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TILING 3 AND 4-DIMENSIONAL EUCLIDEAN SPACES BY LEE SPHERES

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Abstract. The paper addresses the problem if the n-dimensional Euclidean space can be tiled with translated copies of Lee spheres of not necessarily equal radii such that at least one of the Lee spheres has radius at least 2. It will be showed that for $n = 3, 4$ there is no such tiling.

Introduction. It is known that the function $d : R^n \times R^n \to R$ defined by

$$d[(x_1, \ldots, x_n), (y_1, \ldots, y_n)] = |x_1 - y_1| + \cdots + |x_n - y_n|$$

is a metric in R^n. Customarily it is called the octahedron or Manhattan metric. The discrete analogue in coding theory it is referred to as Lee metric. The union of the translated copies an n-dimensional unit cube whose centers are

$$(x_1, \ldots, x_n), \quad |x_1| + \cdots + |x_n| \leq \rho,$$

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is called an n-dimensional Lee sphere of radius ρ centered at $(0, \ldots, 0)$. Here, of course, the x_i's are integer valued. The Lee spheres are particular cases of the so-called cubical clusters defined in [12]. We describe this concept.

Let us fix an orthogonal coordinate system in the n-dimensional space. The standard cube tiling consists of all n-dimensional unit cubes whose centers are the points with integer coordinates. The union of finitely many members of the standard cube tiling is defined to be a cubical cluster in [12]. Let L be a set of vectors. A family of translated copies of a fixed cubical cluster by the elements of L is called a tiling if the interiors of translated clusters are pair-wise disjoint and if their union is equal to the whole n-dimensional space. The tiling is termed integer if the coordinates of the vectors of L are all integers. The tiling is called lattice-like if L forms an abelian group with the operation of the addition of vectors.

In this paper we are interested in tiling the n-dimensional space by translated copies of Lee spheres. An n-dimensional Lee sphere of radius 0 is simply an n-dimensional unit cube. Plainly the n-dimensional space can be tiled with translated copies of an n-dimensional Lee sphere of radius 0. An n-dimensional Lee sphere of radius 1 is a cross shaped body. It is composed of $2n+1$ unit cubes. One central cube and $2n$ arm cubes. E. Molnár [7] showed that the n-dimensional space can be tiled with translated copies of an n-dimensional Lee sphere of radius 1. These tilings are lattice-like integer tilings. In fact it was shown that there are essentially different tilings and the number of the tilings is equal to the number of non-isomorphic abelian groups of order $2n+1$.

In 1-dimension the Lee sphere of radius ρ is a straight line section of length $2\rho+1$. Plainly, the 1-dimensional space can be tiled with translated copies of this Lee sphere. The reader can verify that, the 2-dimensional space can be tiled by translated copies of a 2-dimensional Lee sphere of radius ρ for each ρ. In 1970 S. W. Golomb and L. R. Welch [2] advanced the following conjecture.

Conjecture 1. *If $n \geq 3$ and $\rho \geq 2$, then the n-dimensional space cannot be tiled with translated copies of an n-dimensional Lee sphere of radius ρ.*

Let us call a tiling of translated copies of Lee spheres a mixed tiling if the radii of the spheres are not necessarily equal. One might wonder if Conjecture 1 can be replaced by the following more general conjecture about mixed tilings.

Conjecture 2. *If $n \geq 3$ and $\rho \geq 2$, then the n-dimensional space cannot be tiled with translated copies of n-dimensional Lee spheres of various radii ρ.*

S. Gravier, M. Mollard and C. Payan in [3] verified Conjecture 2 for $n = 3$.

S. Špacapan [10] settled the $n = 4$ case. Later in [11] he presented a computer aided proof for this result. P. Horak [4] using an entirely different method proved the $n = 5$ case. In this note we will give a new proof for the $n = 5$ case of Conjecture 2. Using Špacapan's ideas for each fixed n Conjecture 2 can be reduced to a finite combinatorial problem to the set partitioning or exact cover problem. It is known that the set partitioning problem is an NP complete problem. However, an algorithm proposed by D. E. Knuth [6] is proved to be powerful enough to successfully handle the set partitioning instances corresponding to the $n \leq 5$ cases.

