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SCHUR-SZEGŐ COMPOSITION OF SMALL DEGREE POLYNOMIALS

Vladimir Petrov Kostov

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Abstract. We consider real polynomials in one variable without root at 0 and without multiple roots. Given the numbers of the positive, negative and complex roots of two such polynomials, what can be these numbers for their composition of Schur-Szegő? We give the exhaustive answer to the question for degree 2, 3 and 4 polynomials and also in the case when the degree is arbitrary, the composed polynomials being with all roots real, and one of the polynomials having all roots but one of the same sign.

1. Introduction.

1.1. (Finite) multiplier sequences and complex zero decreasing sequences. In the present paper we consider polynomials in one variable presented in the form $P := \sum_{j=0}^{n} \binom{n}{j} a_j x^j$. The *Schur-Szegő composition*

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(SSC) of the polynomials P and $Q := \sum_{j=0}^{n} \binom{n}{j} b_j x^j$ is defined by the formula $P \mathop{*}_{n} Q := \sum_{j=0}^{n} \binom{n}{j} a_j b_j x^j$. Clearly the SSC is commutative and associative. One can find more details about its properties in the monographies [7] and [8].

The SSC is defined also for entire functions presented by their series at 0. For the functions $f = \sum_{j=0}^{\infty} a_j x^j/j!$ and $g = \sum_{j=0}^{\infty} b_j x^j/j!$ their SSC equals $f * g = \sum_{j=0}^{\infty} a_j b_j x^j/j!$. In what follows we are interested in the situation when all polynomials and entire functions are real.

Definition 1. (1) *A real polynomial in one variable is* hyperbolic *if it has only real roots.*

(2) *A sequence of real numbers* $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ *is called a* multiplier sequence *if for any* n *and for any degree* n *hyperbolic polynomial* $P := \sum_{j=0}^{n} \binom{n}{j} a_j x^j$ *the polynomial* $\Gamma[P] := \sum_{j=0}^{n} \binom{n}{j} a_j \gamma_j x^j$ *is also hyperbolic.*

(3) *A sequence of real numbers* $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ *is called a* complex zero decreasing sequence (CZDS) *if for any* n *and for any real polynomial* P *the number of complex zeros of the polynomial* $\Gamma[P]$ *is not greater than the one of* P.

Clearly a CZDS is a multiplier sequence, but the inverse implication is wrong. One can identify the sequence Γ with the series $\sum_{j=0}^{\infty} \gamma_j x^j/j!$. See more about properties of CZDS in [1].

Definition 2. *A* finite multiplier sequence of length $n+1$ (FMS$(n+1)$) *is a real sequence* $\Gamma_n := (\gamma_0, \ldots, \gamma_n)$ *such that if the polynomial* P *is hyperbolic, then this is also the case of the polynomial* $\Gamma_n[P] := \sum_{j=0}^{n} \binom{n}{j} \gamma_j a_j x^j$.

An FMS$(n+1)$ is a linear operator in the space of polynomials of degree $\leq n$ acting diagonally in the standard monomial basis. The polynomial $\Gamma_n[P]$ is the

SSC of the polynomials P and $\Gamma_n^* := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j$. The following characterization of FMS$(n+1)$ can be found in [2] and [1]:

Theorem 3. *The sequence Γ_n is an FMS$(n+1)$ if and only if all different from 0 roots of the polynomial Γ_n^* are of the same sign.*

In certain situations one has to compose degree n polynomials as degree $n+k$ ones (assuming that there is a k-fold root at infinity); that is, one has to consider the polynomial $P *_{n+k} Q := \sum_{j=0}^{n} \left(\binom{n}{j}^2 / \binom{n+k}{j} \right) a_j b_j x^j$.

In the present paper we are interested in the question: *If the numbers of real negative and real positive roots of P and Q are known, what can these numbers be for $P *_{n+k} Q$?*

The question is a complement to the classical problems to study FMS$(n+1)$, multiplier sequences and CZDS. To be more precise, we limit ourselves to the *generic* case of polynomials without root at 0 and with all roots distinct. (But when the proofs are given we use also nongeneric polynomials.) In this paper we give the exhaustive answer to the question for $n = 2$, 3 and 4 and for any $k \geq 0$, see Theorems 13 and 14 in Subsections 2.1 and 2.2 and Propositions 17, 18, 19 and 20 in Subsection 3.1. We answer the question also in the situation when P and Q are hyperbolic and all roots but one of Q are of the same sign, see Theorem 21. This is a continuation of a result of [6] treating the case when all roots of Q are of the same sign.

1.2. Triples of admissible triples. As the problem to study in this paper is formulated in terms of quantities of real (negative and positive) and complex roots, we need the following formal definition:

Definition 4. (1) *Given a real degree n generic polynomial P we define its corresponding* triple $(a, b; 2c)$, *where a, b and c are the numbers of its negative, positive roots and complex conjugate pairs respectively. For n odd one of the first two components of the triple is odd and the other one is even. For n even we call a triple* odd *(resp.* even*) if its first two components are odd (resp. even).*

(2) *For given n and k a triple of nonnegative integers of the form $(a, b; 2c)$ is* admissible *if $c \in \mathbf{N} \cup 0$ and $a + b + 2c = n$. A* triple of admissible triples (TAT) $T := ((g, l; 2\gamma), (r, s; 2\delta), (u, w; 2\eta))$ *is said to be* realizable *if there exist degree n generic polynomials P and Q such that the triples of P, Q and $P *_{n+k} Q$ equal respectively $(g, l; 2\gamma)$, $(r, s; 2\delta)$ and $(u, w; 2\eta)$ ($P *_{n+k} Q$ is also presumed generic). As $P(0)Q(0) = (P *_{n+k} Q)(0)$, the following restriction is necessary for the realizability*

of a given TAT and is considered to be part of the definition of a TAT:

(1) $$\text{the sum } \ l+s+w \ \text{ is even.}$$

In [6] and [4] composition of hyperbolic polynomials is considered. Denote by $H_{u,v,w}$ the set of degree n hyperbolic polynomials with u negative, w positive roots (counted with multiplicity) and with a v-fold root at 0, where $u+v+w=n$. The following proposition summarizes Propositions 1.4 and 1.5 of [6] and Proposition 1.2 of [4]:

