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MOST GENERAL FRACTIONAL REPRESENTATION FORMULA FOR FUNCTIONS AND IMPLICATIONS

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Abstract. Here we present the most general fractional representation formulae for a function in terms of the most general fractional integral operators due to S. Kalla, [3], [4], [5]. The last include most of the well-known fractional integrals such as of Riemann-Liouville, Erdélyi-Kober and Saigo, etc. Based on these we derive very general fractional Ostrowski type inequalities.

1. Introduction. Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, then the following Montgomery identity holds [10]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P_1(x,t)\,f'(t)\,dt, \tag{1}$$

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where $P_1(x,t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \le t \le x, \\ \dfrac{t-b}{b-a}, & x < t \le b, \end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a,b]. \tag{4}$$

Properties of the above operator can be found in [9].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \ge 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) \left\{ \frac{J_a^{\alpha} f(b)}{b-a} - J_a^{\alpha-1} (P_1(x,b) f(b)) + J_a^{\alpha} \left(P_1(x,b) f'(b)\right) \right\}. \tag{5}$$

When $\alpha = 1$ the last (5) reduces to classic Montgomery identity (1).

Motivated by (5), here we establish a very general fractional representation formula based on the most general fractional integral due to S. Kalla, [3], [4], [5]. The last integral includes almost all other fractional integrals as special cases. We then establish a very general fractional Ostrowski type inequality.

We finish with applications.

2. Main results.

Here let $f : \mathbb{R}_+ \to \mathbb{R}$ differentiable with $f' : \mathbb{R}_+ \to \mathbb{R}$ be integrable. Let also $\Phi : [0,1] \to \mathbb{R}_+$ a general kernel function, which is differentiable with $\Phi' : [0,1] \to \mathbb{R}_+$ being integrable too. For z *in* $(0,1)$ we assume $\Phi(z) > 0$.

Let here the parameters γ, δ be such that $\gamma > -1$ and $\delta \in \mathbb{R}$. Set $\varepsilon := \delta - \gamma - 1$, that is $\delta = \varepsilon + \gamma + 1$.

The most general fractional integral operator was defined by S. Kalla ([3], [4], [5]), see also [7], as follows:

$$I_{\Phi}^{\gamma,\delta} f(x) := x^{\delta} \int_0^1 \Phi(\sigma) \sigma^{\gamma} f(x\sigma) d\sigma, \tag{6}$$

for any $x > 0$, with $I_{\Phi}^{\gamma,\delta} f(0) := 0$.

Here we consider $b > 0$ fixed, and $0 < x < b$. We operate on $[0, b]$.

By convenient change of variable we can rewrite $I_{\Phi}^{\gamma,\delta} f(x)$ as follows:

$$I_{\Phi}^{\gamma,\varepsilon} f(x) := x^{\varepsilon} \int_0^x \Phi\left(\frac{w}{x}\right) w^{\gamma} f(w) dw. \tag{7}$$

That is

$$I_{\Phi}^{\gamma,\varepsilon} f(x) = I_{\Phi}^{\gamma,\delta} f(x), \text{ for any } x > 0. \tag{8}$$

We take $\gamma > 0$ from now on.

We present the following most general fractional representation formula.

Theorem 2. *All as above described. Then*

$$f(x) = b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \left[\frac{1}{b} I_{\Phi}^{\gamma,\delta} f(b) + \gamma I_{\Phi}^{\gamma-1,\delta} (P_1(x,b) f(b))\right.$$

$$\left. + \frac{1}{b} I_{\Phi'}^{\gamma,\delta} (P_1(x,b) f(b)) + I_{\Phi}^{\gamma,\delta} \left(P_1(x,b) f'(b)\right)\right]. \tag{9}$$

P r o o f. We observe that

$$I_{\Phi}^{\gamma,\varepsilon} \left(P_1(x,b) f'(b)\right) = b^{\varepsilon} \int_0^b \Phi\left(\frac{w}{b}\right) w^{\gamma} P_1(x,w) f'(w) dw = \tag{10}$$

$$b^{\varepsilon} \left[\int_0^x \Phi\left(\frac{w}{b}\right) w^{\gamma} \frac{w}{b} f'(w) dw + \int_x^b \Phi\left(\frac{w}{b}\right) w^{\gamma} \left(\frac{w-b}{b}\right) f'(w) dw\right] = \tag{11}$$

$$b^{\varepsilon-1} \left[\int_0^x \Phi\left(\frac{w}{b}\right) w^{\gamma+1} f'(w) dw + \int_x^b \Phi\left(\frac{w}{b}\right) \left(w^{\gamma+1} - b w^{\gamma}\right) f'(w) dw\right] =$$

