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ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATORS ON L^1 MÜNTZ SPACES

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Abstract. This paper discusses the problem of boundedness and compactness for weighted composition operators defined on a Müntz subspace of $L^1([0,1])$. We compute the essential norm of such operators when the symbol φ of the composition operator satisfies a special condition (condition $(\mathcal{B})$). As a corollary, we obtain the exact values of essential norms of composition and multiplication operators. This completes the corresponding results of the first named author in the framework of Müntz subspaces of $C([0,1])$.

1. Introduction and notations. Throughout the paper, $L^1 = L^1([0,1])$ denotes the Banach space of complex-valued measurable functions on $[0,1]$ with the norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|dx < \infty$. In the whole paper φ denotes a measurable self-map of $[0,1]$, we set $E_\varphi = \varphi^{-1}(\{1\})$. The composition operator

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C_φ is defined by $C_\varphi(f) = f \circ \varphi$. Given $\psi \in L^\infty([0,1])$, we shall also consider the multiplication operator $\mathcal{T}_\psi$ defined by $\mathcal{T}_\psi(f) = f \cdot \psi$.

The essential norm of an operator T is its distance to the space of compact operators and is denoted by: $\|T\|_e = \inf \|T - S\|$ where S runs over the class of compact operators.

Let Λ be an increasing sequence of positive numbers satisfying $\sum_{\lambda\in\Lambda} 1/\lambda < \infty$ and consider the closed space M^1_Λ, spanned by the monomials 1 and x^λ, where $\lambda \in \Lambda$. By the famous theorem of Müntz, M^1_Λ is not all of L^1. Except in Prop.2.1. stating Müntz's theorem, we shall assume that the condition $\sum_{\lambda\in\Lambda} 1/\lambda < \infty$ is fulfilled.

In this paper, we show that for functions φ satisfying some specific conditions (for instance condition $(\mathcal{B})$, see definition below), the composition operator C_φ from M^1_Λ to L^1 is well-defined. Under that condition, our main result gives a precise estimate of the essential norm of $\mathcal{T}_\psi \circ C_\varphi$ acting on M^1_Λ in terms of the values of φ and ψ. As a corollary we deduce the exact value of the essential norm of C_φ acting on M^1_Λ, and that the essential norm of $\mathcal{T}_\psi$ (associated to a function ψ, continuous at point 1) acting on M^1_Λ is $|\psi(1)|$.

To know more on the geometry of Müntz spaces, see the monographs of Gurariy and Lusky [9], P. Borwein and T. Erdélyi [4] (see also [1, 2]).

The present work extends some results of the first named author [3]. Several papers appeared recently related to this topic: let us mention [2], [5], [8] and more recently [13]. It is worth mentioning especially the results of [5]: the authors obtain there some interesting and sharp results in the framework of L^1, but in a slightly different direction (they study Carleson's type embeddings). Hence these results are rather distinct from ours, although some of the results of [3] are partially recovered in [5].

2. Preliminary results. In this section we recall some properties of the geometry of Müntz spaces, which we shall use later. We list them as propositions. The Müntz spaces have appeared naturally posterior to Müntz's theorem in 1914 (see [12]) which characterizes a sequence $\Lambda = (\lambda_n)_n$ so that the closed span M^∞_Λ of the monomials 1, x^λ, where $\lambda \in \Lambda$, is not all of $C([0,1])$. The next proposition is an L^1 version of the Müntz Theorem.

Proposition 2.1 ([9, p. 180]). *Let* $\Lambda = (\lambda_k)_{k=0}^\infty$, *where* $0 = \lambda_0 < \lambda_1 < \cdots$, *be an increasing sequence of nonnegative real numbers.*

Then the Müntz space $M(\Lambda) = \mathrm{span}\{x^{\lambda_k} : k = 0, 1, \dots\}$, *associated to* Λ, *is a dense subset of* L^1 *if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

Moreover, if $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$ *and if* $\lambda \notin \Lambda$, *then* $x^\lambda \notin M^1_\Lambda$.