We propose a further generalization of Conjecture 2. If in a tiling of the n-dimensional space with Lee spheres of mixed radii we can use Lee spheres of radius 0, then clearly we can construct a tiling. We simply place the Lee spheres of radii larger than 0 arbitrarily keeping their interiors disjoint. Then the remaining uncovered n-dimensional unit cubes of the standard cube tiling can be covered by Lee spheres of radius 0. The n-dimensional space can also be tiled with Lee spheres of radius 1.

Conjecture 3. *If $n \geq 3$ and $\rho \geq 1$, then the n-dimensional space cannot be tiled with translated copies of n-dimensional Lee spheres of various radii ρ such that one ρ is at least 2.*

We will show that Conjecture 3 holds for $n \leq 4$.

2. The exact cover problem.

Let U be a set and let $B_1, \dots, B_t$ be subsets of U. If $B_1, \dots, B_t$ are pair-wise disjoint and $B_1 \cup \cdots \cup B_t = U$, then we say that the family of subsets $B_1, \dots, B_t$ of U forms a partition or exact cover of U. The following problem is commonly referred to as the set partitioning or exact cover problem.

Problem 1. *Given a finite ground set U and a family of subsets $A_1, \dots, A_r$ of U. Decide if there are $B_1, \dots, B_t \in \{A_1, \dots, A_r\}$ that form a partition of U.*

By the theory of computations, Problem 1 falls in the NP complete complexity class. This can be interpreted loosely such that there are computationally hard instances of Problem 1.

The set U will be identified by $\{1, \dots, s\}$. The family of subsets $A_1, \dots, A_r$ of U can be conveniently described by an r by s incidence matrix A. Let $a_{i,j}$ be the entry of A in the i-th row and j-th column. Now

$$a_{i,j} = \begin{cases} 1, & \text{if} \quad j \in A_i, \\ 0, & \text{if} \quad j \notin A_i. \end{cases}$$

D. E. Knuth [6] proposed an algorithm for solving Problem 1. He also illustrated that the algorithm is capable of solving highly non-trivial size instances. The utility of the algorithm was demonstrated by further applications for instance in [8] and [9].

Knuth pointed out that with slight modifications his algorithm can also be used to solve a little more general problem. Suppose that the set U is partitioned into the subsets U_1 and U_2. We call these subsets primary and secondary subsets of U.

Problem 2. *Given a finite ground set U that is partitioned into primary and secondary subsets U_1 and U_2. Given further a family of subsets $A_1, \ldots, A_r$ of U. Decide if there are subsets $B_1, \ldots, B_t \in \{A_1, \ldots, A_r\}$ such that the sets $B_1 \cap U_1, \ldots, B_t \cap U_1$ form a partition of U_1 and the subsets $B_1 \cap U_2, \ldots, B_t \cap U_2$ are pair-wise disjoint.*

The sets U, U_1, U_2 can be identified by the sets $\{1, \ldots, s\}$, $\{1, \ldots, p\}$, $\{p+1, \ldots, s\}$ respectively. The r by s incidence matrix A associated with the subsets $A_1, \ldots, A_r$ of U in Problem 2 now is partitioned into the submatrices A_1 and A_2. Knuth calls the columns of A_1 and A_2 primary and secondary columns of A. This is why we named the subsets U_1 and U_2 of U primary and secondary subsets. This extended version of the exact cover problem can be called the generalized exact cover problem.

