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ON PERMUTABLE FUZZY SUBGROUPS

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Abstract. A fuzzy subgroup κ of a fuzzy group γ on a group G is said to be *permutable in* γ if $\lambda \odot \kappa = \kappa \odot \lambda$ for every fuzzy subgroup λ of γ. Here $\mu \odot \nu$ stands for the product of two fuzzy groups μ and ν on G, that is $(\mu \odot \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in G \text{ and } x = yz\}$. In this paper, largely extending some previous results, we characterize the permutability of fuzzy subgroups in terms of the level subgroups and the support subgroups. We obtain these results emphasizing the role of the characteristic functions of elements of the group. We also show the remarkable fact that the (abstract) subgroups of a group having a fuzzy group whose fuzzy subgroups are permutable, are permutable as well.

Introduction. Let G be a group with a multiplicative binary operation denoted by juxtaposition and identity e. A fuzzy subset $\gamma : G \to [0, 1]$ is said to be *a fuzzy group on* G (see, for example, [11, §1.2]) if it satisfies the following conditions:

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- (FSG1) $\gamma(xy) \geq \gamma(x) \wedge \gamma(y)$ for all $x, y \in G$, and
- (FSG2) $\gamma(x^{-1}) \geq \gamma(x)$ for every $x \in G$.

Here and everywhere we adopt the usual convention on the operator *wedge* $\wedge$ (and on the operator *vee* $\vee$). If W is a subset of $[0,1]$, we denote by $\wedge W$ the greatest lower bound of W and by $\vee W$ the least upper bound of W. If $W = \{a, b\}$, we simply write $a \wedge b$ and $a \vee b$ for short. We assume that the least upper bound of the empty set is 0, and its greatest lower bound is 1. However we remark that we deliberately replace the standard expression *a fuzzy subgroup of* G by *a fuzzy group on* G in order to avoid misunderstandings and to emphasize that a fuzzy group is in fact a function defined on a group G. For example, if γ, κ are fuzzy groups on G and $\gamma \subseteq \kappa$ happens, we will say that γ *is a fuzzy subgroup of* κ and denote this by $\gamma \preceq \kappa$.

Fuzzy group theory, as well as other fuzzy algebraic structures, was introduced very soon after the beginning of fuzzy set theory. The theory of fuzzy groups was developed by many people who obtained a variety of results and introduced many new concepts. One of the main goals of fuzzy group theory is the study of algebraic properties of an *arbitrary* fuzzy group defined on an abstract group G. Here we have the following situation. There are a lot of studies on the structure of the largest fuzzy group $\chi(G,1)$ on G ($\chi(G,1)$ is *the characteristic function of* G, that is $\mathrm{Im}(\chi(G,1)) = \{1\}$). In particular, many articles were dedicated to such an important inner property known as *to be a normal fuzzy subgroup of* $\chi(G,1)$. Meanwhile, there are significant differences between the case of an *arbitrary* fuzzy group defined on a group G and the case of the largest fuzzy group $\chi(G,1)$. If a fuzzy subgroup λ of a fuzzy group γ defined on a group G possess some given property with respect to γ, then, in general, it does not always possess the same property with respect to $\chi(G,1)$. Further, an arbitrary fuzzy group γ can be considered as a set of fuzzy points, and this set is a semigroup with identity with respect to the multiplication of the fuzzy sets. However, the largest fuzzy group $\chi(G,1)$ has many invertible elements (for example, all fuzzy points $\chi(g,1)$, $g \in G$, are invertible), and this makes possible to use essentially the action of the group G on $\chi(G,1)$. These benefits were very clearly demonstrated in the study of some important properties of normal fuzzy subgroups of $\chi(G,1)$. At the same time, an *arbitrary* fuzzy group defined on a group G may have very few invertible elements, and as a consequence, we have very little tangible results on arbitrary fuzzy groups defined on a group G. Once again, the example of normal subgroups supports this statement. Our goal is to begin a systematic study of the properties of an arbitrary fuzzy group defined on a group G.

One of the key inner properties closely related to the normality is *permutability* or the property *to be a permutable subgroup*. Therefore it seems very natural to study it. To define properly the *permutable fuzzy subgroups* we need to recall the definition of the product of two fuzzy groups.

Given μ and ν two fuzzy groups on G, we define the operation $\odot$ by

$$(\mu \odot \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in G \text{ and } x = yz\}.$$

Thus $(\mu \odot \nu)(x) = \vee\{\mu(y) \wedge \nu(y^{-1}x) \mid y \in G\} = \vee\{\mu(xz^{-1}) \wedge \nu(z) \mid z \in G\}$. As above, we introduce the symbol $\odot$ to avoid misunderstandings with the more standard symbol $\circ$ that is used throughout the book [11] to denote both, the regular product of mappings and the product of fuzzy subgroups.

Let γ and κ be two fuzzy groups on a group G. It is said that γ and κ *permute* if $\gamma \odot \kappa = \kappa \odot \gamma$. At this point, it is worth mentioning that in general the product of two fuzzy groups is not a fuzzy group in general. Actually, the product $\gamma \odot \kappa$ is a fuzzy group if and only if the fuzzy groups γ and κ permute (see [11, §4.3] for example). If $\kappa \preceq \gamma$, we say that κ *is permutable in* γ if $\kappa \odot \lambda = \lambda \odot \kappa$ for each $\lambda \preceq \gamma$. As we shall see, the first examples of permutable fuzzy subgroups are normal fuzzy subgroups. If $\kappa \preceq \gamma$ are fuzzy groups on G as above, it is said that κ *is a normal fuzzy subgroup of* γ if $\kappa(yxy^{-1}) \geq \kappa(x) \wedge \gamma(y)$ for any elements $x, y \in G$ (see [11, §1.4]).

In abstract group theory, the property *to be a permutable subgroup* was the main topic of many prolific papers for a long period of time. These papers were devoted to properties of permutable subgroups in infinite and finite groups. Many of such results can be found in the book [13]. For fuzzy permutable subgroups, the situation is completely different. As for many other properties, the study of this one was initiated for only the largest fuzzy group $\chi(G, 1)$ as some properties of permutable fuzzy subgroups of $\chi(G, 1)$ were obtained. For example, in the papers [4, 5, 6, 7] and in the book [11, §4.3]), one can find some initial results in the case of a finite group G. The investigation of permutable fuzzy subgroups of $\chi(G, 1)$, where G is an arbitrary (not necessary finite) group has been initiated in [2, 12]. As we already noted, the case of arbitrary fuzzy group defined on a group G is significantly different from the case of the largest fuzzy group $\chi(G, 1)$. In abstract group theory, the theory of permutable subgroups is deeply and broadly developed. In this area, one can find quite interesting papers and many important results obtained by prominent algebraists. But in fuzzy groups, the picture is quite different. In the submitted paper, we just initiate the study of permutable fuzzy subgroups.