Proposition 5. (1) *Given two complex degree n polynomials P and Q such that $x_P \neq 0$ and $x_Q \neq 0$ are their roots of degree m_P and m_Q respectively with $\mu^* := m_P+m_Q-n \geq 0$, one has that $-x_P x_Q$ is a root of $P \ {}^*_n \ Q$ of multiplicity μ^*. (If $\mu^* = 0$, then $-x_P x_Q$ is not a root of $P \ {}^*_n \ Q$.)*

(2) *If $P \in H_{n,0,0}$ and $Q \in H_{u,v,w}$, then $P \ {}^*_n \ Q \in H_{u,v,w}$.*

(3) *If $P \ {}^*_n \ Q \in H_{u,v,w}$, then for all k one has $P \, {}^*_{n+k} Q \in H_{u,v,w}$ and $H_{u,v,w} \, {}^*_{n+k} H_{n,0,0} \subset H_{u,v,w}$. If $P \ {}^*_n \ Q$ is not necessarily hyperbolic and has u' negative, w' positive and a v'-fold root at 0, then for all k, $P \, {}^*_{n+k} Q$ has $\geq u'$ negative, $\geq w'$ positive and a v'-fold root at 0.*

(4) *If $x_P \neq 0$ and $x_Q \neq 0$ are roots of P and Q of multiplicity m_P and m_Q respectively, where $m_P + m_Q \geq n+k$, then $-x_P x_Q$ is a root of $P \, {}^*_{n+k} Q$ of multiplicity $m_P + m_Q - n - k$.*

In [6] the condition $x_P \neq 0 \neq x_Q$ is omitted which is not correct.

Parts (2) and (3) of the above proposition already provide (for each $n \geq 2$) a list of TATs which are not realizable. For $n = 2$ (with exceptions for $k = 0$) and $n = 3$ these are the only TATs that are not realizable, see Section 2. For $n = 4$ there are other such TATs, see Theorem 21 and Proposition 18.

The following lemma is self-evident:

Lemma 6. *For $\alpha \neq 0 \neq \beta$ the following equality holds true: $P(\alpha x) \, {}^*_{n+k} Q(\beta x) = (P \, {}^*_{n+k} Q)(\alpha\beta x)$. In particular, for $\alpha = -1$ and/or $\beta = -1$ one obtains $P(-x) \, {}^*_{n+k} Q(-x) = (P \, {}^*_{n+k} Q)(x)$ and $P(-x) \, {}^*_{n+k} Q(x) = P(x) \, {}^*_{n+k} Q(-x) = (P \, {}^*_{n+k} Q)(-x)$.*

Notation 7. For a triple $H := (a, b; 2c)$ we denote by $-H$ the triple $(b, a; 2c)$. It is clear that if H is defined by the polynomial $P(x)$, then $-H$ is defined by $P(-x)$.

Remark 8. (1) For n odd, if a TAT (A, B, C) is realizable, then such are also the TATs $(-A, -B, C)$, $(-A, B, -C)$ and $(A, -B, -C)$ while $(-A, B, C)$, $(A, -B, C)$, $(A, B, -C)$ and $(-A, -B, -C)$ are not. (This results from condition

(1) and from the three sums $g+l$, $r+s$ and $u+w$ being odd.) This observation enables us to consider only part of the TATs.

(2) For n even there are triples G such that $-G=G$. (Example: $n=2$, $(1,1;0)$.) If a TAT is realizable and contains such a triple, then all TATs obtained from it by changing the signs of one, two or all three of its triples are realizable.

(3) For n even there are couples of TATs of the form (A,B,C) and $(A,B,-C)$ (in which $A\neq -A$, $B\neq -B$ and $C\neq -C$). The realizability of each of them has to be (dis)proved independently. (Example: see for $n=2$ the TATs (A,A,A) and $(A,A,-A)$ from Section 2, the first of them is realizable for any k, the second is not.)

Definition 9. *Two TATs are called* equivalent *if they are obtained by permuting of their first two triples and/or by changing the signs of two of their three triples. Equivalent TATs are simultaneously realizable or not (this follows from the commutativity of the SSC and from Lemma* 6*).*

1.3. Some more preliminaries. The fact that some TATs are not realizable is explained sometimes not by parts (2) and (3) of Proposition 5, but by the Descartes rule of signs:

Example 10. For g, γ, r, δ, u and η fixed and n sufficiently large the TAT T is not realizable (see Definition 4). Indeed, by perturbing the polynomials one can assume that P and Q have no zero coefficients. By the Descartes rule of signs P (resp. Q) has at least $n-g-2\gamma$ (resp. $n-r-2\delta$) changes of sign in the sequence of its coefficients. If between two given consecutive coefficients there are sign changes both in P and Q, then there is no sign change in $P*_{n+k}Q$. This means that $P*_{n+k}Q$ has at most $\mu := g+2\gamma+r+2\delta$ such sign changes. By the Descartes rule it must have at least $n-u-2\eta$ changes which for n large enough is more than μ.

The following formulas can be found in [4]:

(2)
$$(P*_{n+k}Q)' = \frac{1}{n+k}(P' *_{n+k-1} Q')$$

(3)
$$\text{If } \ Q = x^q S \ \ (\deg S = n-q), \ \text{ then } \ P*_{n+k}Q = \frac{(n+k-q)!}{(n+k)!}x^q(P^{(q)} *_{n+k-q} S) \ .$$

$$\text{In particular, if } \ q=1 \ , \ \text{ then } \ P*_{n+k}Q = \frac{x}{n+k}(P' *_{n+k-1} S) \ .$$

(4)
$$\text{If } \ \deg Q = n-1 \ , \ \text{ then } \ P *_n Q = (P-(x/n)P') *_{n-1} Q \ .$$

The following theorem is Theorem 1.6 in [6]. In that paper the theorem is formulated only for $k = 0$, for $k > 0$ it follows from the case $k = 0$ automatically, because the polynomials can be considered as limits of one-parameter families of polynomials whose first k coefficients vanish and which have k roots tending to ∞. The roots of P ,Q and $P *_{n+k} Q$ involved in Proposition 5 (i.e. those of the form $-x_P x_Q$, the sum of the multiplicities of x_P and x_Q being $> n+k$) are called A-roots, the remaining roots of P ,Q and $P *_{n+k} Q$ are called B-roots. With one exception – if 0 is a root of P , then it is considered as A-root of $P *_{n+k} Q$. Associate to any hyperbolic polynomial P its *multiplicity vector* (the ordered partition of n defined by the multiplicities of the roots of P in the increasing order). For a root α of P denote by $[\alpha]_-$ (resp. $[\alpha]_+$) the total number of roots of P to the left (resp. to the right) of α and by $\mathrm{sign}(\alpha)$ the sign of α.