3. Reducing tilings. In order to verify Conjecture 2 for a fixed $n \geq 3$ suppose that there is a mixed tiling of the n-dimensional space by translates of n-dimensional Lee spheres of radii at least 2. We would like to point out that without any further reasoning we cannot assume that our counter example is an integer tiling. In fact in [13] it was shown that if $n \geq 2$ and $2n+1$ is not a prime, then there is an n-dimensional non-integer tiling consisting of n-dimensional Lee spheres of radius 1. However, one can use the ideas of [1] to prove the following result.

Theorem 1. *Each n-dimensional mixed tiling consisting of n-dimensional Lee spheres can be modified with certain shifting of the spheres to get an integer tiling.*

P r o o f. We only sketch the proof. Let $e_1, \dots, e_n$ be coordinate unit vectors in the n-dimensional space parallel to the unit cube whose translates form the n-dimensional Lee spheres of the original tiling. We introduce a relation $\sim$ on the Lee spheres of the tiling. Two distinct Lee spheres are in relation $\sim$ if and only if they abut along an $(n-1)$-dimensional face perpendicular to the coordinate unit vector e_1. It turns out that the relation $\sim$ is an equivalence relation. (To guarantee reflexivity we declare each Lee sphere to be in relation $\sim$ with itself.) As a consequence the Lee spheres of the tiling are partitioned into equivalence classes.

It can be shown that each equivalence class can be shifted freely in the direction of the coordinate unit vector e_1 without destroying the tiling. Using suitable shifts we can make the 1-st coordinates of all Lee spheres in the tiling to be an integer number. Repeating the procedure in connection with the other coordinates finally we get an integer tiling. $\square$

In order to verify Conjecture 2 for a fixed n we may assume that there is an integer n-dimensional counter example consisting of n-dimensional Lee spheres with mixed radii. In short the original counter example which had a continuous nature can be reduced to a counter example having a discrete nature. S. Špacapan [11] established that for a fixed n we can further reduce this discrete problem to a finite problem.

Theorem 2. *If there is an n-dimensional counter example for Conjecture* 2, *then there is a bounded domain C of the n-dimensional space and finitely many Lee spheres of radii at most $2n-1$ such that the centers of the spheres are in the bounded domain C and the restricted parts of the Lee spheres form a tiling of the domain C.*

P r o o f. Let us consider an n-dimensional integer tiling T that is counter example for Conjecture 2. Pick a Lee sphere S_0 from this tiling. Let $(c_1, \dots, c_n)$ be the center and let ρ be the radius of S_0. Clearly $\rho \geq 2$. The n-dimensional unit cube centered at $(c_1+\rho, c_2, \dots, c_n)$ belongs to S_0. Consider an n-dimensional cube C of side length 5 whose center is $(c_1+\rho, c_2, \dots, c_n)$. Plainly C is composed of 5^n n-dimensional unit cubes. One may introduce a new coordinate system such that the center of C became $(2, \dots, 2)$. Now C consists of all n-dimensional unit cubes whose centers are

$$(a_1, \dots, a_n), \quad 0 \leq a_1 \leq 4, \dots, 0 \leq a_n \leq 4.$$

The center of S_0 is $(0, 2, \dots, 2)$.

The tiling T may contain Lee spheres that are disjoint to the interior of C. We simply discard these spheres. Pick a Lee sphere S_1 from the remaining spheres. We replace S_1 by a Lee sphere S_1'. If the center of S_1 is in C then set S_1' to be S_1. Since the central cube of C is covered by S_0, if S_1 is not identical to S_0, then there are only finitely many choices for the radius ρ of S_1. Namely, $\rho < 2n$ must hold. Let us turn finally to the case when the center $(d_1, \ldots, d_n)$ of S_1 is not in C. This means that $d_i < 0$ for some i or $d_j > 4$ for some j. If $d_i < 0$, then replace d_i by 0 and replace ρ by $r - |d_i|$. If $d_j > 4$, then replace d_j by 4 and replace ρ by $\rho - (d_j - 4)$. The center of the modified Lee sphere S_1' is in C and $S_1' \subseteq S_1$. In fact $S_1 \cap C = S_1' \cap C$. There are only finitely many Lee spheres whose center are in C and whose radii are at most $2n - 1$. Now consider the modified Lee spheres and note that the restrictions of the modified Lee spheres to C form a tiling of C. $\square$

We would like to point out that S. Špacapan experimented with various high dimensional boxes in place of C. In 4-dimension he found useful the box whose first dimension is equal to 4 and the other dimensions are all equal to 3.