Let λ be a fuzzy subset of a set X, $a \in [0, 1]$. We define

$$L_a(\lambda) = \{x \in X \mid \lambda(x) \geq a\}.$$

The subset $L_a(\lambda)$ is said to be the *a–level set* (or *a–cut* of λ. We recall that if λ is a fuzzy group on G, then either $L_a(\lambda)$ is a subgroup of G or $L_a(\lambda) = \emptyset$. Actually, $L_a(\lambda)$ is a subgroup of G for every $a \leq \lambda(e)$. Moreover, it is known that the level sets allow to characterize fuzzy subgroups in the following way: *λ is a fuzzy group on G if and only if $L_a(\lambda)$ is a subgroup of G for each $a \leq \lambda(e)$* (see [11, Theorem 1.2.6] for example). Thus λ *is a fuzzy group on G if and only if every non-empty level of λ is a subgroup of G.*

The main result of our paper characterizes permutable fuzzy subgroups through level subgroups.

Theorem A. *Let G be a group and γ be a fuzzy group on G. A fuzzy subgroup κ of γ is permutable in γ if and only if $L_a(\kappa)$ is a permutable subgroup of $L_a(\gamma)$ for every $a \leq \kappa(e)$.*

Note that this theorem generalizes the main results of the paper [2]. In this paper, the partial case when κ is permutable in $\chi(G, 1)$ and satisfies some additional condition (the so-called *sup property*) has been considered.

We deduce some consequences from Theorem A.

Corollary A1. *Let G be a group and γ be a fuzzy group on G. If a fuzzy subgroup κ of γ is permutable in γ, then* $\mathbf{Supp}(\kappa)$ *is permutable in* $\mathbf{Supp}(\gamma)$.

Here and everywhere, as usual, the *support* of a fuzzy group λ on G is the subgroup of G given by

$$\mathbf{Supp}(\lambda) = \{x \in G \mid \lambda(x) > 0\}.$$

Corollary A2. *Let G be a group and γ be a fuzzy group on G. If a fuzzy subgroup κ of γ is normal in γ, then κ is a permutable subgroup of γ.*

Corollary A3. *Let G be a group and γ be a fuzzy group on G. If λ and κ are permutable fuzzy subgroups of γ, then the fuzzy subgroup $\langle \lambda, \kappa \rangle$ is permutable in γ.*

Here $\langle \lambda, \kappa \rangle$ is the fuzzy group on G *generated by* λ *and* κ. In general, let $\mathfrak{S}$ be a family of fuzzy subsets of the group G. Let

$$\mathfrak{M} = \{\gamma \mid \gamma \text{ is a fuzzy subgroup of } G \text{ and } \lambda \subseteq \gamma \text{ for every } \lambda \in \mathfrak{S}\}.$$

Then $\bigcap \mathfrak{M}$ is said to be *the fuzzy subgroup of* G *generated by* $\mathfrak{S}$. We denote this fuzzy subgroup by $\langle \mathfrak{S} \rangle$ ([11, §1.2]).

One of the main consequences of Theorem A is Theorem B which establishes some connections between permutable fuzzy subgroups of a fuzzy group on a group G and permutable subgroups of G. The following result follows from this.

Corollary B1. *Let G be a group and γ be a fuzzy group on G. Then every fuzzy subgroup of γ is permutable in γ if and only if every subgroup of* $\mathbf{Supp}(\gamma)$ *is permutable in* $\mathbf{Supp}(\gamma)$.

This is not the only result of such nature in fuzzy group theory. For example, every fuzzy subgroup of $\chi(G, 1)$ is normal if and only if every subgroup of G is normal in G (see [11, Theorem 4.1.3]). Such kind of results generates an illusion of easiness of transferring of any result of abstract group theory to fuzzy groups defined on G. This illusion was supported by the *metatheorem of T. Head* [9]. T. Head defined connections among the sets $\mathcal{P}(X)$ (the power set of X), $\mathcal{F}(X)$ (the set of all fuzzy subsets of X), and the crisp power set $\mathcal{C}(X)$ of X (the set of characteristic functions of all subsets of X). Extending the result of T. Head, A. Weinberger [15] shows that for any set X, there is a set Y such that the lattice of fuzzy subsets of X is isomorphic to a sublattice of the classical subsets of Y. Moreover, if X is infinite, then it is possible to choose $Y = X$. Employing this result, A. Weinberger [14, 15] proved that the lattice of fuzzy normal subgroups of $\chi(G, 1)$ is modular (see also [10]), a result formerly proved in [1, 3]. However, the situation is much more sophisticated. In the first, the isomorphism of lattices of two algebraic structures does not always imply the similarity of the properties of these structures. It is appropriate here to recall the classical Whitman Theorem stating that every lattice is isomorphic to a sublattice of a subgroup lattice of some group. Secondly, the abstract groups and the fuzzy groups are different algebraic structures. A fuzzy group on a group G as a set of fuzzy points is a semigroup, and as we mentioned above, the set of its invertible elements can be quite small. Therefore, there is no complete identification between the concepts that arises in abstract group theory and its fuzzy counterparts. For example, there is a notion of nilpotency in group theory, ring theory, and Lie ring theory. Nevertheless, each of these structures possesses their own characteristics, and not all of the properties pertaining to one of these structures can be transferred to others. We will show this below by constructing some example. In the fourth, as we mentioned above, in the articles [14, 15, 10], the authors by employing the *metatheorem* obtained another proof yielding that the lattice of normal fuzzy subgroups of $\chi(G,1)$ is modular. But both these proofs (taking into account all the preliminaires and pre-constructions) are not easier than the proof of N. Ajmal and K.V. Thomas. And the last remark. We did not want to go beyond classical fuzzy group theory. Therefore, we examined the situation when we consider mappings of a group G in $[0,1]$. However, at this stage, we did not employ specific properties of real numbers of $[0,1]$. The essential fact is that the image of a group is a complete lattice.

1. Some remarks about properties of abstract groups and fuzzy groups. As we mentioned above, some results of abstract group theory have corresponding analogs in fuzzy group theory. We begin with the constructing of the following example showing that this case not always legitimate. Recall some needed notions from group theory.

Let p be a prime and for, each natural number $n \geq 1$, let $G_n = \langle a_n \rangle$ a cyclic group of order p^n If $n \geq 1$, let $\theta_n : G_n \to G_{n+1}$ be the monomorphism given by $a_n\theta_n = a_{n+1}^p$. We think of G_n as a subgroup of G_{n+1}, and hence we construct the group $C_{p^\infty} = \bigcup_{n\geq 1} G_n$ as the union of the ascending chain

$$G_1 \leq G_2 \leq \cdots \leq G_n \leq G_{n+1} \leq \cdots$$

of cyclic p–groups of orders p, p^2, ... The above group C_{p^∞} is said to be *a quasicyclic p–group* or *a Prüfer group of type* p^∞. It is not hard to see, that every proper subgroup of $C_{p\infty}$ is finite. In particular, it satisfies the minimal condition for all subgroups.

A group G is called *a Chernikov group* if G includes a normal subgroup D of finite index which is a direct product of finitely many Prüfer p–subgroups. Such groups were named in honour of S.N. Chernikov who made an extensive study of groups with the minimum condition. In particular, S.N. Chernikov proved [8] that a locally soluble group satisfying the minimal condition for all subgroups is a Chernikov group.

Consider now a fuzzy group defined on a quasicyclic p–group G.