Theorem 11. *For any hyperbolic polynomial P and for any $Q \in H_{n,0,0}$ the multiplicity vector of $P *_{n+k} Q$ is uniquely determined by Proposition 5 and the following conditions:*

(i) For any A-root $\alpha \neq 0$ of P and any A-root β of Q one has $[\alpha\beta]_- = [\alpha]_- + [\beta]_{\mathrm{sign}(\alpha)}$.

(ii) Every B-root of $P *_{n+k} Q$ is simple.

Notation 12. For n and k fixed and P a degree n generic polynomial, the polynomial $P^R := x^{n+k} P(1/x)$ is said to be obtained from P by *reversion* (i.e. by reading P from the back). Clearly $\deg(P^R/x^k) = n$ and

– P and P^R/x^k define one and the same triple (if $P(h) = 0$, then $(P^R/x^k)(1/h) = 0$);

– $(P *_{n+k} Q)^R = (P^R) *_{n+k} (Q^R)$ (recall that $\binom{n+k}{j} = \binom{n+k}{n+k-j}$).

2. The cases $n = 2$ and $n = 3$.

2.1. The case $n = 2$. For $n = 2$ the possible triples are

$$A = (2, 0; 0), \quad -A = (0, 2; 0), \quad B = -B = (1, 1; 0) \quad \text{and} \quad C = -C = (0, 0; 2).$$

It is clear that condition (1) does not hold true exactly when one or three of the triples of a given TAT equal B (the others being equal to $\pm A$ and/or $\pm C$). Hence there are no such TATs.

In [5] (see Proposition 1.3 therein) the following result is proved:

Theorem 13. (1) *For $n = 2$, $k = 0$ the TAT (C, C, A) is not realizable while for $n = 2$, $k > 0$ it is.*

(2) *The following TATs are realizable for any* $k \geq 0$

$$\begin{array}{cccc} (A,A,A) & (A,B,B) & (A,C,A) & (A,C,C) \\ (B,B,A) & (B,B,C) & (B,C,B) & (C,C,C) \end{array}$$

while (A,A,C) *and* $(A,A,-A)$ *are not.*

(3) *Up to equivalence these are all TATs for* $n=2$.

2.2. The case $n = 3$. For $n = 3$ one has to consider 18 TATs up to equivalence. Indeed, by abuse of notation we consider as possible triples

$$A = (3,0;0), \quad B = (2,1;0), \quad C = (1,0;2)$$

(obtained from A, B and C for $n = 2$ by adding a unit to the first component) and $-A$, $-B$ and $-C$. Up to equivalence, a TAT can begin with AA, AB, AC, BB, BC or CC and end with $\pm A$, $\pm B$ or $\pm C$. If it begins with AA, AC, BB or CC, then the last triple can be only A, $-B$ or C, see condition (1). This gives 12 possibilities. If the TAT begins with AB or BC, then the last triple can be $-A$, B or $-C$ which adds 6 more possibilities. Further we consider these 18 cases.

Theorem 14. *For* $n = 3$ *the following* 14 *TATs are realizable for any* $k \geq 0$*:*

$$(A,A,A) \quad (A,B,B) \quad (A,C,A) \quad (A,C,-B) \quad (A,C,C) \quad (B,B,A) \quad (B,B,-B)$$

$$(B,B,C) \quad (B,C,-A) \quad (B,C,B) \quad (B,C,-C) \quad (C,C,A) \quad (C,C,-B) \quad (C,C,C)$$

while the following 4 *are not realizable for any* $k \geq 0$*:*

$$(A,A,-B), \qquad (A,A,C), \qquad (A,B,-A), \qquad (A,B,-C).$$

P r o o f. The TATs are not listed in alphabetic order because groups of similar cases are considered together.

1) – 6) The TATs (A,A,A) and (A,B,B) are realizable while $(A,A,-B)$, (A,A,C), $(A,B,-A)$ and $(A,B,-C)$ are not, see parts (2) and (3) of Proposition 5.

7) To prove that all other remaining TATs are realizable, for some of them we construct triples of polynomials P, Q and $P \ast_{3+k} Q$ directly. E.g. if $P = Q = x^3 + x + 1$, then $P' = Q' = 3x^2 + 1 > 0$, $(P \ast_{3+k} Q)'$ is of the form $\xi x^2 + \zeta$, $\xi > 0$, $\zeta > 0$ (hence $(P \ast_{3+k} Q)' > 0$) and $P(0) = Q(0) = (P \ast_{3+k} Q)(0) = 1 > 0$.

Thus P, Q and $P^*_{3+k}Q$ have each a real negative root and a complex conjugate pair. Therefore this triple of polynomials realizes the TAT (C,C,C).

8) – 9) For $P=(x+1)(x+2)(x+3)=x^3+6x^2+11x+6$ and $Q=x^3+x+1$ (respectively $P=(x+1)(x-2)(x-3)=x^3-4x^2+x+6$ and $Q=x^3+x+1$) in the same way one sees that $(P^*_{3+k}Q)'>0$ and $P(0)>0$, $Q(0)>0$, $(P^*_{3+k}Q)(0)>0$. Hence this triple of polynomials realizes the TAT (A,C,C) (resp. $(-B,C,C)$ which is equivalent to $(B,C,-C)$).

10) – 11) To realize other TATs one can use deformations. For instance, if $P=(x+1)(x+2)(x+3)$ and $Q=(x+1)(x+2)^2$ (resp. $Q=(x+1)(x-2)^2$), then for any $k\geq 0$ the polynomial $P^*_{3+k}Q$ has three negative distinct roots (resp. one negative and two distinct positive ones, see Proposition 5 and Theorem 11). Deform Q (i.e. replace it by $(x+1)((x\pm 2)^2+\varepsilon)$, where $\varepsilon>0$ is small). Hence the roots of $P^*_{3+k}Q$ remain distinct and negative (resp. one negative and two distinct positive), Q has a complex conjugate pair and the triple of polynomials realizes the TAT (A,C,A) (resp. $(A,C,-B)$).

