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SUFFICIENT CONDITION FOR STRONGLY STARLIKE AND CONVEX FUNCTIONS

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Abstract. In this paper, we obtain sufficient conditions for analytic functions $f(z)$ in the open unit disk Δ to be strongly starlike and strongly convex of order β and type α. Some interesting corollaries of the results presented here are also discussed.

1. Introduction and preliminaries. Let $\mathcal{H}(\Delta)$ be the class of functions that are analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_p$ be the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in Δ. In particular set $\mathcal{A}_1(1) = \mathcal{A}$.

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A function $f \in \mathcal{A}_p$ is said to be starlike of order γ $(0 \le \gamma < p)$ in Δ if and only if

$$\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \Delta.$$

We denote by $\mathcal{S}_p^*(\gamma)$, the subclass of $\mathcal{A}_p$ consisting of all functions $f(z)$ which are starlike of order γ in Δ. We note that $\mathcal{S}_1^*(0) \equiv \mathcal{S}^*$, is the class of starlike functions. Furthermore, a function f is said to be convex of order γ, if and only if $zf' \in \mathcal{S}_p^*(\gamma)$. We denote this class by $\mathcal{K}_p^*(\gamma)$. These classes are subclasses of the class of univalent functions [1]. For a function $f \in \mathcal{A}_p$, we say that it is strongly starlike of order $\beta(0 < \beta \le 1)$ and type $\alpha(0 \le \alpha \le p)$ if

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}, \quad z \in \Delta.$$

The corresponding class is denoted by $\mathcal{SS}_p^*(\beta, \alpha)$. In particular, $\mathcal{SS}_1^*(1, 0) \equiv \mathcal{S}^*$. Also, $\mathcal{SS}_p^*(\beta, 0)$, is the class of strongly starlike of order β. If $f \in \mathcal{A}_p$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}, \quad z \in \Delta,$$

for some $\alpha(0 \le \alpha \le p)$ and $\beta(0 < \beta \le 1)$, then f is said to be strongly convex of order β and type α in Δ. We denote this by $\mathcal{SK}_p(\beta, \alpha)$. For more information about these calsses see Liu [2] and Nunokawa [5]. In particular, $\mathcal{SK}_1(1, 0) \equiv \mathcal{K}$, is the class of convex functions. Note that $\mathcal{SK}_p(\beta, 0)$ is class strongly convex functions of order β. The classes $\mathcal{SS}_p^*(\beta, 0)$ and $\mathcal{SK}_p(\beta, 0)$ are studied extensively by Mocanu [3] and Nunokawa [5]. It is obvious that $f \in \mathcal{A}$ belongs to $\mathcal{SK}_p(\beta, \alpha)$ if and only if $zf' \in \mathcal{SS}_p^*(\beta, \alpha)$. In the present paper, we give some conditions for $f \in \mathcal{A}_p$ to be in the classes $\mathcal{SS}_p^*(\beta, \alpha)$ and $\mathcal{SK}_p(\beta, \alpha)$.

2. Main results. For proving our results we need the following lemma due to F. Rønning et al. [7].

Lemma 2.1. *Let* $b \in \mathcal{H}(\Delta) \cap C^0(\bar{\Delta})$, $b(0) = 0$, $\sup_{z\in\Delta} |b(z)| = 1$ *and* $c = \sup_{z\in\Delta} \int_0^1 |b(tz)|dt$. *For* $0 < \beta \le 1$ *let*

$$\lambda(\beta) = \frac{\sin(\pi\beta/2)}{\sqrt{1 + 2c\cos(\pi\beta/2) + c^2}}.$$

If $f \in \mathcal{A}$ *and*

$$| f'(z) - 1 | \le \lambda(\beta) \mid b(z) \mid, \quad z \in \Delta,$$

then f *is strongly starlike of order* β*. Additionally, if*

$$b(t) = \max_{0\le\varphi\le 2\pi} \mid b(te^{i\varphi}) \mid, \quad 0 \le t \le 1,$$

then the constant $\lambda(\beta)$ cannot be replaced by any larger number without violating the conclusion.

