, $c \le 1$ *and* $\tau_i > \sigma_i$ *for* $i = 1, 2$. *If the differential inequality*

$$y''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+c^\beta}\left(\frac{P_2(t)}{4^{\gamma-1}}\right)^{\eta_1} Q_2^{\eta_2}(t)y(t-\tau-\sigma_2) \le 0 \tag{2.22}$$

where $\eta_1 = \dfrac{\alpha-\beta}{\gamma-\beta}$, $\eta_2 = \dfrac{\gamma-\alpha}{\gamma-\beta}$, *and* $\tau = \max\{\tau_1, \tau_2\}$ *has no positive increasing solution, then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, and $x(t-\sigma_1) > 0$ for all $t \ge t_1$. From equation (1.1)

$$z''(t) = -q(t)x^\beta(t-\tau_1) - p(t)x^\gamma(t+\tau_2) \le 0.$$

for $t \ge t_0$. Then as in the proof of Theorem 2.3, we have $z'(t) > 0$ for $t \ge t_1$. Define

$$y(t) = z(t) + b^\beta z(t-\sigma_1) + c^\beta z(t+\sigma_2).$$

Since $z(t) > 0$ and $z'(t) > 0$, we have $y(t) > 0,\ y'(t) > 0$ and

$$\begin{aligned} y''(t) &= z''(t) + b^\beta z''(t-\sigma_1) + c^\beta z''(t+\sigma_2) \\ y''(t) &+ Q_1(t)\left[x^\beta(t-\tau_1) + b^\beta x^\beta(t-\sigma_1-\tau_1) + c^\beta x^\beta(t+\sigma_2-\tau_1)\right] \\ &+ P_1(t)\left[x^\gamma(t+\tau_2) + b^\beta x^\gamma(t-\sigma_1+\tau_2) + c^\beta x^\gamma(t+\sigma_2+\tau_2)\right] \le 0. \end{aligned}$$

Using Lemma 2.2, and $0 < \beta < 1 < \gamma$ and $c \le 1$, we get

$$\begin{aligned} y''(t) &+ Q_1(t)\left[x(t-\tau_1) + bx(t-\sigma_1-\tau_1) + cx(t+\sigma_2-\tau_1)\right]^\beta \\ &+ P_1(t)\left[x^\gamma(t+\tau_2) + b^\gamma x^\gamma(t-\sigma_1+\tau_2) + c^\gamma x^\gamma(t+\sigma_2+\tau_2)\right] \le 0. \end{aligned}$$

Now using Lemma 2.1 and $c \le 1,\ \gamma > 1$, we have

$$\begin{aligned} &y''(t) + Q_1(t) z^{\beta/\alpha}(t-\tau_1) \\ &\quad + P_1(t)\left[\frac{1}{2^{\gamma-1}}\left(x(t+\tau_2) + bx(t+\tau_2-\sigma_1)\right)^\gamma + \frac{c^\gamma}{2^{\gamma-1}} x^\gamma(t+\tau_2+\sigma_2)\right] \le 0. \end{aligned}$$

Again using Lemma 2.1, we obtain

$$\begin{aligned} &y''(t) + Q_1(t) z^{\beta/\alpha}(t-\tau_1) \\ &\qquad + \frac{P_1(t)}{4^{\gamma-1}}\left[x(t+\tau_2) + bx(t+\tau_2-\sigma_1) + cx(t+\tau_1+\sigma_2)\right]^\gamma \le 0. \end{aligned}$$

$$y''(t) + Q_1(t) z^{\beta/\alpha}(t-\tau_1) + \frac{P_1(t)}{4^{\gamma-1}} z^{\gamma/\alpha}(t+\tau_2) \le 0.$$

or

$$y''(t) + Q_1(t) z^{\beta/\alpha}(t-\tau) + \frac{P_1(t)}{4^{\gamma-1}} z^{\gamma/\alpha}(t-\tau) \le 0 \tag{2.23}$$

Define $u_1 = \eta_1^{-1}\frac{P_1(t)}{4^{\gamma-1}} z^{\gamma/\alpha}(t-\tau)$ and $u_2 = \eta_2^{-1} Q_1(t) z^{\beta/\alpha}(t-\tau)$. Using arithmetic-geometric mean inequality $u_1\eta_1 + u_2\eta_2 \ge u_1^{\eta_1} u_2^{\eta_2}$, we have

$$\begin{aligned} y''(t) &\ge \left(\frac{P_1(t)}{\eta_1 4^{\gamma-1}} z^{\gamma/\alpha}(t-\tau)\right)^{\eta_1}\left(\frac{Q_1(t) z^{\beta/\alpha}(t-\tau)}{\eta_2}\right)^{\eta_2} \\ &= \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{P_1(t)}{4^{\gamma-1}}\right)^{\eta_1} Q_1^{\eta_2}(t) z(t-\tau). \end{aligned} \tag{2.24}$$