12) To realize the TAT (B,C,B) set $P:=x(x^2-1)$, $Q:=x(x^2+1)$. Hence $P^*_{3+k}Q$ is of the form $\alpha x^3-\beta x$ with $\alpha>0$ and $\beta>0$. That is, it has a negative, a zero and a positive root. Perturb then P and Q into $(x+\varepsilon)(x^2-1)$ and $(x+\varepsilon)(x^2+1)$ respectively. The constant term of $P^*_{3+k}Q$ becomes $-\varepsilon^2$, i.e. its zero root becomes negative.

13) To realize the TATs (B,B,A), $(B,B,-B)$, (B,B,C) and $(-B,C,A)$ (equivalent to $(B,C,-A)$) we use the observation that a real monic degree 2 polynomial has a negative and a positive root if and only if it is of the form x^2+gx-h, where $h>0$. In the case of (B,B,A) we look for P and Q of the form $P=x(x^2+g_1x-h_1)$, $Q=x(x^2+g_2x-h_2)$ with $g_i>0$ and $h_i>0$, $i=1,2$. Hence for any k fixed $P^*_{3+k}Q$ is of the form $x(\rho x^2+g_3x+h_3)$, where $\rho>0$, $g_3>0$, $h_3>0$. Fix g_1 and g_2 and let $h_i\to 0$, $i=1,2$. Hence $h_3\to 0$ and for h_1 and h_2 small enough one has $g_3^2-4\rho h_3>0$, i.e. $P^*_{3+k}Q$ has a root at 0 and two negative distinct roots. Fix such h_1 and h_2. Perturb then P and Q respectively into $(x+\varepsilon)(x^2+g_1x-h_1)$ and $(x+\varepsilon)(x^2+g_2x-h_2)$. The constant term of $P^*_{3+k}Q$ equals $h_1h_2\varepsilon^2>0$, i.e. its zero root becomes negative.

14) For $(B,B,-B)$ one looks for Q of the form $x(x^2-g_2x-h_2)$, so before the deformation $P^*_{3+k}Q$ has a zero and two positive roots and the rest of the reasoning is analogous.

15) For (B,B,C) we set again $P:=x(x^2+g_1x-h_1)$, $Q:=x(x^2+g_2x-h_2)$, then we fix h_1 and h_2 and let $g_i\to 0$, $i=1,2$, so that $g_3^2-4\rho h_3<0$.

16) For $(-B,C,A)$ set $P:=x(x^2-g_1x+h_1)$ and $Q:=x(x^2-g_2x+h_2)$

with $g_1^2-4h_1>0$, $g_2^2-4h_2<0$. Hence $P\ {}^*_{3+k}\ Q$ is of the form $x(\rho x^2+g_3x+h_3)$. To obtain the condition $g_3^2-4\rho h_3>0$ (for k fixed) one can fix g_1, g_2 and h_2 and let $h_1\to 0$. Perturb then P and Q into $(x+\varepsilon)(x^2-g_1x+h_1)$ and $(x+\varepsilon)(x^2-g_2x+h_2)$. The zero root of $P\ {}^*_{3+k}\ Q$ becomes negative.

17) – 18) To realize the TAT (C,C,A) (respectively $(C,C,-B)$) consider polynomials $P:=x(x^2+g_1x+h_1)$, $Q:=x(x^2+g_2x+h_2)$ (respectively $Q:=x(x^2-g_2x+h_2)$) with

$$g_i^2-4h_i<0, \qquad i=1,2. \tag{5}$$

One has $P\ {}^*_{3+k}\ Q = x\left(x^2/\binom{3+k}{3} \pm g_1g_2x/\binom{3+k}{2} + h_1h_2/(3+k)\right)$.

The two roots of the quadratic factor are real if and only if

$$g_1^2g_2^2/\binom{3+k}{2}^2 - 4h_1h_2/\binom{3+k}{3}(3+k)>0, \quad \text{i.e.}\quad g_1^2g_2^2-\frac{6(2+k)}{1+k}h_1h_2>0. \tag{6}$$

It is possible to satisfy all inequalities (5) and (6) by choosing g_i close to 2 and h_i close to 1. Indeed, in this case the term $g_1^2g_2^2$ is close to 16, the product h_1h_2 is close to 1 and the fraction $6(2+k)/(1+k)$ belongs to $(6,12]$. □

3. The case $n=4$.

3.1. The TATs for $n=4$. In the case $n=4$ there exist the following triples:

$$D:=(4,0;0),\quad E:=(3,1;0),\quad F:=(2,2;0),$$
$$G:=(2,0;2),\quad H:=(1,1;2),\quad K:=(0,0;4)$$

and $-D$, $-E$, $-G$ (noticing that $F=-F$, $H=-H$ and $K=-K$). The triples $\pm D$, F, $\pm G$ and K are even while $\pm E$ and H are odd. Condition (1) implies that each TAT contains either two odd triples or none of them.

When triples are considered as 3-vectors, a triple corresponding to $n=4$ can be presented as a sum of two triples corresponding to $n=2$. In this sense we use the following addition table:

$$D=A+A,\quad \pm E=\pm A+B,\quad F=B+B=A+(-A),$$
$$G=A+C,\quad H=B+C,\quad K=C+C.$$

We treat one of the cases in detail in the following example. For the rest of the cases we explain only how to present a TAT with $n=4$ as a sum of two TATs with $n=2$. The reasoning in the other cases is completely analogous.