Proposition 1.1. *Let G be a group and let*

$$\langle e\rangle = L_0 \leq L_1 \leq \cdots \leq L_n \leq L_{n+1} \leq \cdots \bigcup_{n\geq 1} L_n = G$$

be a countable ascending chain of subgroups. Let $\{r_1, \cdots, r_n \cdots\}$ be a subset of $[0,1]$ such that $r_n \geq r_{n+1}$ for every $n \geq 1$ (the case $r_n = r_{n+1}$ could happen). Define the function $\gamma : G \to [0,1]$ by $\gamma(g) = r_{k(g)}$, where $k(g)$ is the least number such that $g \in L_{k(g)}$. Then γ is a fuzzy group on G and $L_n \leq L_{r_n}(\gamma)$ for every $n \geq 1$.

Proof. Pick $x, y \in G$. We have $\gamma(x) = a_{k(x)}$ and $\gamma(y) = a_{k(y)}$ and, without loss of generality, we can assume that $k(x) \leq k(y)$ so that $x, y \in L_{k(y)}$.

Suppose first that $k(x) < k(y)$. Then $y \notin L_{k(y)-1}$, and hence $y \notin L_{k(x)}$. It follows that $xy \in L_{k(y)}$ but $xy \notin L_{k(x)}$. Then $\gamma(xy) = a_{k(y)}$ and so $\gamma(x) \geq \gamma(y)$ and $\gamma(x) \wedge \gamma(y) = \gamma(y)$. Hence $\gamma(xy) = a_{k(y)} = a_{k(y)} \wedge a_{k(x)} = \gamma(x) \wedge \gamma(y)$. Suppose now that $k(x) = k(y)$. In this case either $xy \in L_{k(x)}$ and $xy \notin L_{k(x)-1}$ or $xy \in L_m$ for some $m < k(x)$. In first the case $\gamma(xy) = a_{k(x)} = a_{k(x)} \wedge a_{k(x)} = \gamma(x) \wedge \gamma(y)$. In the second case, $\gamma(xy) = a_m \geq a_{k(x)} = a_{k(x)} \wedge a_{k(x)} = \gamma(x) \wedge \gamma(y)$. Therefore we see that γ satisfies (FSG1).

Clearly $k(x^{-1}) = k(x)$ and then $\gamma(x) = \gamma(x^{-1})$. This holds for every element $x \in G$ so that γ satisfies (FSG2).

The inclusions $L_n \leq L_{a_n}(\gamma)$ are a fair consequence of the choice of γ. □

Let now G be a quasicyclic p–group, where p is a prime. Then G has an ascending chain of cyclic subgroups

$$\langle e\rangle \leq \langle a_1\rangle \leq \cdots \leq \langle a_n\rangle \leq \langle a_{n+1}\rangle \leq \cdots \bigcup_{n\geq 1} \langle a_n\rangle = G,$$

where $a_1^p = e$ and $a_{n+1}^p = a_n$ for every $n \geq 1$. By Proposition 1.1, we may construct fuzzy groups $\gamma_n : G \to [0,1]$ as follows.

Fix $0 < d < 1$. Define the function γ_1 by $\gamma_1(x) = d_1 = d$ for every $x \in G$. Let $d_2 = \dfrac{d_1}{2}$. Define the function γ_2 by

$$\gamma_2(x) = \begin{cases} d_2, & \text{if } x \in \langle a_1 \rangle \\ \dfrac{d_2}{2}, & \text{otherwise} \end{cases}$$

Let $d_3 = \dfrac{d_2}{4}$. Define the function γ_3 by

$$\gamma_3(x) = \begin{cases} d_3, & \text{if } x \in \langle a_1 \rangle \\ \dfrac{d_3}{2}, & \text{if } x \in \langle a_2 \rangle \setminus \langle a_1 \rangle \\ \dfrac{d_3}{4}, & \text{otherwise} \end{cases}$$

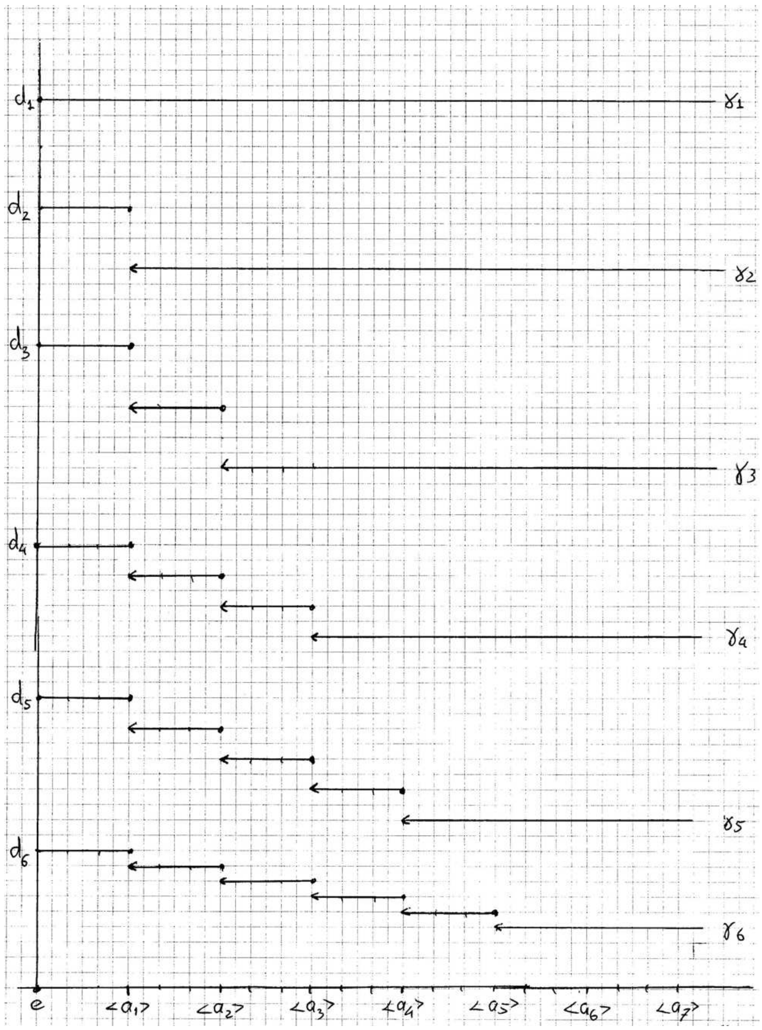
Suppose that we have already defined the functions γ_n for all $n < k$. Let $d_k = \dfrac{d_{k-1}}{2^{k-1}}$ and define the function γ_k by

$$\gamma(x) = \begin{cases} d_k, & \text{if } x \in \langle a_1 \rangle \\ \dfrac{d_k}{2}, & \text{if } x \in \langle a_2 \rangle \setminus \langle a_1 \rangle \\ \dfrac{d_k}{4}, & \text{if } x\langle a_3 \rangle \setminus \langle a_2 \rangle \\ \cdots & \cdots\cdots\cdots \\ \dfrac{d_k}{2^{k-1}}, & \text{if } x\langle a_k \rangle \setminus \langle a_{k-1} \rangle \\ \dfrac{d_k}{2^k}, & \text{otherwise} \end{cases}$$

Thus we obtain an infinite family $\{\gamma_n \mid n \geq 1\}$ of fuzzy groups on G. By the construction, $\gamma_{n+1} \preceq \gamma_n$, γ_n and γ_{n+1} are not equivalent for all $n \geq 1$. Therefore we obtain an infinite descending chain

$$\gamma_1 \preceq \gamma_2 \preceq \cdots \preceq \gamma_n \preceq \cdots$$

of pairwise non-equivalent fuzzy groups defined on the group G. But the group G has no infinite descending chain of subgroups.