Since $z'(t) > 0$, we see that

$$\begin{aligned} y(t-\tau) &= z(t-\tau) + b^\beta z(t-\tau-\sigma_1) + c^\beta z(t-\tau+\sigma_2) \\ &\leq (1+b^\beta+c^\beta) z(t-\tau+\sigma_2). \end{aligned} \tag{2.25}$$

Using the inequality (2.25) in (2.24), we obtain

$$y''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+c^\beta}\left(\frac{P_1(t)}{4^{\gamma-1}}\right)^{\eta_1} Q_1^{\eta_2}(t) y(t-\tau-\sigma_2) \leq 0.$$

Therefore $y(t)$ is a positive increasing solution of the inequality (2.22), which is a contradiction. This completes the proof. □

Theorem 2.16. *Assume that* $a(t) \equiv 1, \beta > 1,\ 0 < \gamma < 1$ *with* $\dfrac{\beta}{\alpha} > 1 > \dfrac{\gamma}{\alpha}$, $b \geq 1$, $c \geq 1$ *and* $\tau_i > \sigma_i$ *for* $i = 1, 2$. *If the differential inequality*

$$y''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+c^\beta}\left(\frac{Q_1(t)}{4^{\beta-1}}\right)^{\eta_1} P_1^{\eta_2}(t) y(t-\tau-\sigma_2) \leq 0 \tag{2.26}$$

where $\eta_1 = \dfrac{\alpha-\gamma}{\beta-\gamma}$, $\eta_2 = \dfrac{\beta-\alpha}{\beta-\gamma}$, *and* $\tau = \max\{\tau_1, \tau_2\}$ *has no positive increasing solution, then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, and $x(t-\sigma_1) > 0$ for all $t \geq t_1$. Then as in the proof of Theorem 2.3 we have $z'(t) > 0$ for $t \geq t_1$. Define

$$y(t) = z(t) + b^\beta z(t-\sigma_1) + c^\beta z(t+\sigma_2).$$

Since $z(t) > 0$ and $z'(t) > 0$, we have $y(t) > 0$, $y'(t) > 0$ and

$$\begin{aligned} y''(t) &= z''(t) + b^\beta z''(t-\sigma_1) + c^\beta z''(t+\sigma_2) \\ y''(t) &+ \frac{Q_1(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\tau_1) + P_1(t) z^{\gamma/\alpha}(t+\tau_2) \leq 0. \end{aligned}$$

or

$$y''(t) + \frac{Q_1(t)}{4^{\beta-1}} z^{\beta/\alpha}(t-\tau) + P_1(t) z^{\gamma/\alpha}(t-\tau) \leq 0. \tag{2.27}$$

Define $u_1 = \eta_1^{-1}\frac{Q_1(t)}{4^{\beta-1}}z^{\beta/\alpha}(t-\tau)$ and $u_2 = \eta_2^{-1}P_1(t)z^{\gamma/\alpha}(t-\tau)$. Using arithmetic-geometric mean inequality $u_1\eta_1 + u_2\eta_2 \geq u_1^{\eta_1}u_2^{\eta_2}$, we have

$$y''(t) + \left(\frac{Q_1(t)}{\eta_1 4^{\beta-1}}z^{\beta/\alpha}(t-\tau)\right)^{\eta_1}\left(\frac{P_1(t)z^{\gamma/\alpha}(t-\tau)}{\eta_2}\right)^{\eta_2} \leq 0$$

$$y''(t) + \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{Q_1(t)}{4^{\beta-1}}\right)^{\eta_1} P_1^{\eta_2}(t)z(t-\tau) \leq 0. \tag{2.28}$$

Since $z'(t) > 0$, we see that

$$\begin{aligned} y(t-\tau) &= z(t-\tau) + b^\beta z(t-\tau-\sigma_1) + c^\beta z(t-\tau+\sigma_2) \\ &\leq (1 + b^\beta + c^\beta)z(t-\tau+\sigma_2). \end{aligned} \tag{2.29}$$

Using the inequality (2.29) in (2.28), we have

$$y''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+c^\beta}\left(\frac{Q_1(t)}{4^{\beta-1}}\right)^{\eta_1} P_1^{\eta_2}(t)y(t-\tau-\sigma_2) \leq 0.$$

Therefore $y(t)$ is a positive increasing solution of the inequality (2.26), which is a contradiction. This completes the proof of the theorem. □