Example 15. Explain how to realize the TAT (F, F, D) for any k fixed. Observe first that this TAT can be considered as a sum of two TATs (B, B, A) (relative to degree 2 polynomials) according to the above addition table. Consider degree 2 generic polynomials P and Q such that the triple P, Q, $P \underset{4+k}{*} Q$ realizes the TAT (B, B, A). (With respect to the degree 2 of P and Q, in their SSC it is the number $k+2$ that is playing the role of k, i.e. it would be more correct to write $P \underset{2+(2+k)}{*} Q$.) Hence this TAT is also realized by the triple P^R/x^{k+2}, Q^R/x^{k+2}, $(P^R \underset{4+k}{*} Q^R)/x^{k+2}$. Indeed, for P^R/x^{k+2} and Q^R/x^{k+2} this follows from part (2) of Remark 8. If $P = x^2 + gx - h$, $Q = x^2 + px - q$, then

$$P \underset{4+k}{*} Q = x^2 + gpx/(4+k) + hq/\binom{4+k}{2} \quad \text{and}$$

$$(P^R \underset{4+k}{*} Q^R)/x^{k+2} = hqx^2 + gpx/(4+k) + 1/\binom{4+k}{2} = ax^2 + bx + c.$$

The discriminants of these polynomials and their coefficients of x are the same, so they define the same triple.

Present the polynomials P^R, Q^R and $P^R \underset{4+k}{*} Q^R$ in the form

$$x^k x^2(x^2 + gx - h) \ \ , \ \ x^k x^2(x^2 + px - q) \ \text{ and } \ x^k x^2(ax^2 + bx + c) \ \text{ respectively.}$$

Deform the first two of them as follows:

$$P^R = x^k(x^2 + \varepsilon g'x - \varepsilon^2 h')(x^2 + gx - h), \qquad Q^R = x^k(x^2 + \varepsilon p'x - \varepsilon^2 q')(x^2 + px - q).$$

Hence each of the polynomials P^R/x^k and Q^R/x^k realizes for $\varepsilon > 0$ the triple F. After the deformation $P^R \underset{4+k}{*} Q^R$ still has two negative roots close to the ones before the deformation.

Make the change of variable $x \mapsto \varepsilon x$. Hence P^R and Q^R become respectively

$$\varepsilon^{k+2}x^k(x^2 + g'x - h')(\varepsilon^2x^2 + \varepsilon gx - h) \quad \text{and} \quad \varepsilon^{k+2}x^k(x^2 + p'x - q')(\varepsilon^2x^2 + \varepsilon px - q)$$

and their SSC $P^R \underset{4+k}{*} Q^R$ takes the form $\varepsilon^{2k+4}x^k(a'(\varepsilon)x^2 + b'(\varepsilon)x + c'(\varepsilon))(\varepsilon^2x^2 + \varepsilon bx + c)$ with

$$a'(0) = hq/\binom{4+k}{2}, \qquad b'(0) = hqg'p'/\binom{4+k}{3} \quad \text{and} \quad c'(0) = hqh'q'/\binom{4+k}{4}.$$

Fixing g' and p' and choosing then h' and q' small enough we obtain that after the deformation $(P^R \underset{4+k}{*} Q^R)/x^k$ has two distinct negative roots close to 0 (the discriminant of the polynomial $a'(0)x^2 + b'(0)x + c'(0)$ is positive and $b'(0) > 0$ hence these conditions hold for nearby positive values of ε as well). Thus the deformed triple of polynomials realizes the TAT (F, F, D).

Remark 16. (1) In the example we had $(F, F, D) = (B, B, A)+(B, B, A)$. We showed how to realize the TAT (F, F, D) if one can realize the TAT (B, B, A). In the same way one can construct triples of polynomials realizing a given TAT T for degree $n \geq 4$ polynomials, if one can construct such triples for two TATs (one for degree $n - 2$ and one for degree 2 polynomials) whose sum is T. In general, these two TATs are different, unlike the case treated in the example, but the reasoning remains exactly the same.

(2) Recall that we present TATs with $n = 4$ and a given k as sums of two TATs with $n = 2$ and $k + 2$ playing the role of k. Hence when the TATs $(C, C, \pm A)$ are involved in the construction, they are used for $k = 2, 3, \ldots$, i.e. the nonrealizable TATs $(C, C, \pm A)$ with $k = 0$ (see Subsection 2.1) are not used.

As in the case $n = 3$, when we consider TATs up to equivalence, we find out that if they begin with DD, DF, DG, DK, FF, FG, FK, GG, GK, KK, EE, EH or HH, then their last triple can be only $\pm D$, F, $\pm G$ or K, and if they begin with DE, DH, EF, EG, EK, FH, GH or HK, then it can be only $\pm E$ or H. The sign of the third triple of a TAT is of no importance if and only if at least one of the triples of the TAT is equal to F, H or K (see Remark 8).

Propositions 17 and 18 explain which TATs beginning with D are realizable and which are not:

Proposition 17. (1) *For $n = 4$ the answer to the question whether a TAT beginning with DD or DE or DF is realizable is given by Proposition* 5.

(2) *The following TATs are realizable:*

$$(D, G, D) \quad (D, G, F) \quad (D, G, G) \quad (D, G, K) \quad (D, H, E)$$
$$(D, H, H) \quad (D, K, D) \quad (D, K, F) \quad (D, K, G) \quad (D, K, K)$$

P r o o f. The first statement of the proposition is self-evident. Consider the TATs beginning with DG. One can write $(D, G, \ ) = (A, A, \ ) + (A, C, \ )$ ignoring for the moment the triples in third position. They can be $F = A+(-A)$ or $G = A + C$ or $-G = -A + C$ or $D = A + A$ or $-D$ or K. The first possibility implies that the TAT $(D, G, F) = (A, A, A) + (A, C, -A)$ is realizable (both TATs (A, A, A) and $(A, C, -A)$ are realizable, see Section 2; the realizability of the TAT (D, G, F) is proved then by analogy with Example 15). Similarly, the TATs $(D, G, G) = (A, A, A) + (A, C, C)$ and $(D, G, D) = (A, A, A) + (A, C, A)$

are realizable. It is shown in [5] that the TAT (D, G, K) is realizable. The TAT $(D, G, -D)$ is not. (If it were, then $(D, -G, D)$ would also be realizable. However by the Descartes rule of signs the realizing polynomials P and $P^*_{n+k}Q$ have no sign change in the sequence of the coefficients, while Q has at least two of them which is impossible.)