2. Some properties of fuzzy subgroups. We start by developing some properties of the product of fuzzy groups. If X is a set, for every subset Y of X and every $a \in [0,1]$, we recall that the fuzzy subset $\chi(Y,a)$ (*the characteristic function of* Y) is defined by:

$$\chi(Y,a)(x) = \begin{cases} a & \text{if} \quad x \in Y \\ 0 & \text{if} \quad x \notin Y \end{cases} \quad \text{where } x \in X.$$

Clearly $\chi(H,a)$ is a fuzzy group on G for every subgroup H of G. If $Y = \{y\}$, then we will write shorter $\chi(y, a)$. A fuzzy subset $\chi(y, a)$ is called *a fuzzy point* (or *a fuzzy singleton*).

Let L be a subgroup of G and γ be a fuzzy subgroup on G. We define the function $L|^{\gamma} : G \to [0,1]$ by the following rule:

$$L|^{\gamma}(x) = \begin{cases} \gamma(x) & \text{if } \ x \in L \\ 0 & \text{if } \ x \notin L \end{cases} \quad \text{where } x \in G.$$

If $x, y \in G$, it is easy to check that $L|^{\gamma}(xy) \geq L|^{\gamma}(x) \wedge L|^{\gamma}(y)$. It follows that $L|^{\gamma}$ is a fuzzy group on G.

Proposition 2.1. *Let G be a group. Then we have*

(1) *The operation $\odot$ of fuzzy groups on G is associative;*

(2) *The function $\chi(e, 1)$ is the identity element of the operation $\odot$. Moreover, if γ is a fuzzy group on G and $\lambda \preceq \gamma$, then $\lambda \odot \chi(e, \gamma(e)) = \chi(e, \gamma(e)) \odot \lambda = \lambda$; and*

(3) *If λ is a fuzzy group on G, $x, y \in G$ and $a, b \in [0,1]$, then*
   (a) *$(\chi(y, a) \odot \lambda)(x) = a \wedge \lambda(y^{-1}x)$;*
   (b) *$(\lambda \odot \chi(y, a))(x) = a \wedge \chi(xy^{-1})$; and*
   (c) *$(\chi(y, a) \odot \chi(u, b))(yu) = a \wedge b$ and $(\chi(y, a) \odot \chi(u, b))(x) = 0$ if $u \in G$ and $x \neq yu$. That is, $\chi(y, a) \odot \chi(u, b) = \chi(yu, a \wedge b)$. Thus, $\chi(y, a) \odot \chi(u, a) = \chi(yu, a)$.*

Proof. (1) and (2) It suffices to apply the results of [11, §1.2].

(3) If $y \neq z \in G$, then $\chi(y, a)(z) = 0$. Therefore we have $(\chi(y, a) \odot \lambda)(x) =$

$$= \vee\{(\chi(y, a)(z) \wedge \lambda(z^{-1}x)) \mid z \in G\} = \chi(y, a)(y) \wedge \lambda(y^{-1}x) = a \wedge \lambda(y^{-1}x).$$

In particular, if $u \in G$, $(\chi(y, a) \odot \chi(u, b))(x) = a \wedge \chi(u, b)(y^{-1}x)$. Since $\chi(u, b)(y^{-1}x)) = b$ provided $y^{-1}x = u$ and $\chi(u, b)(y^{-1}x)) = 0$ otherwise,

$$(\chi(y, a) \odot \chi(u, b))(x) = \begin{cases} a \wedge b & \text{if } \ x = yu \\ 0 & \text{if } \ x \neq yu \end{cases},$$

and hence we obtain (a) and (c). The proof of (b) is similar. □

Corollary 2.2. *Let γ be a fuzzy group on a group G. If λ and κ are permutable fuzzy subgroups of γ, then so is $\lambda \odot \kappa$.*

Proof. As we mentioned above the product $\lambda \odot \kappa$ is a fuzzy subgroup of γ since its factors are permutable. Let μ be an arbitrary fuzzy subgroup of γ. Applying Proposition 2.1, we have $(\lambda \odot \kappa) \odot \mu = \lambda \odot (\kappa \odot \mu) = \lambda \odot (\mu \odot \kappa) = (\lambda \odot \mu) \odot \kappa = (\mu \odot \lambda) \odot \kappa = \mu \odot (\lambda \odot \kappa)$. □

Unlike from abstract groups, $\lambda \odot \kappa$ cannot include λ and κ. In fact, $(\lambda \odot \kappa)(e) = \vee\{\lambda(y) \wedge \kappa(z) \mid y, z \in G, e = yz\} = \vee\{\lambda(y) \wedge \kappa(y^{-1}) \mid y \in G\} = \vee\{\lambda(y) \wedge \kappa(y) \mid y \in G\}$. Since λ and κ are fuzzy groups, $\lambda(e) \geq \lambda(y)$ and $\kappa(e) \geq \kappa(y)$ for every element $y \in G$. It follows that $\lambda(y) \wedge \kappa(y) \leq \lambda(e) \wedge \kappa(e)$ for every $y \in G$, and hence $(\lambda \odot \kappa)(e) = \lambda(e) \wedge \kappa(e)$.

We also remark that if $\lambda \odot \kappa = \kappa \odot \lambda$, then $\langle \lambda, \kappa \rangle = \lambda \cup \kappa \cup (\lambda \odot \kappa)$.

We now are developing a new criterion of being a fuzzy subgroup.

Lemma 2.3. *Let γ be a fuzzy group on a group G. If $\lambda, \kappa \subseteq \gamma$ are fuzzy subsets of G, then $\lambda \odot \kappa \subseteq \gamma$. In particular, $\gamma \odot \gamma \preceq \gamma$.*

Proof. By definition, $(\lambda \odot \kappa)(x) = \vee\{\lambda(y) \wedge \kappa(z) \mid y, z \in G$ and $x = yz\}$. Since $\lambda, \kappa \preceq \gamma$ and the latter is a fuzzy group, $\lambda(y) \wedge \kappa(z) \leq \gamma(y) \wedge \gamma(z) \leq \gamma(yz)$. Thus $(\lambda \odot \kappa)(x) \leq \vee\{\gamma(yz) \mid y, z \in G$ and $x = yz\} = \gamma(x)$. □

Proposition 2.4. *A fuzzy subset γ of a group G is a fuzzy subgroup if and only if the following assertion holds:*

- *$(FSG3)$ $\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma$ for every $x, y \in$ **Supp**(γ), and*
- *$(FSG4)$ $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$ for every $x \in G$.*

Proof. Let γ be a fuzzy group on G. Since γ includes $\chi(x, \gamma(x))$ and $\chi(y, \gamma(y))$ if $x, y \in$ **Supp**(γ), by Lemma 2.3, $\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma$. Moreover, if $x \in$ **Supp**(γ), $\chi(x^{-1}, \gamma(x))(x^{-1}) = \gamma(x) \leq \gamma(x^{-1})$ since γ is a fuzzy group. Since $\chi(x^{-1}, \gamma(x))(y) = 0$ provided $y \neq x^{-1}$, $\chi(x^{-1}, \gamma(x))(y) \leq \gamma(y)$ for every $y \in G$. This means that $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$.