We note that Lemma 2.1, without the sharpness part, was previously obtained by Ponnusamy and Singh in [6].

Theorem 2.1. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left| \left(\frac{f(z)}{z} \right)^{\frac{1}{p-\alpha}} \left(z^{\frac{1-\alpha}{p-\alpha}} \frac{f'(z)}{f(z)} - \alpha z^{\frac{1-p}{p-\alpha}} \right) - (p-\alpha) \right| < (p-\alpha)\lambda(\beta) \mid b(z) \mid, \quad (z \in \Delta),$$

for some real values of $\alpha (0 \leq \alpha < p)$, then $f(z) \in \mathcal{SS}_p^*(\beta, \alpha)$, where $\lambda(\beta)$ and $b(z)$ are defined in Lemma 2.1.

P r o o f. Let us define a function $g(z)$ by

$$g(z) = \left(\frac{f(z)}{z^\alpha} \right)^{\frac{1}{p-\alpha}} = z + \frac{a_{p+1}}{p-\alpha} z^{n+1} + \cdots, \tag{2.1}$$

for $f(z) \in \mathcal{A}_p$. Then it is easy to see that $g(z) \in \mathcal{A}$.

Differentiating from (2.1), we find that

$$\frac{g'(z)}{g(z)} = \frac{1}{p-\alpha} \left(\frac{f'(z)}{f(z)} - \frac{\alpha}{z} \right), \tag{2.2}$$

which gives

$$\mid g'(z) - 1 \mid = \frac{1}{p-\alpha} \left| \left(\frac{f(z)}{z} \right)^{\frac{1}{p-\alpha}} \left(z^{\frac{1-\alpha}{p-\alpha}} \frac{f'(z)}{f(z)} - \alpha z^{\frac{1-p}{p-\alpha}} \right) - (p-\alpha) \right|. \tag{2.3}$$

Now, by using the condition given with the Theorem 2.1, we get

$$\mid g'(z) - 1 \mid \leq \lambda(\beta) \mid b(z) \mid .$$

Hence, using the Lemma 2.1, we find that $g(z)$ is strongly starlike of order β. That is

$$\left| \arg \frac{zg'(z)}{g(z)} \right| < \frac{\pi\beta}{2}.$$

Also From (2.2), we have

$$\frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha} \left(\frac{zf'(z)}{f(z)} - \alpha \right).$$

Since $g(z)$ is strongly starlike of order β, thus

$$\left| \arg \left[\frac{1}{p-\alpha} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right] \right| < \frac{\pi\beta}{2},$$

or

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}.$$

Therefore $f(z) \in \mathcal{SS}_p^*(\beta, \alpha)$. $\square$

Setting $b(z) = z$, $\alpha = 0$ and $p = \beta = 1$ in Theorem 2.1, we obtain the following result:

Remark 2.1 [4]. If $f(z) \in \mathcal{A}$ satisfies

$$\left|f'(z) - 1\right| < \frac{2}{\sqrt{5}} = 0.894427\ldots, \quad (z \in \Delta),$$

then $f \in S^*$.

Putting $\alpha = 0$, we have:

Corollary 2.1. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|(f(z))^{\frac{1}{p}-1}\left(f'(z)\right) - p\right| < p\lambda(\beta) \mid b(z) \mid, \quad (z \in \Delta),$$

then f is strongly starlike of order β.

Setting $b(z) = z$, $p = 1$ and $\alpha = \beta = \frac{1}{2}$, we have:

Corollary 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{f(z)}{z}\right)^2\left(\frac{zf'(z)}{f(z)} - \frac{1}{2}\right) - \frac{1}{2}\right| < 0.2527247\ldots, \quad (z \in \Delta),$$

then $f(z)$ is strongly starlike of order $\frac{1}{2}$ and type $\frac{1}{2}$.