Proposition 18. *The TAT $(D, G, -G)$ is not realizable for any $k \geq 0$.*

The proposition is proved in Subsection 3.3. (The equality $(D, G, -G) = (A, A, -A) + (A, C, C)$ is of no use because the TAT $(A, A, -A)$ is not realizable. This however does not imply that the TAT $(D, G, -G)$ is also not realizable – we haven't shown that if a TAT with $n = 4$ is realizable, then its realizability can be proved as in Example 15, by presenting the TAT as a sum of two TATs with $n = 2$.) The rest of the TATs beginning with D are realizable:

$$\begin{array}{ll} (D,H,E) = (A,B,B) + (A,C,A) & (D,H,H) = (A,B,B) + (A,C,C) \\ (D,K,D) = (A,C,A) + (A,C,A) & (D,K,F) = (A,C,A) + (A,C,-A) \\ (D,K,G) = (A,C,A) + (A,C,C) & (D,K,K) = (A,C,C) + (A,C,C)\ . \end{array}$$

Proposition 17 is proved. □

Further when listing realizable TATs we fix their first two triples and change only the last one. In this sense the first of the equalities listed in Proposition 19 expresses each of the three TATs (E, E, D), (E, E, F) and (E, E, G) as a sum of two TATs with $n = 2$.

Proposition 19. *The following table explains why some TATs with $n = 4$ are realizable for any $k \geq 0$:*

$$\begin{array}{lll}
(E,E,\; D\;\; F\;\; G) & = (A,A,\; A\;\; A\;\; A) & + (B,B,\; A\;\; -A\;\; C) \\
(E,F,\; E\;\; H) & = (B,B,\; A\;\; C) & + (A,B,\; B\;\; B) \\
(E,G,\pm E\;\; H) & = (B,A,\; B\;\; B) & + (A,C,\pm A\;\; C) \\
(E,H,\; D\;\; F\;\; G\;\; K) & = (B,B,\; A\;\; A\;\; A\;\; C) & + (A,C,\; A\;\; -A\;\; C\;\; C) \\
(F,F,\; D\;\; F\;\; G\;\; K) & = (B,B,\; A\;\; A\;\; A\;\; C) & + (B,B,\; A\;\; -A\;\; C\;\; C) \\
(F,G,\; D\;\; F\;\; G) & = (A,A,\; A\;\; A\;\; A) & + (-A,C,\; A\;\; -A\;\; C) \\
(F,H,\; E\;\; H) & = (B,B,\; A\;\; C) & + (B,C,\; B\;\; B) \\
(F,K,\; D\;\; F\;\; G\;\; K) & = (A,C,\; A\;\; A\;\; A\;\; C) & + (-A,C,\; A\;\; -A\;\; C\;\; C) \\
(G,G,\pm D\;\; F\;\; \pm G\;\; K) & = (A,C,\pm A\;\; A\;\; \pm A\;\; C) & + (C,A,\pm A\;\; -A\;\; C\;\; C) \\
(G,H,\; E\;\; H) & = (A,C,\; A\;\; C) & + (C,B,\; B\;\; B) \\
(G,K,\; F\;\; G\;\; K) & = (A,C,\; A\;\; A\;\; C) & + (C,C,-A\;\; C\;\; C) \\
(H,H,\; F\;\; G\;\; K) & = (B,B,\; A\;\; A\;\; C) & + (C,C,-A\;\; C\;\; C) \\
(K,K,\; F\;\; K) & = (C,C,\; A\;\; C) & + (C,C,-A\;\; C)
\end{array}$$

The proposition can be proved in the same way as Example 15. The TATs beginning with E, F, G, H or K that are missing on the above list are:

$$(E,E,-D),\quad (E,E,-G),\quad (E,E,K),\quad (F,G,K),$$
$$(G,K,D),\quad (H,H,D),\quad (K,K,D),\quad (K,K,G).$$

Proposition 20. *The first three of them are not realizable while the remaining five are.*

Proof. For the first statement see Theorem 21 below.

To realize the TAT (F,G,K) set $P := (x^2-1)^2 = x^4-2x^2+1$ and $Q := (x+1)^2((x-1)^2+a) = P + a(x^2+2x+1)$ with $a>0$. For $a=0$ the polynomial $P *_{4+k} Q$ is even and with positive coefficients, hence without real roots. For $a>0$ small enough it is also without real roots.

The TAT (G,K,D) is realized by $P := (x+1)(x+2)((x+3)^2+b)$ and $Q := ((x+1)^2+b)((x+2)^2+b)$, where $b>0$ is small enough. By Theorem 11 for $b=0$ (hence for $b>0$ small enough as well) the polynomial $P *_{4+k} Q$ has four negative distinct roots.

To realize the TAT (H,H,D) consider $P := (x+c)(x+1)((x+2)^2+c) =: Q$. For $c=0$ the polynomial $P *_{4+k} Q$ has three negative distinct roots and a root at 0; this follows from formula (3) and Theorem 11. For $c>0$ small enough its constant term is positive hence it has four negative distinct roots.

The TAT (K,K,D) is realized by $P = Q = ((x+1)^2+d)((x+2)^2+d)$. (By Theorem 11, for $d=0$ the polynomial $P *_{4+k} Q$ has four distinct negative roots, hence this is the case for $d>0$ small enough as well.)

To realize (K,K,G) set first $P = Q = (x+\varepsilon)^2(x^2+1)$. Hence $P *_{4+k} Q$ equals

$$\frac{x^4}{\binom{4+k}{4}} + \frac{4\varepsilon^2 x^3}{\binom{4+k}{3}} + \frac{(1+\varepsilon^2)^2 x^2}{\binom{4+k}{2}} + \frac{4\varepsilon^2 x}{4+k} + \varepsilon^4$$
$$= (\kappa x^2 + \tau\varepsilon^2 x)^2 + \frac{(1+2\varepsilon^2+\rho\varepsilon^4)x^2}{\binom{4+k}{2}} + \frac{4\varepsilon^2 x}{4+k} + \varepsilon^4.$$

with $\kappa>0$, $\tau>0$ and $\rho\in\mathbf{R}$ readily defined. The right-hand side is the sum of a quadratic polynomial in x and of the square of such. Hence it has not more than two real roots. As

$$(P *_{4+k} Q)(0) = \varepsilon^4 > 0 \quad\text{and}\quad (P *_{4+k} Q)(-(3+k)\varepsilon^2) = -(2+k)\varepsilon^4/(4+k) + o(\varepsilon^4) < 0$$

for $\varepsilon > 0$ small enough, $P_{4+k}^{\,*}Q$ has exactly two negative distinct roots (see the last displayed formula). Fix $\varepsilon > 0$ and choose $0 < \varepsilon' \ll \varepsilon$ such that for $P = Q = ((x+\varepsilon)^2 + \varepsilon')(x^2+1)$, $P_{4+k}^{\,*}Q$ still defines the triple G. $\square$

The following theorem (proved in the next subsection) contains a statement which holds true for any $n \geq 2$.