Conversely, let γ satisfy (FSG3) and (FSG4). Pick $x, y \in G$. To show (FGS 1), There is no loss of generality if we assume $x, y \in$ **Supp**(γ). For, if $\gamma(x) = 0$ for example, we clearly have $\gamma(xy) \geq 0 = \gamma(x) \wedge \gamma(y)$. Let $x, y \in$ **Supp**(γ). By (FSG3), $\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma$. By Proposition 2.1,

$$\gamma(x) \wedge \gamma(y) = (\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)))(xy) \leq \gamma(xy),$$

and (FSG1) follows. By (FGS 4), $(\chi(x^{-1}, \gamma(x)))(y) \leq \gamma(y)$ for every $y \in G$. In particular, $\gamma(x) = (\chi(x^{-1}, \gamma(x))(x^{-1}) \leq \gamma(x^{-1})$, and (FSG 2) follows. □

We shall need other characterizations of the normality. First, we show

Lemma 2.5. *Let γ be a fuzzy subgroup on a group G. Then $(\chi(x, a) \odot \gamma \odot \chi(x^{-1}, a))(w) = a \wedge \gamma(x^{-1}wx)$ for every $x, w \in G$ and $a \in [0, 1]$.*

Proof. By Proposition 2.1, $(\chi(x, a) \odot (\gamma \odot \chi(x^{-1}, a))(w) =$

$$= \vee\{\chi(x, a)(u) \wedge (\gamma(v) \wedge \chi(x^{-1}, a))(z) \mid u, v, z \in G, uvz = w\} =$$

$= \chi(x,a)(x) \wedge \gamma(x^{-1}wx) \wedge \chi(x^{-1},a)(x^{-1}) = a \wedge \gamma(x^{-1}wx) \wedge a = a \wedge \lambda(x^{-1}wx)$,

as required. □

Proposition 2.6. *Let γ, κ be fuzzy groups on a group G such that $\kappa \preceq \gamma$. Then κ is a normal fuzzy subgroup of γ if and only if*

$$\chi(x,\gamma(x)) \odot \kappa \odot \chi(x^{-1},\gamma(x)) \preceq \kappa$$

for every element $x \in G$.

Proof. Suppose first that κ is normal in γ. Given $x, y \in G$, by Lemma 2.5

$$(\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y))(x) = \gamma(y) \wedge \kappa(y^{-1}xy).$$

Let $u = y^{-1}xy$. Then $x = y(y^{-1}xy)y^{-1} = yuy^{-1}$ and so

$$(\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y))(yuy^{-1}) = \gamma(y) \wedge \kappa(u).$$

Since $\kappa(u) \wedge \gamma(y) \leq \kappa(yuy^{-1})$, we obtain

$$(\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y))(yuy^{-1}) \leq \kappa(yuy^{-1}),$$

that is

$$(\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y))(x) \leq \kappa(x).$$

Since this holds for every element $x \in G$, $\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y)) \preceq \kappa$.

Conversely, suppose that $\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\lambda(y)) \preceq \kappa$ for every $y \in G$. Pick $x \in G$. We define $z = yxy^{-1}$ so that $x = y^{-1}zy$. We have

$$(\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y)))(z) \leq \kappa(z).$$

By Lemma 2.5,

$$(\chi(y,\gamma(y)) \odot \kappa \odot \chi(y^{-1},\gamma(y))(z) = \gamma(y) \wedge \kappa(y^{-1}zy).$$

Then $\gamma(y) \wedge \kappa(y^{-1}zy) \leq \kappa(z)$, that is $\gamma(y) \wedge \kappa(x) \leq \kappa(yxy^{-1})$, as required. □

We quote the well-known characterization of the normality connecting it with level subgroups (see [11, Theorem 1.4.3])

Proposition 2.7. *A fuzzy subgroup κ of a fuzzy group γ on a group G is normal in γ if and only if $L_a(\kappa)$ is normal in $L_a(\gamma)$ for every $a \leq \kappa(e)$.*

We now analyze the structure of the product $\gamma \odot \kappa$.

Proposition 2.8. *Let γ, κ be fuzzy subsets of a group G. Then*

$$\gamma \odot \kappa = \bigcup\{\chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)\}.$$

P r o o f. By definition, we have

$$(\gamma \odot \kappa)(x) = \vee\{\gamma(y) \wedge \kappa(z) \mid y, z \in G, x = yz\}.$$

If $y \notin \mathbf{Supp}(\gamma)$ or $z \notin \mathbf{Supp}(\kappa)$, it readily follows that $\gamma(y) \wedge \kappa(z) = 0$. Thus

$$(\gamma \odot \kappa)(x) = \vee\{\gamma(y) \wedge \kappa(z) \mid y \in \mathbf{Supp}(\lambda), z \in \mathbf{Supp}(\kappa), x = yz\}.$$

Let $\xi := \bigcup\{\chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)\}$. We have that $\chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) = \chi(yz, (\gamma(y) \wedge \kappa(z))$ by Proposition 2.1. We note that, if $x = yz$, then $\chi(yz, (\gamma(y) \wedge \kappa(z))(x) = \gamma(y) \wedge \kappa(z)$. Otherwise, $\chi(yz, (\gamma(y) \wedge \kappa(z))(x) = 0$. Therefore, if $x \in G$, we have

$$\xi(x) = \vee\{(\chi(yz, (\gamma(y) \wedge \kappa(z)))(x) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa), x = yz\} =$$

$$= \vee\{\gamma(y) \wedge \kappa(z) \mid y \in \mathbf{Supp}(\lambda), z \in \mathbf{Supp}(\kappa), x = yz\} = (\gamma \odot \kappa)(x).$$

Hence $\gamma \odot \kappa = \cup\{\chi(y, \gamma(y)) \odot \chi(z, \kappa(z) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\kappa)\}$. □

Corollary 2.9. *Let G be a group.*

(1) *If γ, λ, κ are fuzzy subsets of G and $\lambda \subseteq \kappa$, then $\gamma \odot \lambda \subseteq \gamma \odot \kappa$ and $\lambda \odot \gamma \subseteq \kappa \odot \gamma$; and*

(2) *If γ and $\{\lambda_a \mid a \in A\}$ are fuzzy subsets of G, then $\gamma \odot (\bigcup_{a \in A} \lambda_a) = \bigcup_{a \in A} (\gamma \odot \lambda_a)$ and $(\bigcup_{a \in A} \lambda_a) \odot \lambda = \bigcup_{a \in A} (\lambda_a \odot \gamma)$.*

((1) We have

$$\lambda = \bigcup\{\chi(x, \lambda(x)) \mid x \in \mathbf{Supp}(\lambda)\} \text{ and } \kappa = \bigcup\{\chi(x, \kappa(x)) \mid x \in \mathbf{Supp}(\kappa)\}.$$

Since $\lambda \subseteq \kappa$, we have $\chi(x, \lambda(x))(x) = \lambda(x) \leq \kappa(x) = \chi(x, \kappa(x))(x)$. On the other hand $\chi(x, \lambda(x))(y) = 0 = \chi(x, \kappa(x))(y)$ if $y \neq x$. Then $\chi(x, \lambda(x)) \subseteq \chi(x, \kappa(x))$ for every $x \in G$. By Proposition 2.8

$$\gamma \odot \lambda = \bigcup\{\chi(y, \gamma(y)) \odot \chi(z, \lambda(z)) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\lambda)\} \subseteq$$

$$\subseteq \bigcup\{\chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) \mid y \in \mathbf{Supp}(\lambda), z \in \mathbf{Supp}(\kappa)\} = \gamma \odot \kappa.$$

The proof of the other inclusion is similar.