$$b^{\varepsilon-1}\left[\Phi\left(\frac{x}{b}\right)x^{\gamma+1}f(x)-\int_0^x f(w)\,d\left(\Phi\left(\frac{w}{b}\right)w^{\gamma+1}\right)-\right.$$

$$\left.\Phi\left(\frac{x}{b}\right)\left(x^{\gamma+1}-bx^{\gamma}\right)f(x)-\int_x^b f(w)\,d\left(\Phi\left(\frac{w}{b}\right)\left(w^{\gamma+1}-bw^{\gamma}\right)\right)\right]=$$

(12)

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-\int_0^x f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)w^{\gamma+1}+(\gamma+1)\Phi\left(\frac{w}{b}\right)w^{\gamma}\right]dw-\right.$$

$$\left.\int_x^b f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)\left(w^{\gamma+1}-bw^{\gamma}\right)+\Phi\left(\frac{w}{b}\right)\left((\gamma+1)w^{\gamma}-b\gamma w^{\gamma-1}\right)\right]dw\right]=$$

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-\frac{1}{b}\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma+1}dw-\right.$$

$$(\gamma+1)\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma}dw-\int_0^b f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)\left(w^{\gamma+1}-bw^{\gamma}\right)+\right.$$

$$\left.\Phi\left(\frac{w}{b}\right)\left((\gamma+1)w^{\gamma}-b\gamma w^{\gamma-1}\right)\right]dw+\int_0^x f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)\left(w^{\gamma+1}-bw^{\gamma}\right)+\right. \tag{13}$$

$$\left.\left.\Phi\left(\frac{w}{b}\right)\left((\gamma+1)w^{\gamma}-b\gamma w^{\gamma-1}\right)\right]dw\right]=$$

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-\frac{1}{b}\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma+1}dw+\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw-\right.$$

$$(\gamma+1)\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma}dw+b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw-$$

$$\left.\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw-b\gamma\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw\right]=:(\eta)\,. \tag{14}$$

We notice that

$$-\frac{1}{b}\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma+1}dw=-\left[\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)\frac{w}{b}w^{\gamma}dw+\right.$$

$$\left.\int_x^b f(w)\,\Phi'\left(\frac{w}{b}\right)\frac{(w-b)}{b}w^{\gamma}dw+\int_x^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw\right]= \tag{15}$$

$$-\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma}dw-\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw$$

$$+\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw.$$

Furthermore we have

$$-\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma}dw=-\gamma\left[b\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)\frac{w}{b}w^{\gamma-1}dw+\right.$$

$$\left.b\int_x^b f(w)\,\Phi\left(\frac{w}{b}\right)\frac{(w-b)}{b}w^{\gamma-1}dw+b\int_x^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw\right]= \tag{16}$$

$$-b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma-1}dw-b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw$$

$$+b\gamma\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw.$$

Putting together (10), (14), (15), (16) we obtain

$$I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b)\,f'(b)\right)=(\eta)=$$

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma}dw-\right. \tag{17}$$

$$\left.\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma}dw-b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma-1}dw\right]=$$

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-\frac{1}{b^{\varepsilon}}I_{\Phi'}^{\gamma,\varepsilon}\left(P_1(x,b)\,f(b)\right)\right.$$

$$\left.-\frac{1}{b^{\varepsilon}}I_{\Phi}^{\gamma,\varepsilon}f(b)-\gamma b^{1-\varepsilon}I_{\Phi}^{\gamma-1,\varepsilon}\left(P_1(x,b)\,f(b)\right)\right]= \tag{18}$$

$$b^{\varepsilon}x^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-\frac{1}{b}I_{\Phi'}^{\gamma,\varepsilon}\left(P_1(x,b)\,f(b)\right)-\frac{1}{b}I_{\Phi}^{\gamma,\varepsilon}f(b)-\gamma I_{\Phi}^{\gamma-1,\varepsilon}\left(P_1(x,b)\,f(b)\right).$$

That is

$$I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b)\,f'(b)\right)=b^{\varepsilon}x^{\gamma}\Phi\left(\frac{x}{b}\right)f(x)-$$

$$\frac{1}{b} I_{\Phi'}^{\gamma,\varepsilon} \left(P_1(x,b) f(b)\right) - \frac{1}{b} I_{\Phi}^{\gamma,\varepsilon} f(b) - \gamma I_{\Phi}^{\gamma-1,\varepsilon} \left(P_1(x,b) f(b)\right). \tag{19}$$