Theorem 21. *Suppose that the generic hyperbolic degree n polynomials P and Q define the triples $U := (g, l; 0)$ and $V := (n-1, 1; 0)$. Then $P_{n+k}^{\,*}Q$ defines one of the triples $W_1 := (g-1, l+1; 0)$, $W_2 := (g+1, l-1; 0)$ or $W_3 := (g-1, l-1; 2)$. All three TATs (U, V, W_i), $i = 1, 2, 3$, are realizable for any $k \geq 0$.*

3.2. Proof of Theorem 21. We prove the theorem for $k = 0$. For $k \in \mathbf{N}$ it results from part (3) of Proposition 5.

Set $U' := (g-1, l-1; 0)$, $V' := (n-2, 0; 0)$, $T := (U', V', U')$. The TAT T is realizable by generic degree $n-2$ polynomials, see part (2) of Proposition 5. The realizability of the three TATs (U, V, W_i) follows from the equalities

$$(U, V, W_1) = T + (B, B, -A), \qquad (U, V, W_2) = T + (B, B, A) \quad \text{and} \quad (U, V, W_3) = T + (B, B, C)$$

and from the method of constructing polynomials P and Q realizing a given TAT as explained in Example 15. Set $Q := (x-a)S$, $\deg S = n-1$, $a > 0$ (all roots of S are negative). Hence

$$P\,{}_n^{*}\,Q = \frac{x}{n}(P'\,{}_{n-1}^{\;*}S) - a(P\,{}_n^{*}\,S) = \frac{x}{n}\left(P'\,{}_{n-1}^{\;*}S\right) - a\left(\left(P - \frac{x}{n}P'\right)_{n-1}^{*} S\right).$$

(The first equality follows from formula (3), the second from formula (4)). Consider the one-parameter family of polynomials

$$T(x,\lambda) := \frac{\lambda}{n}(P'\,{}_{n-1}^{\;*}S) - a\left(\left(P - \frac{x}{n}P'\right)_{n-1}^{*} S\right) = -\left(\left(\frac{\lambda}{n}P' + a\left(P - \frac{x}{n}P'\right)\right)_{n-1}^{*} S\right),$$

where $\lambda \leq 0$. Hence $P\,{}_n^{*}\,Q = T(x,x)$. When $|\lambda|$ is large enough (say, $\lambda \leq \lambda_0$), the roots of $T(x,\lambda)$ are close to the ones of $P'\,{}_{n-1}^{\;*}S$. Among them there are at least $g-1$ negative and at least $l-1$ positive ones (because this is the case of the roots of P', see part (2) of Proposition 5); they belong all to some closed interval

J_0 (for all $\lambda \leq \lambda_0$). The last root is also real and its sign depends on the signs of the leading coefficient and the constant term of $T(x, \lambda)$. It also belongs to J_0.

All coefficients of $T(x, \lambda)$ are linear functions in λ. Hence there exists at most one value of λ (denoted by λ_1) for which the leading coefficient of $T(x, \lambda)$ changes sign. When λ passes through this value, a root of $T(x, \lambda)$ changes sign by passing through ∞. Hence for λ close to λ_1 (say, $\lambda \in I_1 := [\lambda_1 - \delta_1, \lambda_1 + \delta_1]$, $\delta_1 > 0$) the other roots remain in some closed interval J_1.

There exists at most one value of λ (denoted by λ_2) for which the constant term of $T(x, \lambda)$ changes sign. When λ passes through this value, a root of $T(x, \lambda)$ changes sign by passing through 0. Hence for λ close to λ_2 (say, $\lambda \in I_2 := [\lambda_2 - \delta_2, \lambda_2 + \delta_2]$, $\delta_2 > 0$) all roots of $T(x, \lambda)$ (without exception) remain in some closed interval J_2.

Finally, for $\lambda \in \overline{[\lambda_0, 0] \backslash (I_1 \cup I_2)}$ all roots of $T(x, \lambda)$ remain in some closed interval J_3. Thus the closed interval $J := J_0 \cup J_1 \cup J_2 \cup J_3$ contains at least $n-2$ roots of $T(x, \lambda)$. More exactly, as the roots depend continuously on λ, one can assign to each root an index. The interval J contains at least $n-2$ roots, with one and the same indices, for all $\lambda \leq 0$. The missing index is of the root that changes sign by passing through ∞.

The roots of $T(x, \lambda)$ depending continuously on λ, they can be viewed as graphs of continuous functions defined on $(-\infty, 0]$. The half-line $\{x = \lambda, \lambda \leq 0\}$ (in the plane (x, λ)) meets at least $g-2$ of these graphs. Hence $P \overset{*}{_n} Q$ has at least $g-2$ negative distinct roots. One can consider instead of $P \overset{*}{_n} Q$ the polynomial $P(-x) \overset{*}{_n} Q$ and conclude in the same way that it has at least $l-2$ negative distinct roots, hence $P \overset{*}{_n} Q$ has at least $l-2$ positive distinct ones.

There is a root of $P \overset{*}{_n} Q$ whose sign depends on the sign of the constant term and which remains for all $\lambda \leq 0$ in J. The half-line $\{x = \lambda, \lambda \leq 0\}$ intersects the graph of its absolute value taken with the minus sign which implies the existence of at least $g + l - 3 = n - 3$ distinct real roots of $P \overset{*}{_n} Q$. Hence, of at least $n-2$ of them, i.e. $P \overset{*}{_n} Q$ has at most one complex conjugate pair.