Thanks to this result and since our framework is Müntz proper subspaces of L^1, *we shall assume in the sequel of the paper that the condition* $\sum_{\lambda \in \Lambda \setminus \{0\}} 1/\lambda < \infty$ *is fulfilled.*

The next proposition due to Clarkson and Erdös [6] (see also [9], p.81) and Schwartz [15, 16], gives us a characterization of Müntz spaces which reveals both the originality and richness of these spaces, see also [7] for the full version of this proposition.

Proposition 2.2 ([6, 15, 16]). *Assume the gap condition* $\inf\{\lambda_{k+1} - \lambda_k : k \in \mathbb{N}\} > 0$ *holds. Then, for every function* $f \in L^1$ *we have:*

The function f *belongs to* M^1_Λ *if and only if there exists a sequence* $(a_k)_{k \in \mathbb{N}}$ *such that, for every* $x \in [0, 1)$, *we have* $f(x) = \sum_{k=0}^{\infty} a_k x^{\lambda_k}$.

If the gap condition does not hold, then every function $f \in L^1$ *belonging to the closure of span*$\{x^{\lambda_k}; k = 0, 1, \dots\}$ *can still be represented as an analytic function on* $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$ *restricted to* $(0, 1)$.

Note that the preceding two propositions are still valid for M^∞_Λ (respectively M^p_Λ, $1 \le p < \infty$) the closure of $M(\Lambda)$ in $C[0, 1]$ (respectively $L^p[0, 1]$) and were first proved for the case of M^∞_Λ.

In the sequel, we shall write $\|p\|_K = \sup_{t \in K} |p(t)|$, where $K \subset [0, 1)$ is compact.

Proposition 2.3 (See [4, p. 185, E.8.a]). *For every* $\varepsilon \in (0, 1)$, *there is a constant* $\gamma(\varepsilon, \Lambda)$ *depending only on* ε *and* $(\lambda_i)_{i=0}^{\infty}$ *such that*

$$\|p\|_{[0,1-\varepsilon]} \le \gamma(\varepsilon, \Lambda) \int_{1-\varepsilon}^{1} |p(x)| dx$$

for every $p \in$ *span*$\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Proposition 2.3 (Bounded Bernstein-Type Inequality. See [4, p. 178, E.3.d.]). *For every* $\varepsilon \in (0,1)$*, there is a constant* c_ε *depending only on* ε*, and* $(\lambda_i)_{i=0}^\infty$ *(but not on the number of terms in* p*) such that*

$$\|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon \|p\|_{L_1}$$

for every $p \in span\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$*.*

Actually the version given in [4] uses a majorization with $\|p\|_{L_2}$. Nevertheless, it is easy to adapt the proof to obtain the version given above.

Combining Proposition 2.3 and the Arzela-Ascoli theorem (see, for example, [14]), we deduce the next useful corollary.

Corollary 2.5. *Given a sequence* $(f_n)_{n\geq 1}$ *in the unit ball of* M^1_Λ*, there is a subsequence of* $(f_n)_{n=1}^\infty$ *uniformly converging on every compact subset of* $[0,1)$*.*

Proof. Let $(f_n)_{n=1}^\infty \subset \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ such that $\|f_n\|_1 \leq 1$. Let $\varepsilon > 0$. By the preceding proposition, $(f_n)_n$ is bounded and equicontinuous on $[0, 1-\varepsilon]$. Then by the Arzela-Ascoli theorem, it has a uniformly convergent subsequence on $[0, 1-\varepsilon]$.
By induction, we construct infinite sets S_n of integers, $\mathbb{N} \supset S_1 \supset S_2 \supset \cdots$, such that $(f_n)_n$ converges uniformly on $\left[0, 1 - \frac{1}{j}\right]$ when $n \to \infty$ in S_j. Now using the diagonal process, we obtain an infinite set S such that $(f_n)_n$ converges uniformly on every compact subset of $[0,1[$ when $n \to \infty$ in S. $\square$