Solving the last (19) for $f(x)$ we get

$$f(x) = b^{-\varepsilon} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \left[\frac{1}{b} I_{\Phi}^{\gamma,\varepsilon} f(b) + \gamma I_{\Phi}^{\gamma-1,\varepsilon} \left(P_1(x,b) f(b)\right) + \right.$$

$$\left. \frac{1}{b} I_{\Phi'}^{\gamma,\varepsilon} \left(P_1(x,b) f(b)\right) + I_{\Phi}^{\gamma,\varepsilon} \left(P_1(x,b) f'(b)\right)\right], \tag{20}$$

proving the claim. $\square$

Next we establish a very general fractional Ostrowski type inequality.

Theorem 3. *Here all as in Theorem* 2. *Then*

$$\left| f(x) - b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \left[\frac{1}{b} I_{\Phi}^{\gamma,\delta} f(b) + \right.\right.$$

$$\left.\left. \gamma I_{\Phi}^{\gamma-1,\delta} \left(P_1(x,b) f(b)\right) + \frac{1}{b} I_{\Phi'}^{\gamma,\delta} \left(P_1(x,b) f(b)\right)\right]\right| \leq$$

(21)

$$b^{-1} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \|\Phi\|_{\infty,[0,1]} \left\|f'\right\|_{\infty,[0,b]} \left[\frac{\left(2x^{\gamma+2} - b^{\gamma+2}\right)}{\gamma+2} + \frac{b\left(b^{\gamma+1} - x^{\gamma+1}\right)}{\gamma+1}\right].$$

P r o o f. We observe that

$$\left| I_{\Phi}^{\gamma,\delta} \left(P_1(x,b) f'(b)\right)\right| = \left| I_{\Phi}^{\gamma,\varepsilon} \left(P_1(x,b) f'(b)\right)\right| = \tag{22}$$

$$b^{\varepsilon} \left| \int_0^b \Phi\left(\frac{w}{b}\right) w^{\gamma} P_1(x,w) f'(w)\, dw \right| \leq b^{\varepsilon} \int_0^b \Phi\left(\frac{w}{b}\right) w^{\gamma} \left|P_1(x,w)\right| \left|f'(w)\right| dw \leq$$

$$b^{\varepsilon} \|\Phi\|_{\infty,[0,1]} \left\|f'\right\|_{\infty,[0,b]} \int_0^b w^{\gamma} \left|P_1(x,w)\right| dw = \tag{23}$$

$$b^{\varepsilon} \|\Phi\|_{\infty,[0,1]} \left\|f'\right\|_{\infty,[0,b]} \left[\frac{1}{b} \int_0^x w^{\gamma+1} dw + \frac{1}{b} \int_x^b w^{\gamma} (b-w)\, dw\right] =$$

$$b^{\varepsilon-1} \|\Phi\|_{\infty,[0,1]} \left\|f'\right\|_{\infty,[0,b]} \left[\frac{2x^{\gamma+2}}{\gamma+2} + \frac{b}{\gamma+1}\left(b^{\gamma+1} - x^{\gamma+1}\right) - \frac{b^{\gamma+2}}{\gamma+2}\right]. \tag{24}$$

That is we derived

$$\left| I_{\Phi}^{\gamma,\delta}\left(P_1(x,b)\, f'(b)\right)\right| \leq$$

$$(25)\qquad b^{\delta-\gamma-2}\,\|\Phi\|_{\infty,[0,1]}\,\|f'\|_{\infty,[0,b]}\left[\frac{\left(2x^{\gamma+2}-b^{\gamma+2}\right)}{\gamma+2}+\frac{b\left(b^{\gamma+1}-x^{\gamma+1}\right)}{\gamma+1}\right].$$

The claim is proved. □

3. Applications.

We mention

Definition 4. *Let $\alpha > 0$, $\beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for $f \in C(\mathbb{R}_+)$ is defined by* ([12], *see also* [6, p. 19], [11]):

$$(26)\qquad I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right) f(\tau)\,d\tau,$$

where the function ${}_2F_1$ in (26) *is the Gaussian hypergeometric function defined by*

$$(27)\qquad {}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{t^n}{n!},$$

and $(a)_n$ is the Pochhammer symbol $(a)_n = a(a+1)\dots(a+n-1)$, $(a)_0 = 1$; where $c \neq 0,-1,-2,\dots$.