3. The exact cover reformulation. The content of this section is to describe how the veracity of Conjecture 2 can be tested by the generalized exact cover problem.

Claim 1. *The existence of a tiling in Theorem* 2 *can be expressed in terms of the existence of a set partition.*

The 5^n n-dimensional unit cubes of C play the role of the elements of the ground set U. The restrictions of the modified Lee spheres to C will correspond to the family of subsets $A_1, \ldots, A_r$. If for a fixed n an exhaustive inspection reveals that there are no subsets $B_1, \ldots, B_t \in \{A_1, \ldots, A_r\}$ that form a partition of U, then we can conclude that Conjecture 2 holds for this particular n. On the other hand if there are $B_1, \ldots, B_t \in \{A_1, \ldots, A_r\}$ that form a partition of U, then one cannot conclude that the whole n-dimensional space can be tiled by Lee spheres of radii at least 2. First of all there is no guarantee that a tiling of C can be extended to a tiling of the whole space. Secondly the tiling of C may contain reduced Lee spheres of radii less than 2. If for example S_1 contains only one cube in C then S_1' will be a single n-dimensional unit cube that is a Lee sphere of radius 0.

In the generalized version of the exact cover problem the ground set U is partitioned into the primary and secondary subsets U_1 and U_2. We will take advantage of this possibility. The n-dimensional unit cubes of C whose centers are

$$(a_1, \ldots, a_n), \quad 1 \leq a_1 \leq 3, \ldots, 1 \leq a_n \leq 3$$

will play the role of U_1 and the remaining unit cubes of C will play the role of U_2. For the sake of a simpler terminology we partition C into the subsets C_1 and C_2 corresponding to U_1 and U_2.

Let us consider an n-dimensional integer counter example tiling T for Conjecture 2. As before we have a Lee sphere S_0 that cover the central unit cube of C. We may assume that the center of S_0 is $(0, 2, \ldots, 2)$ and that the radius of S_0 is 2. We construct a subset A_0 of U associated with S_0. Namely, A_0 will consists of all unit cubes of C that are contained by S_0. Pick another Lee sphere S_1 in T. If the center of S_1 in the primary cube C_1, then as S_1 is a Lee sphere in the counter example T, the radius of S_1 is at least 2. As the central unit cube of C is covered by S_0, the radius of S_1 is at most $n-1$. There are $|C_1|$ choices for the center of S_1 as the centers of S_1 must come from C_1. There are $n-2$ choices for the radius of S_1 as the radius of S_1 is in the range $2, \ldots, n-1$. We construct a subsets A_i for each choice of the center and the radius of S_1. Namely, A_i consists of all cubes of C contained in S_1. We discard A_i if it is not disjoint to A_0 and keep it if it is disjoint to A_0.

Assume next that the center of S_1 is in the secondary cube C_2. As the central unit cube of C is covered by S_0, the radius of S_1 is at most $2n-1$. There are $|C_2|$ choices for the center of S_1 as the centers of S_1 are in C_2. There are $2n-1$ choices for the radius of S_1 as the radius of S_1 is in the range $1, \ldots, 2n-1$. We construct a subsets A_j for each choice of the center and the radius of S_1. Namely, A_j consists of all unit cubes of C contained in S_1. We keep only those A_j subsets that are disjoint to A_0.