3. Examples. In this section we present some examples to illustrate the main results.

Example 3.1. Consider the differential equation

$$(x(t) + bx(t-\sigma_1) + cx(t+\sigma_2))'' + \frac{q}{t}x(t-\tau_1) + \frac{p}{t}x(t+\tau_2) = 0, \quad t \geq 1 \tag{3.1}$$

where b, c, q and p are positive constants and $\tau_1 - \sigma_1 > 0$.

$$a(t) = 1,\ b(t) = b,\ c(t) = c,\ q(t) = \frac{q}{t},\ p(t) = \frac{p}{t} \ \text{ and } \ \alpha = \beta = \gamma = 1.$$

Therefore,

$$\begin{aligned} Q_1(t) &= \min\left\{\frac{q}{t}, \frac{q}{t-\sigma_1}, \frac{q}{t+\sigma_2}\right\} = \frac{q}{t+\sigma_2} \\ P_1(t) &= \min\left\{\frac{p}{t}, \frac{p}{t-\sigma_1}, \frac{p}{t+\sigma_2}\right\} = \frac{p}{t+\sigma_2} \\ Q^*(t) &= Q(t) + P(t) = \frac{p+q}{t+\sigma_2} \\ R(t) &= \int_1^t dt = t - 1. \end{aligned}$$

and

(i) Let $\tau_1 > \sigma_1$. Now

$$\begin{aligned}\liminf_{t\to\infty} \int\limits_{t+\sigma_1-\tau_1}^{t} Q^*(s)R(s-\tau_1)ds &= \liminf_{t\to\infty} \int\limits_{t+\sigma_1-\tau_1}^{t} \frac{p+q}{s+\sigma_2}(s-\tau_1-1)ds \\ &= (\tau_1-\sigma_1)(p+q).\end{aligned}$$

Therefore, if

$$(p+q)(\tau_1-\sigma_1) > \frac{(1+b+c)}{e},$$

then equation (3.1) is oscillatory due to Corollary 2.4.

(ii) Let $\tau < \sigma_1$.

$$\liminf_{t\to\infty} \int\limits_{t}^{t+\sigma_1-\tau_1} Q^*(s+\sigma_1)\left(\int\limits_{t_1}^{s+\sigma_1} \frac{1}{a(v-\tau_1)}dv\right) ds =$$

$$\liminf_{t\to\infty} \int\limits_{t}^{t+\sigma_1-\tau_1} \frac{(p+q)(s+\sigma_1-\tau_1)}{s+\sigma_2+\sigma_1}ds = (p+q)(\sigma_1-\tau_1)$$

Therefore, if

$$(p+q)(\sigma_1-\tau_1) > \frac{(1+b+c)}{e},$$

then equation (3.1) is oscillatory due to Corollary 2.8.

Example 3.2.

$$\begin{aligned}(3.2)\quad \left(\frac{1}{t}\left((x(t)+bx(t-\sigma_1)+cx(t+\sigma_2))'\right)^3\right)' \\ +\frac{q}{t}x(t-\tau_1)+\frac{p}{t}x(t+\tau_2)=0,\ t\geq 1,\end{aligned}$$

where b, c, q and p are positive constants and $\tau_1-\sigma_1>0$. Here $a(t)=\frac{1}{t}$, $\alpha=3$, $\gamma=\beta=1$. Then

$$Q^*(t)=\frac{p+q}{t+\sigma_2} \quad \text{and} \quad R(t)=\int\limits_{1}^{t} sds=\frac{t^2-1}{2}.$$

Since $\int\limits_1^\infty Q^*(s)R^{\beta/\alpha}(s-\tau_1)ds = \int\limits_1^\infty \frac{p+q}{s+\sigma_2}\left(\frac{(s-\tau_1)^2-1}{2}\right)^{1/3} ds = \infty$, every solution of equation (3.2) is oscillatory due to Corollary 2.5.

Example 3.3. Consider the differential equation

$$\left(\frac{1}{t}(x(t)+bx(t-1)+cx(t+2))'\right)' + q\exp\left(e^{2(t+1)}\right)x^3(t-2)+px^3(t+3)=0 \tag{3.3}$$

where $t \geq 1,\ b, c, q$ and p are positive constants. Here $\alpha = 1, \beta = \gamma = 3, a(t) = \frac{1}{t}$, $q(t) = q\exp\left(e^{2(t+1)}\right)$, $p(t) = p$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\tau_1 = 2$, $\tau_2 = 3$. Choose $\lambda = 2$. Then $\lambda > \frac{1}{\tau_1-\sigma_1}\log(\beta/\alpha)$. Also

$$\liminf_{t\to\infty}\left[Q^*(t)R^{\beta/\alpha}(t-\tau_1)\exp\left(e^{-\lambda t}\right)\right] = \liminf_{t\to\infty}\left[(qe^{e^{2t}}+p)\frac{(t-2)^3(t-1)^3}{2^3}e^{-e^{2t}}\right] > 0.$$

Hence equation (3.3) is oscillatory due to Corollary 2.6.