(2) Let $\lambda = \bigcup_{a\in A} \lambda_a$. By Proposition 2.8,

$$\gamma \odot \lambda = \bigcup \{\chi(y, \gamma(y)) \odot \chi(z, \lambda(z)) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\lambda)\}.$$

Clearly $L := \mathbf{Supp}(\bigcup_{a\in A} \lambda_a) = \bigcup_{a\in A} \mathbf{Supp}(\lambda_a)$. Thus we have $\bigcup_{a\in A}(\gamma \odot \lambda_a) =$

$$= \bigcup_{a\in A}(\bigcup \{\chi(y, \gamma(y)) \odot \chi(z, \lambda_a(z))) \mid y \in \mathbf{Supp}(\gamma), z \in \mathbf{Supp}(\lambda_a)\} =$$

$$\bigcup \{(\bigcup_{a\in A} \chi(y, \gamma(y)) \odot \chi(z, \lambda_a(z)) \mid y \in \mathbf{Supp}(\gamma), z \in L\}).$$

By Proposition 2.1, $\chi(y, \gamma(y)) \odot \chi(z, \lambda_a(z)) = \chi(yz, \gamma(y) \wedge \lambda_a(z))$. But

$$(\bigcup_{a\in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)))(g) = \vee\{(\chi(yz, \gamma(y) \wedge \lambda_a(z))(g)) \mid a \in A\}$$

for every $g \in G$. In particular,

$$(\bigcup_{a\in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)))(yz) = \vee\{(\chi(yz, \gamma(y) \wedge \lambda_a(z))(yz)) \mid a \in A\} =$$

$$= \vee\{(\chi(y) \wedge \lambda_a(z)) \mid a \in A\} = \gamma(y) \wedge (\vee\{\lambda_a(z)) \mid a \in A\} = \gamma(y) \wedge \lambda(z)$$

and

$$(\bigcup_{a\in A} \chi(yz, \gamma(y) \wedge \lambda_a(z)))(g) = \vee\{(\chi(yz, \gamma(y) \wedge \lambda_a(z))(g)) \mid a \in A\} = 0$$

provided $g \neq yz$. Then $\bigcup_{a\in A} \chi(yz, \gamma(y)\lambda_a(z)) = \chi(yz, \gamma(y) \wedge \lambda(z))$. Hence

$$\bigcup_{a\in A}(\gamma \odot \lambda_a) = \bigcup \{(\bigcup_{a\in A} \chi(y, \gamma(y)) \odot \chi(z, \lambda_a(z))) \mid y \in \mathbf{Supp}(\gamma), z \in L\} =$$

$$= \bigcup \{\chi(yz, \gamma(y) \wedge \lambda(z)) = \gamma \odot \lambda.$$

The second equation can be obtained in a similar way. □

The next result characterizes permutable fuzzy subgroups in terms of the characteristic function χ of a cyclic group which shall need as a specific formula. Let γ be a fuzzy group on G, ν a fuzzy subgroup and $x \in G$ satisfying $\chi(x, \gamma(x)) \subseteq \nu$. By Lemma 2.3, $\chi(x, \gamma(x))^n \subseteq \nu$ for every integer $n \geq 1$. Since $\chi(x, \gamma(x))^n = \chi(x^n, \gamma(x))$ by Proposition 2.1, we may apply Lemma 2.3 again to obtain that $\chi(x^{-1}, \gamma(x)) \subseteq \nu$, and therefore $\chi(x^{-1}, \gamma(x))^n \subseteq \nu$ for every integer $n \geq 1$. Since Proposition 2.1 ensures that $\chi(x^{-1}, \gamma(x))^n = \chi(x^{-n}, \gamma(x))$, we have that $\chi(x^n, \gamma(x)) \subseteq \nu$ for every non-zero integer n. Since $\gamma(x) = \chi(x, \gamma(x))(x) \leq \nu(x)$ and $\nu(x) \leq \nu(e)$, we deduce that $\gamma(x) \leq \nu(e)$ for every fuzzy subgroup ν containing $\chi(x, \gamma(x))$. We note that the function $\xi := \chi(\langle x \rangle, \gamma(x))$ is a fuzzy subgroup on G. Since $x \in \langle x \rangle$, $\chi(x, \gamma(x)) \subseteq \xi$. If n is a non-zero integer, then $\xi(x^n) = \gamma(x) = \chi(x^n, \gamma(x)) \leq \nu(x^n)$. Moreover, as we showed above, $\xi(e) = \gamma(x) \leq \nu(e)$. Hence $\xi \preceq \nu$ for every fuzzy subgroup ν that contains $\chi(x, \gamma(x))$. Thus

$$\xi = \chi(\langle x \rangle, \gamma(x)) = \langle \chi(x, \gamma(x)) \rangle.$$

Theorem 2.10. *Let γ be a fuzzy group on a group G and $\kappa \preceq \gamma$ be a fuzzy subgroup. Then κ is permutable in γ if and only if $\chi(\langle x \rangle, \gamma(x)) \odot \kappa = \kappa \odot \chi(\langle x \rangle, \gamma(x))$ for every $x \in \mathbf{Supp}(\gamma)$.*

Proof. Let κ be permutable in γ. If $x \in \mathbf{Supp}(\gamma)$, then $\chi(x, \gamma(x)) \subseteq \gamma$ and therefore $\langle \chi(x, \gamma(x)) \rangle \preceq \gamma$. Therefore $\langle \chi(x, \gamma(x)) \rangle \odot \kappa = \kappa \odot \langle \chi(x, \gamma(x)) \rangle$. We have already showed that $\langle \chi(x, \gamma(x)) \rangle = \langle \chi(\langle x \rangle, \gamma(x)) \rangle$ and so $\chi(\langle x \rangle, \gamma(x)) \odot \kappa = \kappa \odot \chi(\langle x \rangle, \gamma(x))$, as required.