The quantity of positive (resp. negative) roots is at least $l-2$ (resp. $g-2$). Using condition (1) (for $P \overset{*}{_n} Q$ and $P(-x) \overset{*}{_n} Q$) one concludes that it can be $l-1$ or $l+1$ (resp. $g-1$ or $g+1$). (It cannot be $l+3$ (resp. $g+3$) because $l+3+g-2 > n$ (resp. $l-2+g+3 > n$).) This gives as possibilities only the three TATs claimed by the theorem. $\square$

3.3. Proof of Proposition 18. The proof is based on the same ideas as the ones used in the proof of Theorem 21. It is carried out only for $k = 0$. Suppose that the polynomial $P = (x+a)S$ has all roots negative and distinct. Here $-a < 0$ is one of the roots and $\deg S = 3$. Suppose that the polynomial Q

defines the triple $G = (2, 0; 2)$. Denote by $\alpha < \beta < 0$ the real negative roots of Q.

We show that the polynomial $P \, {}^*_4 \, Q$ has at least one negative root. For $k \in \mathbf{N}$ it follows then from part (3) of Proposition 5 that this is true also for $P^*_{4+k}Q$. One has (see formulas (3) and (4))

$$P \, {}^*_4 \, Q = (x+a) S \, {}^*_4 \, Q = \frac{x}{4} \left(S \, {}^*_3 \, Q'\right) + a \left(S \, {}^*_3 \, \left(Q - \frac{x}{4} Q'\right)\right).$$

Consider the one-parameter family of polynomials (depending on $\lambda \leq 0$)

$$L(x, \lambda) := \frac{\lambda}{4} \left(S \, {}^*_3 \, Q'\right) + a \left(S \, {}^*_3 \, \left(Q - \frac{x}{4} Q'\right)\right).$$

Obviously, $P \, {}^*_4 \, Q = L(x, x)$. We assume without loss of generality that all coefficients of Q are nonzero. (Indeed, if we suppose that the TAT $(D, G, -G)$ is realizable by some generic polynomials P and Q, then it is realizable by all nearby couples of polynomials as well.) Observe that all coefficients of P and S are positive. Set $K(x, \lambda) := \dfrac{\lambda}{4} Q' + a(Q - (x/4)Q')$. We consider two cases:

Case 1. $Q'(0) < 0$. The quantity $K(0, \lambda)$ is a linear function in λ. If $Q'(0) < 0$, then it is positive for $\lambda \leq 0$ (this follows from $Q(0) > 0$). On the other hand $K(\zeta, \lambda) = aQ(\zeta) < 0$, where ζ is the root of Q' belonging to (α, β). Hence $K(., \lambda)$ has a root $\nu \in (\zeta, 0)$.

For $|\lambda|$ large enough (say $|\lambda| \geq h_0$), $K(x, \lambda)$ is a degree 3 polynomial with negative leading coefficient. Hence except ν it has a positive root and another negative root $\kappa \in (-\infty, \zeta)$; the root ν remains bounded for $|\lambda| \geq h_0$.

For $\lambda \leq 0$ and close to 0, $K(x, \lambda)$ has a leading coefficient of the same sign as the coefficient w of x^3 in Q. If the latter is negative, then the leading coefficient of $K(x, \lambda)$ is negative for $\lambda \leq 0$ and $K(x, \lambda)$ has two negative and one positive roots. The root ν is bounded for $\lambda \leq 0$.

If $w > 0$ (and hence the leading coefficient w' of $K(x, \lambda)$ changes sign for some $\lambda = \tilde{\lambda}$), then the leftmost root of $K(x, \lambda)$ goes to $-\infty$ for $\lambda = \tilde{\lambda} - 0$ and then a positive root emerges from $+\infty$ for $\lambda = \tilde{\lambda} + 0$. As w' is a linear function in λ, such a sign change can take place only once. As $K(0, \lambda) > 0$ for $\lambda \geq 0$, the root $\nu < 0$ is well-defined, continuous and bounded in $\lambda \leq 0$.

The respective coefficients of the degree 3 polynomials $L(x, \lambda)$ and $K(x, \lambda)$ have the same signs. Hence $L(x, \lambda)$ has also a root $\nu^* < 0$, well-defined, continuous and bounded in $\lambda \leq 0$.

Hence in the plane (x, λ) the half-line $\{x = \lambda, \lambda \leq 0\}$ intersects the graph of ν^* as a function of λ (say for $\lambda = \lambda^* < 0$). This means that $L(\lambda^*, \lambda^*) = 0$. Hence the triple defined by $L(x, x)$ is not $-G$.

Case 2. $Q'(0) > 0$. As in Case 1 one sees that if $w < 0$, then the leading coefficient of $K(x, \lambda)$ remains negative for $\lambda \leq 0$. For $\lambda = -aQ(0)/Q'(0)$ the root ν becomes 0 and for $\lambda \in (-aQ(0)/Q'(0), 0]$ it is positive. Hence the root κ is negative and bounded for all $\lambda \leq 0$ and $L(., \lambda)$ has for all $\lambda \leq 0$ a root $\kappa^* < 0$, continuous as a function in λ and bounded. The half-line $\{x = \lambda, \lambda \leq 0\}$ intersects the graph of κ^* for some $\lambda = \lambda^{**} < 0$ which means that $L(\lambda^{**}, \lambda^{**}) = 0$, i.e. $L(x, x)$ does not define the triple $-G$.

If $w > 0$, then all coefficients of Q (hence of $P \mathop{*}_4 Q$) except the one of x^2 are positive. If $x_0 > 0$, then $(P \mathop{*}_4 Q)(x_0) > (P \mathop{*}_4 Q)(-x_0)$. Suppose that $P \mathop{*}_4 Q$ defines the triple $-G$. Then it has a couple of distinct positive roots and if x_0 is between them, then $(P \mathop{*}_4 Q)(x_0) < 0$. Hence $(P \mathop{*}_4 Q)(-x_0) < 0$, i.e. $P \mathop{*}_4 Q$ must have also two negative roots (because $(P \mathop{*}_4 Q)(0) > 0$). This shows that $P \mathop{*}_4 Q$ does not define the triple $-G$ – a contradiction. $\square$

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Université de Nice
Laboratoire de Mathématiques
Parc Valrose
06108 Nice Cedex 2, France
e-mail: `kostov@math.unice.fr`

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