We conclude this paper with the following remark.

Remark 3.4. It would be interesting to study the oscillatory behavior of equation(1.1) when $\int_{t_0}^\infty \frac{1}{a(t)}dt < \infty$;

Acknowledgements. The authors thank the reviewer for his/her valuable suggestions and useful comments that have led to the present improved version of the original manuscript.

REFERENCES

[1] R. P. Agarwal, S. R. Grace. Oscillation theorems for certain neutral functional differential equations. *Comput. Math. Appl.* **38** (1999), 1–11.

[2] B. Baculikova. Oscillation criteria for second order nonlinear differential equations. *Arch. Math.* **42** (2006), 141–149.

[3] J. G. Dong. Oscillation behaviour of second order nonlinear neutral differential equations with deviating arguments. *Comput. Math. Appl.* **59** (2010), 3710–3717.

[4] J. Dzurina. On the second order functional differential equations with advanced and retarded arguments. *Nonlinear Times Digest* **1** (1994), 179–187.

[5] J. Dzurina, S. Kulasar. Oscillation criteria for second order neutral functional differential equations. *Publ. Math. Debrecen.* **59** (2001), 25–33.

[6] J. Dzurina, J. Busa, E. A. Airyan. Oscillation criteria for second-order differential equations of neutral type with mixed arguments. *Differential Equations* **38** (2002), 137–140.

[7] J. Dzurina, I. P. Stavoulakis. Oscillation criteria for second-order delay differential equations. *Appl. Math. Comput.* **140** (2003), 445–453.

[8] J. Dzurian, D. Hudakova. Oscillation of second order neutral delay differential equations. *Math. Bohem.* **134** (2009), 31–38.

[9] L. H. Erbe, Q. Kong. Oscillation results for second order neutral differential equations. *Funkcial. Ekvac.* **35** (1992), 545–555.

[10] S. R. Grace. Oscillation criteria for n-th order neutral functional differential equations. *J. Math. Anal. Appl.* **184** (1994), 44–55.

[11] S. R. Grace. On the oscillations of mixed neutral equations. *J. Math. Anal. Appl.* **194** (1995), 377–388.

[12] S. R. Grace. Oscillations of mixed neurtal functional-differential equations. *Appl. Math. Comput.* **68** (1995), 1–13.

[13] I. Gyori, G. Ladas. Theory of Delay Differential Equations with Applications. Oxford, Clarendon Press, 1991.

[14] Z. Han, T. Li, S. Sun, W. Chen. On the oscillation of second-order neutral delay differential equations. *Adv. Differential Equations* **2010** (2010), Article ID 289340, 8 pages.

[15] G. S. Ladde, V. Lakshimikantham, B. G. Zhang. Oscillation Theory of Differential Equations with Deviating Arguments. New York, Marcel Dekker, 1987.

[16] L. Liu, Y. Bai. New oscillation criteria for second-order nonlinear neutral delay differential equations. *J. Comput. Appl. Math.* **231** (2009), 657–663.

[17] Ch. G. Philos. Oscillation theorems for linear differential equations of second order. *Arch. Math.* **53** (1989), 482–492.

[18] S. H. Saker. Oscillation of second order neutral delay differential equations of Emden-Fowler type. *Acta Math. Hungar* **100** (2003), 37–62.

[19] S. Sun, T. Li, Z. Han, Y. Sun. Oscillation of second-order neutral functional differential equations with mixed nonlinearities. *Abstr. Appl. Anal.* **2011** (2011), Article ID 927690, 15 pages.

[20] *S. H. Tang, C. Gao, E. Thandapani, T. Li.* Oscillation theorem for second order netural differential equations of mixed type. *Far East J. Math. Sci.* (2011) (to appear).

[21] X. Tang. Oscillation for the first order super linear delay difference equations. *J. London Math. Soc.* **65** (2002), 115–122.

[22] R. Xu, F. Meng. Oscillation criteria for second order quasi-linear neutral delay differential equations. *Appl. Math. Comput.* **192** (2007), 216–222.

[23] J. R. Yan. Oscillation of higher order neutral differential equations of mixed type. *Isreal J. Math.* **115** (2000), 125–136.

Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Chennai -600 005, India
e-mail: ethandapani@yahoo.co.in
renurama68@gmail.com

Received February 28, 2012
Revised November 6, 2012