Conversely, suppose $\chi(\langle x \rangle, \gamma(x)) \odot \kappa = \kappa \odot \chi(\langle x \rangle, \gamma(x))$ for every $x \in \mathbf{Supp}(\gamma)$, and let $\lambda \preceq \gamma$ be an arbitrary fuzzy subgroup. By Proposition 2.8,

$$\lambda \odot \kappa = \bigcup \{\chi(y, \lambda(y)) \odot \chi(z, \kappa(z)) \mid y \in \mathbf{Supp}(\lambda), z \in \mathbf{Supp}(\kappa)\}.$$

We have

$$\chi(y, \lambda(y)) \odot \chi(z, \kappa(z)) \subseteq \langle \chi(y, \lambda(y)) \rangle \odot \kappa = \chi(\langle y \rangle, \lambda(y)) \odot \kappa = \kappa \odot \chi(\langle y \rangle, \lambda(y)).$$

Since $\chi(\langle y \rangle, \lambda(y)) = \langle \chi(y, \lambda(y)) \rangle \preceq \lambda$, by Corollary 2.9, $\kappa \odot \chi(\langle y \rangle, \lambda(y)) \subseteq \kappa \odot \lambda$. Therefore, $\chi(y, \lambda(y)) \odot \chi(z, \kappa(z)) \subseteq \kappa \odot \lambda$ for every $y \in \mathbf{Supp}(\lambda)$ and $z \in \mathbf{Supp}(\kappa)$. Hence

$$\lambda \odot \kappa = \bigcup \{\chi(y, \lambda(y)) \odot \chi(z, \kappa(z)) \mid y \in \mathbf{Supp}(\lambda), z \in \mathbf{Supp}(\kappa)\} \subseteq \kappa \odot \lambda.$$

Similarly, $\kappa \odot \lambda \subseteq \lambda \odot \kappa$, which proves that $\kappa \odot \lambda = \lambda \odot \kappa$. $\square$

3. Proof of the main results and their corollaries.

Proof of Theorem A. Suppose that κ is permutable in γ. By Theorem 2.10, $\chi(\langle x\rangle,\gamma(x))\odot\kappa=\kappa\odot\chi(\langle x\rangle,\gamma(x))$ for every element $x\in L_a(\gamma)$. Pick $y\in L_a(\kappa)$. By Proposition 2.8,

$$\chi(\langle x\rangle,\gamma(x))\odot\kappa=\bigcup\{\chi(u,\gamma(x))\odot\chi(z,\kappa(z))\mid z\in\mathbf{Supp}(\kappa),u\in\langle x\rangle\}=$$

$$=\bigcup\{\chi(x^k,\gamma(x))\odot\chi(z,\kappa(z))\mid z\in\mathbf{Supp}(\kappa),k\in\mathbb{Z}\},$$

and therefore

$$\chi(y,\kappa(y))\odot\chi(x,\gamma(x))\subseteq\chi(x^k,\gamma(x))\odot\chi(z,\kappa(z))$$

for some $k\in\mathbb{Z}$ and $z\in\mathbf{Supp}(\kappa)$. By Proposition 2.1, $\chi(yx,\kappa(y)\wedge\gamma(x))\subseteq\chi(x^kz,\gamma(x)\wedge\kappa(z))$. It follows that $yx=x^kz$ and $\gamma(x)\wedge\kappa(y)\le\gamma(x)\wedge\kappa(z)$. Since $x\in L_a(\gamma)$ and $y\in L_a(\kappa)$, we have that $\gamma(x)\wedge\kappa(y)\ge a$. We note that if $\gamma(x)\le\kappa(z)$ happens, then $\kappa(z)\ge a$ holds and then $z\in L_a(\kappa)$. Otherwise, $\gamma(x)>\kappa(z)$, and then $\kappa(z)=\gamma(x)\wedge\kappa(z)\ge a$ and so we deduce again that $z\in L_a(\kappa)$. In other words, for every element $y\in L_a(\kappa)$, there exists another element $z\in L_a(\kappa)$ such that $yx=x^kz$ for some integer k. Therefore $L_a(\kappa)\langle x\rangle\le\langle x\rangle L_a(\kappa)$. Proceeding in the same way, we obtain that $\langle x\rangle L_a(\kappa)\le L_a(\kappa)\langle x\rangle$ and so we have the equality $\langle x\rangle L_a(\kappa)=L_a(\kappa)\langle x\rangle$. Since this holds for every $x\in L_a(\gamma)$, we deduce that $L_a(\kappa)$ is permutable in $L_a(\gamma)$.

Conversely, suppose that $L_a(\kappa)$ is a permutable subgroup of $L_a(\gamma)$ for every $a\le\kappa(e)$. We want to show that $\chi(\langle x\rangle,\gamma(x))\odot\kappa=\kappa\odot\chi(\langle x\rangle,\gamma(x))$ for each $x\in\mathbf{Supp}(\gamma)$. If $x\in\mathbf{Supp}(\gamma)$, pick $y\in\mathbf{Supp}(\gamma)$ and let $\kappa(y)=b$ and $\gamma(x)=a$. By Proposition 2.1, $\chi(x,\gamma(x))\odot\chi(y,\kappa(y))=\chi(xy,\gamma(x)\wedge\kappa(y))$.

If $a\ge b$ occurs, then $x\in L_b(\gamma)$ and $y\in L_b(\kappa)$. Since $L_b(\kappa)$ is permutable in $L_b(\gamma)$, $L_b(\kappa)$ and $\langle x\rangle$ are permutable. It follows that there exists an element $z\in L_b(\kappa)$ such that $xy=zx^k$ for some integer k. By Proposition 2.1, $\chi(z,\kappa(z))\odot\chi(x^k,\gamma(x))=\chi(zx^k,\kappa(z)\wedge\gamma(x))$. Thus we have

$$\chi(xy,\gamma(x)\wedge\kappa(y))(xy)=\gamma(x)\wedge\kappa(y)=a\wedge b=b$$

and

$$\chi(xy,\gamma(x)\wedge\kappa(y))(u)=0$$

provided $u\neq xy$. Moreover

$$\chi(zx^k,\kappa(z)\wedge\gamma(x))(xy)=\chi(zx^k,\kappa(z)\wedge\gamma(x))(zx^k)=\kappa(z)\gamma(x)=\kappa(z)\wedge a$$

and

$$\chi(zx^k, \kappa(z) \wedge \gamma(x))(u) = 0$$

provided $u \neq zx^k = xy$. Since $z \in L_b(\kappa)$, $\kappa(z) \geq b$ and so $\kappa(z) \wedge a \geq b$. Thus

$$\chi(xy, \gamma(x) \wedge \kappa(y))(v) \leq \chi(zx^k, \kappa(z) \wedge \gamma(x))(v)$$

for every $v \in G$. Hence $\chi(xy, \gamma(x) \wedge \kappa(y)) \subseteq \chi(zx^k, \kappa(z) \wedge \gamma(x))$.

If we have that $b > a$, then $y \in L_b(\kappa) \leq L_a(\kappa)$. Since $x \in L_a(\gamma)$ and $L_a(\kappa)$ is permutable in $L_a(\gamma)$, there exists an element $z \in L_a(\kappa)$ such that $xy = zx^k$ for some integer k. As above we have

$$\chi(x, \gamma(x)) \odot \chi(y, \kappa(y)) = \chi(xy, \gamma(x) \wedge \kappa(y)),$$

$$\chi(z, \kappa(z)) \odot \chi(x^k, \gamma(x)) = \chi(zx^k, \kappa(z) \wedge \gamma(x))$$

$$\chi(xy, \gamma(x) \wedge \kappa(y))(xy) = \gamma(x) \wedge \kappa(y) = a \wedge b = a$$

$$\chi(xy, \gamma(x) \wedge \kappa(y))(u) = 0$$

provided $u \neq xy$. Moreover

$$\chi(zx^k, \kappa(z) \wedge \gamma(x))(xy) = \chi(zx^k, \kappa(z) \wedge \gamma(x))(zx^k) = \kappa(z) \wedge \gamma(x) = \kappa(z) \wedge a$$

and

$$\chi(xy, \gamma(x) \wedge \kappa(y))(u) = 0$$

provided $u \neq xy$. Since $z \in L_a(\kappa)$, $\kappa(z) \geq a$ and so $\kappa(z) \wedge a \geq a$. Thus

$$\chi(xy, \gamma(x) \wedge \kappa(y))(v) \leq \chi(zx^k, \kappa(z) \wedge \gamma(x))(v)$$

for every $v \in G$. Therefore in every case

$$\chi(xy, \gamma(x) \wedge \kappa(y)) \subseteq \chi(zx^k, \kappa(z) \wedge \gamma(x)).$$