We have now an instance of the generalized exact cover problem. The ground set U consists of all the unit cubes of C. The primary and secondary subsets U_1, U_2 contain the unit cubes of C_1, C_2 respectively. The given family of subsets of U corresponds to the subsets A_0, A_i, A_j constructed above.

It is plain that if there is a counter example for Conjecture 2, then this instance of the exact cover problem has a solution. In other words checking that the exact cover problem does not have any solution verifies Conjecture 2.

5. The computations. The main results of this section can be summarized in the following two theorems

Theorem 3. *Conjecture* 2 *holds for dimensions* 3, 4, *and* 5.

Theorem 4. *Conjecture* 3 *holds for dimensions* 3 *and* 4.

Proof of Theorem 3. As we have pointed out earlier the result in Theorem 3 has already been proven. The novelty of our approach is that the result has an independent computer assisted proof which uses the exact cover algorithm. It seems likely that the technique used in [4] is suitable to settle further cases of Conjecture 3. The point we wanted to make in this paper is that an independent computer aided proof using the exact cover algorithm is available. The existing exact cover algorithms are capable of solving highly nontrivial instances.

We describe the exact cover searches to verify Conjecture 2 for dimensions 3, 4, 5 again. This time we focus on the computational aspects. The side length of the n-dimensional test cube C is chosen to be 5 and so it consists of 5^n n-dimensional unit cubes whose centers are

$$(a_1, \dots, a_n), \quad 0 \leq a_1 \leq 4, \dots, 0 \leq a_n \leq 4.$$

The side length of the primary test cube C_1 is chosen to be 3 and consequently C_1 consists of 3^n n-dimensional unit cubes whose centers are

$$(a_1, \dots, a_n), \quad 1 \leq a_1 \leq 3, \dots, 1 \leq a_n \leq 3.$$

The center of the initial Lee sphere S_0 is $(0, 2, \dots, 2)$ and the radius of S_0 is 2.

The incidence matrix of the exact cover problem was constructed row by row in the following manner. We picked an n-dimensional unit cube of C_1 and we picked an integer ρ in the range $2, \dots, 5n$. We marked each n-dimensional unit cube of C whose distance from the fixed cube was at most ρ. These cubes form $S_i \cap C$, where S_i is a Lee sphere of the tentative tiling centered at the fixed cube and having radius ρ. In this way we get a row of the incidence matrix of the exact cover problem. From the $3^n(5n-1)$ possible choices for S_i we kept only those for which $S_0 \cap S_i = \emptyset$ holds.

Similarly, we picked an n-dimensional unit cube of $C_2 = C \setminus C_1$ and an integer ρ from the range $1, \dots, 5n$. We marked each n-dimensional unit cube of C whose distance from the fixed cube did not exceed ρ. These cubes form $S_j \cap C$, where S_j is a Lee sphere of the tentative tiling centered at the fixed cube and having radius ρ. In this way we get a row of the exact cover problem. However, from the possible $(5^n - 3^n)(5n)$ choices for S_j we kept only those for which $S_0 \cap S_j = \emptyset$ holds.

Table 1. Conjecture 2

n	r	s	p	v	w
3	153	106	21	99	0
4	1 199	592	73	4 109	0
5	8 449	3 074	233	777 289	0

The details are summarized in Table 1. Here n is the dimension of the space. The incidence matrix of the exact cover problem has r rows and s columns. The number of the primary columns is p. The number of the nodes of the search tree is v and the number of the solutions found is w. We would like to emphasize that these results are not new. However, an independent computation employing a well tested standard algorithm increases our confidence in a computer assisted proof. □

Proof of Theorem 4. Let us turn our attention to the exact cover search to verify Conjecture 3 for dimensions 3, 4. This is the new result of this note. In dimension 3 the side length of the test cube C is chosen to be 9. Therefore C consists of 9^3 unit cubes whose centers are

$$(a_1, a_2, a_3), \quad 0 \leq a_1 \leq 8,\ 0 \leq a_2 \leq 8,\ 0 \leq a_3 \leq 8.$$