Note 5. *Given that $a+b<c$, ${}_2F_1$ converges on $[-1,1]$, see* [2]. *Furthermore we have*

$$(28)\qquad \frac{d\ {}_2F_1(a,b;c;t)}{dt} = \left(\frac{ab}{c}\right)\ {}_2F_1(a+1,b+1;c+1;t),$$

which converges on $[-1,1]$ when $1+a+b<c$. So when $1+a+b<c$, then both (27) *and* (28) *converge on $[-1,1]$. Therefore when $\eta > 1+\beta$ we get that both ${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$ and its derivative with respect to τ: $\left(\frac{(\alpha+\beta)\eta}{t\alpha}\right)$ ${}_2F_1\left(\alpha+\beta+1,-\eta+1;\alpha+1;1-\frac{\tau}{t}\right)$, converge on $[0,1]$; notice here $0 \leq 1-\frac{\tau}{t} \leq 1$, $t>0$.*

Remark 6. The integral operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by

$$(29)\qquad J_0^{\alpha}\{f(x)\} = I_{0,t}^{\alpha,-\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau \quad (\alpha>0),$$

and
(30)

$$I^{\alpha,\eta}\{f(t)\} = I_{0,t}^{\alpha,0,\eta}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\eta} f(\tau)\, d\tau \quad (\alpha > 0,\ \eta \in \mathbb{R}).$$

Remark 7. By a simple change of variable $\left(w = \frac{\tau}{t}\right)$ we get

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\beta}}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1}\ {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-w)\, f(tw)\, dw. \tag{31}$$

Similarly we find

$$J_0^{\alpha}\{f(t)\} = \frac{t^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} f(tw)\, dw, \tag{32}$$

and

$$I^{\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} w^{\eta} f(tw)\, dw. \tag{33}$$

Remark 8 ([8])**.** The above Saigo fractional integral (26) and its special cases of Riemann-Liouville and Erdélyi-Kober fractional integrals (29), (30), are all examples of the S. Kalla ([5]) generalized fractional integral in the reduced form

$$K_{\Phi}^{\gamma} f(x) = x^{-\gamma-1} \int_0^x \Phi\left(\frac{w}{x}\right) w^{\gamma} f(w)\, dw = \int_0^1 \Phi(\sigma)\, \sigma^{\gamma} f(x\sigma)\, d\sigma, \tag{34}$$

where $x > 0$, $\gamma > -1$ and Φ continuous arbitrary Kernel function.

Notice that (by (6) and (34))

$$I_{\Phi}^{\gamma,\delta} f(x) = x^{\delta} K_{\Phi}^{\gamma} f(x), \tag{35}$$

for any $x > 0$, where $\gamma > -1$ and $\delta \in \mathbb{R}$.

So for $b > 0$ we get

$$I_{\Phi}^{\gamma,\delta} f(b) = b^{\delta} K_{\Phi}^{\gamma} f(b). \tag{36}$$

Next we restrict ourselves to $\gamma > 0$. By Theorem 2 and (36) we obtain the following general fractional representation formula

Theorem 9. *It holds*

$$f(x) = b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \left[b^{\delta-1} K_{\Phi}^{\gamma} f(b) + \gamma b^{\delta} K_{\Phi}^{\gamma-1} \left(P_1(x,b) f(b)\right) + \right.$$

$$\left. b^{\delta-1} K_{\Phi'}^{\gamma} \left(P_1(x,b) f(b)\right) + b^{\delta} K_{\Phi}^{\gamma} \left(P_1(x,b) f'(b)\right) \right]. \tag{37}$$

We finish the following very general fractional Ostrowski type inequality, a direct application of (21) and (36).

Theorem 10. *All as in Theorem* 3. *Then*

$$\left| f(x) - b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \left[b^{\delta-1} K_{\Phi}^{\gamma} f(b) + \right.\right. \tag{38}$$

$$\left.\left. \gamma b^{\delta} K_{\Phi}^{\gamma-1} \left(P_1(x,b) f(b)\right) + b^{\delta-1} K_{\Phi'}^{\gamma} \left(P_1(x,b) f(b)\right) \right] \right| \leq$$

$$b^{-1} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \|\Phi\|_{\infty,[0,1]} \left\|f'\right\|_{\infty,[0,b]} \left[\frac{\left(2x^{\gamma+2} - b^{\gamma+2}\right)}{\gamma+2} + \frac{b\left(b^{\gamma+1} - x^{\gamma+1}\right)}{\gamma+1} \right].$$

Comment 11. One can apply (37) and (38) for the Riemann-Liouville and Erdélyi-Kober fractional integrals, as well as many other fractional integrals. To keep article short we omit this task.

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