It follows that

$$\chi(\langle x \rangle, \gamma(x)) \odot \kappa \subseteq \kappa\chi(\langle x \rangle, \gamma(x)).$$

Proceeding in the same way, we obtain the reverse inclusion, that is

$$\kappa \odot \chi(\langle x \rangle, \gamma(x)) \subseteq \chi(\langle x \rangle, \gamma(x)) \odot \kappa,$$

and therefore

$$\chi(\langle x \rangle, \gamma(x)) \odot \kappa = \kappa \odot \chi(\langle x \rangle, \gamma(x)).$$

Thus it suffices to apply Theorem 2.10 to see that κ is permutable in γ. $\square$

Proof of Corollary A1. Let $K = \mathbf{Supp}(\kappa)$ and $L = \mathbf{Supp}(\gamma)$. It is suffices to prove that $K\langle x\rangle = \langle x\rangle K$ for every element $x \in L$. Pick $y \in K$ and Let $a = \gamma(x) \wedge \kappa(y)$ so that $x \in L_a(\gamma)$ and $y \in L_a(\kappa)$. By Theorem A, $L_a(\kappa)$ is permutable in $L_a(\gamma)$. Then there exists an element $z \in L_a(\kappa)$ such that $yx = x^k z$ for some integer k. Since $z \in \mathbf{Supp}(\kappa) = K$, $K\langle x\rangle \leq \langle x\rangle K$. Similarly, we obtain the reverse inclusion and then $K\langle x\rangle = \langle x\rangle K$ holds. □

Proof of Corollary A2. Indeed $L_d(\kappa)$ is normal in $L_d(\gamma)$ for every $d \leq \kappa(e)$ by Proposition **??**. Remark that every normal subgroup of an abstract group is permutable. Hence $L_d(\kappa)$ is a permutable subgroup of $L_d(\gamma)$ for every $d \leq \kappa(e)$. By Theorem 2.10, κ is a permutable fuzzy subgroup of γ. □

Proof of Corollary A3. Since λ and κ are permutable fuzzy subgroups of γ, we have that $\langle\lambda,\kappa\rangle = \lambda \cup \kappa \cup (\lambda \odot \kappa)$. Let μ be an arbitrary fuzzy subgroup of γ. Applying Corollaries 2.9 and 2.2,

$$\langle\lambda,\kappa\rangle \odot \mu = (\lambda \cup \kappa \cup (\lambda \odot \kappa)) \odot \mu = (\lambda \odot \mu) \cup (\kappa \odot \mu) \cup (\lambda \odot (\kappa) \odot \mu) =$$

$$= (\mu \odot \lambda) \cup (\mu \odot \kappa) \cup (\mu \odot (\lambda \odot \kappa)) = \mu \odot \langle\lambda,\kappa\rangle,$$

as required. □

Theorem B. *Let G be a group and γ be a fuzzy group on G. Suppose that L is a subgroup of* $\mathbf{Supp}(\gamma)$*. Then L is permutable in* $\mathbf{Supp}(\gamma)$ *if and only if the fuzzy subgroup $L|^{\gamma}$ of γ is permutable in γ.*

Proof. If $L|^{\gamma}$ is permutable in γ, by Corollary A.1, $L = \mathbf{Supp}(L|^{\gamma})$ is permutable in $\mathbf{Supp}(\gamma)$.

Conversely, suppose that L is a permutable subgroup of $\mathbf{Supp}(\gamma)$. There is no loss of generality if we assume that $G = \mathbf{Supp}(\gamma)$. Let $\kappa = L|^{\gamma}$. Pick $x \in G$ and let k be an integer. Since L is permutable in G, for every $y \in L$, there exists some $z \in L$ and integer t such that $x^k y = zx^t$. Therefore we have

$$\chi(x^k,\gamma(x)) \odot \chi(y,\kappa(y)) = \chi(x^k y,\gamma(x) \wedge \kappa(y)) = \chi(x^k y,\gamma(x) \wedge \gamma(y)),$$

$$\chi(z,\kappa(z)) \odot \chi(x^t,\gamma(x)) = \chi(zx^t,\kappa(z) \wedge \gamma(x)) = \chi(zx^t,\gamma(z) \wedge \gamma(x)).$$

Since $x^k y = zx^t$, $x^k y x^{-t} = z$ and therefore

$$\gamma(z) = \gamma(x^k y x^{-t}) \geq \gamma(x^k) \wedge \gamma(y) \wedge \gamma(x^{-t}) \geq \gamma(x^k) \wedge \gamma(y) \wedge \gamma(x^t) \geq$$

$$\geq \gamma(x) \wedge \gamma(y)\gamma(x) \geq \gamma(x) \wedge \gamma(y).$$

It follows that

$$\chi(x^k,\gamma(x)) \odot \chi(y,\kappa(y)) \subseteq \chi(z,\kappa(z)) \odot \chi(x^t,\gamma(x))$$

and then we have the inclusion

$$\chi(\langle x\rangle, \gamma(x)) \odot \kappa \subseteq \kappa \odot \chi(\langle x\rangle, \gamma(x)).$$

We also have the other inclusion and so $\chi(\langle x\rangle, \gamma(x)) \odot \kappa = \kappa \odot \chi(\langle x\rangle, \gamma(x))$. By Theorem 2.10, $\kappa = L|^{\gamma}$ is a permutable fuzzy subgroup of γ. □

The above result extends the one in [2, Theorem 3.2].

P r o o f o f C o r o l l a r y B1. There is no loss of generality if we assume that $G = \mathbf{Supp}(\gamma)$.

Suppose that every fuzzy subgroup of γ is permutable in γ. Let L be a subgroup of G. Then the fuzzy subgroup $L|^{\gamma}$ is permutable in γ. By Theorem B, L is permutable in G.

Conversely, let every subgroup of G be permutable, and pick λ a fuzzy subgroup of γ. If $a \leq \lambda(e)$, since $L_a(\lambda)$ is permutable in G, it is permutable in $L_a(\gamma)$. This holds for each $a \leq \lambda(e)$ and then Theorem A shows that λ is a permutable fuzzy subgroup of γ. □

Our last consequence has a different nature. We were able to give the structure of an abstract group G such that there exists a fuzzy group on G whose fuzzy subgroups are permutable. It follows from Theorem B and Corollary A4 that the subgroups of such G are permutable and then the structure of G is known (see [13, Lemma 2.4.10, Theorems 2.4.11 and 2.4.14]).

Corollary B2. *Let G be a group and γ be a fuzzy subgroup on G whose fuzzy subgroups are permutable. Then one of the following cases appear Here, as usual, $\Pi(G)$ denotes the set of prime divisors of the orders of the periodic elements of G.*

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