In the next theorem by using Lemma 2.1 we obtain conditions for $f(z) \in \mathcal{A}_p$ to be strongly convex of order β and type α in Δ.

Theorem 2.2. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{[f'(z)]^{\alpha-p+1}}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}[zf''(z) + (1-\alpha)f'(z)] - (p-\alpha)\right| < (p-\alpha)\lambda(\beta) \mid b(z) \mid, \quad (z \in \Delta),$$

for some real values of $\alpha (0 \le \alpha < p)$, then $f(z) \in \mathcal{SK}_p(\beta, \alpha)$, where $\lambda(\beta)$ and $b(z)$ are defined in Lemma 2.1.

P r o o f. Let us define a function $g(z)$ by

$$g(z) = \int_0^z \left(\frac{f'(t)}{pt^{p-1}}\right)^{\frac{1}{p-\alpha}} dt = z + \frac{p+n}{p(p-\alpha)(n+1)} a_{p+n} z^{n+1} + \cdots. \tag{2.4}$$

Further, let

$$h(z) = zg'(z) = z\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}} = z + \frac{p+n}{p(p-\alpha)}a_{p+n}z^{n+1}\cdots. \tag{2.5}$$

We see that $g(z)$ and $h(z)$ belongs to $\mathcal{A}$. Differentiating from (2.5), we have

$$h'(z) = \frac{1}{p-\alpha}\left(\frac{[f'(z)]^{\alpha-p+1}}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}[zf''(z) + (1-\alpha)f'(z)].$$

Hence by hypothesi of Theorem 2.2 we observe that

$$\mid h'(z) - 1 \mid = \frac{1}{p-\alpha}\left|\left(\frac{[f'(z)]^{\alpha-p+1}}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}[zf''(z) + (1-\alpha)f'(z)] - (p-\alpha)\right|$$
$$< \lambda(\beta) \mid b(z) \mid .$$

Therefore, application of the Lemma 2.1 gives us that

$$h(z) = zg'(z) \in \mathcal{SS}_p(\beta, 0) \Longrightarrow g(z) \in \mathcal{SK}_p(\beta, 0).$$

Since

$$\frac{zg''(z)}{g'(z)} = \frac{1}{p-\alpha}\left(\frac{zf''(z)}{f'(z)} - (p-1)\right), \tag{2.6}$$

therefore

$$\left|\arg\left(1 + \frac{g''(z)}{g'(z)}\right)\right| = \left|\arg\frac{1}{p-\alpha}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}, \quad (z \in \Delta),$$

or

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi\beta}{2}.$$

Which imply that $f(z)$ is strongly convex of order β and type α. This completes the proof. □

Setting $b(z) = z, \alpha = 0$ and $p = \beta = 1$ in Theorem 2.2, we have:

Corollary 2.3. *If $f \in \mathcal{A}$ satisfies*

$$\left|zf''(z) + f'(z) - 1\right| < \frac{2}{\sqrt{5}} = 0.894427\ldots, \quad (z \in \Delta),$$

then $f \in \mathcal{K}$.

If we take $\alpha = 0$ in Theorem 2.2, then we have:

Corollary 2.4. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{[f'(z)]^{1-p}}{pz^{p-1}}\right)^{\frac{1}{p}}[zf''(z) + f'(z)] - p\right| < p\lambda(\beta) \mid b(z) \mid, \quad (z \in \Delta),$$

then f is strongly convex of order β.

If we take $p = 1$ and $\alpha = \beta = \frac{1}{2}$, then we have:

Corollary 2.5. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| f'(z)[zf''(z) + \frac{1}{2}f'(z)] - 1 \right| < 0.2527247\ldots, \quad (z \in \Delta),$$

then f is strongly convex of order $\frac{1}{2}$ and type $\frac{1}{2}$.

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