The side length of the test cube C_1 is chosen to be 7 and so C_1 is constructed of 7^3 unit cubes whose centers are

$$(a_1, a_2, a_3), \quad 1 \leq a_1 \leq 7,\ 1 \leq a_2 \leq 7,\ 1 \leq a_3 \leq 7.$$

We picked an integer x from the range $0, \ldots, 4$. The center of the initial Lee sphere S_0 is $(4-x, 4, 4)$ and the radius of S_0 is $2+x$. The incidence matrix of the exact cover problem was constructed in the following way. We picked a unit cube of C and an integer ρ in the range

$$\begin{cases} 1, \ldots, 2+x & \text{if} \quad x \leq 3, \\ 1, \ldots, 3 \cdot 9 & \text{if} \quad x = 4. \end{cases}$$

We marked each unit cube of C whose distance from the chosen unit cube was at most ρ. This provided us with $S_i \cap C$, where S_i is a Lee sphere of the tentative tiling centered at the chosen unit cube having radius ρ. In other words this provided us with a row of the exact cover problem. Out of the $(9^3)(2+x) + (9^3)(3 \cdot 9)$ choices of S_i we kept only those for which $S_0 \cap S_i = \emptyset$ holds.

This construction is motivated by the following consideration. Suppose there is a counter example tiling T for Conjecture 3. The radii of the Lee spheres in T do not necessarily admit a maximum value. But if they do, then we distinguish two cases depending on whether this maximum value is at most 5 or if it is larger than 5. If the maximum of the radii of the Lee spheres in T is 2, then we identify S_0 with a Lee sphere in T of radius 2. In this case T does not contain any Lee sphere of radius larger than 2. If the maximum of the radii of the Lee spheres in T is 3, then we identify S_0 with a Lee sphere of radius 3. (Reducing the first coordinate of the center by one in the same time.) We repeat this argument until the largest among the radii among the Lee spheres in T reaches 5. This correspond to the $x = 0, \ldots, 3$ cases. If T contains a Lee spheres of radius larger than 5, then we use Špacapan's reduction and identify S_0 with a Lee sphere of radius 6. This corresponds to the $x = 4$ case. The details of the computation are summarized in Table 2.

Table 2. Conjecture 3 in dimension 3

x	r	s	p	v	w
0	1 166	704	318	1 021	0
1	1 405	666	281	1 478	0
2	1 377	606	237	322	0
3	1 369	546	217	277	0
4	1 493	518	217	228	0

In 4-dimension the side length of the 4-dimensional test cube C is chosen to be 6. This means that C is composed of 6^4 copies of 4-dimensional unit cubes whose centers are

$$(a_1, \ldots, a_4), \quad 0 \leq a_1 \leq 5, \ldots, 0 \leq a_4 \leq 5.$$

The side length of the test cube C_1 is chosen to be 4 and consequently C_1 consists of 4^4 copies of 4-dimensional unit cubes whose centers are

$$(a_1, \ldots, a_4), \quad 1 \leq a_1 \leq 4, \ldots, 1 \leq a_4 \leq 4.$$

In order to construct a particular row of the incidence matrix of the exact cover problem we picked an integer x from the range $0, 1, 2$. The center of the initial Lee sphere S_0 is selected to be $(2 - x, 2, 2, 2)$ and the radius of S_0 was taken to be

$2+x$. We picked a 4-dimensional unit cube of C and an integer ρ from the range

$$\begin{cases} 1,\dots,2+x & \text{if} \quad x \le 1, \\ 1,\dots,4\cdot 6 & \text{if} \quad x = 2. \end{cases}$$

The details of the computation are summarized in Table 3.

Table 3. Conjecture 3 in dimension 4

x	r	s	p	v	w
0	1 510	1 255	219	13 029 751	0
1	1 803	1 174	189	3 099 107	0
2	2 374	1 045	184	832 656	0

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