Corollary 2.6. *Let* $(f_n)_{n=1}^\infty \subset M^1_\Lambda$ *be a convergent sequence to* f*, then* $(f_n)_n$ *converges uniformly to* f *on every compact subset of* $[0,1[$*.*

3. Bounded operators. In this section, we consider the composition operators defined on Müntz spaces M^1_Λ. Recently (see [2]), the first named author studied these operators acting on M^∞_Λ and gave a precise estimate of the essential norm of weighted composition operators acting on M^∞_Λ in terms of the values of φ and ψ (see Theorem 5.1, [2]). A wide literature is interested in these operators. They were studied in the case of Banach spaces like Hardy spaces, Bergman spaces, Bergman-Orlicz spaces and Hardy-Orlicz spaces studied in [10],[11]. We are interested in the continuity, compactness and the computation of essential norm of these operators.

In general a composition operator does not map a Müntz space into itself (actually, except in very special cases, it nearly never happens). For this reason we shall consider operators from Müntz spaces to the whole space $L^1([0,1])$. It turns out that a Müntz space is mapped (via a composition operator) into another Müntz space. This phenomenon is specified in [2] (see Lemma 3.1, 3.2, 3.3 and Theorem 3.4). These results were proved for Müntz spaces M_Λ^∞ but are still valid on M_Λ^1.

We first give some simple examples of smooth functions φ with various behavior.

Example 3.1. If φ is $\mathcal{C}^1$-diffeomorphism from $[0,1]$ (onto itself), the operator C_φ is bounded and satisfies:

$$\|C_\varphi(f)\|_1 \leq \|1/\varphi'\|_\infty \|f\|_1 \qquad \text{and} \quad \|C_\varphi(f)\|_1 \geq \frac{\|f\|_1}{\|\varphi'\|_\infty}.$$

Clearly C_φ is not a compact operator.

Example 3.2. Let $\varphi_0(t) = 1 - t$. Then C_{φ_0} is an isometry. Indeed:

$$\|C_{\varphi_0}(f)\|_1 = \int_0^1 |f(1-t)|\, dt = \int_0^1 |f(u)| du = \|f\|_1$$

(we could also observe that this follows from the preceding remark as well).

Proposition 3.3. *If $C_\varphi : M_\Lambda \to L^1$ is well defined on M_Λ^1, then $m_\varphi(\{1\}) = 0$, where m_φ is the pull back measure of the Lebesgue measure m associated to φ.*

Proof. Consider the function $f(x) = \sum_{n=1}^{\infty} x^{\lambda_n}$. Thanks to the Müntz condition on Λ, we have

$$\sum_{n=1}^{\infty} \|x^{\lambda_n}\|_1 = \sum_{n=1}^{\infty} \frac{1}{\lambda_n + 1} < \infty$$

hence $f \in M_\Lambda^1$.

Suppose that $C_\varphi : M_\Lambda^1 \mapsto L^1$, then $\|C_\varphi(f)\|_1 < \infty$. On the other hand, we have

$$\|C_\varphi(f)\|_1 \geq \int_{\{\varphi^{-1}(\{1\})\}} \sum_{n=1}^{\infty} \varphi(x)^{\lambda_n} dx = \infty . m_\varphi(\{1\}).$$

This requires that $m_\varphi(\{1\}) = 0$. □

Generally, the condition $m_\varphi(\{1\}) = 0$ is not sufficient. In fact, even the condition $\operatorname{card}\varphi^{-1}(\{1\}) < \infty$ is not sufficient to get that C_φ is well-defined: this follows from Example 3.7 and Lemma 3.6.

Lemma 3.4. *Let $\varphi : [0,1] \longrightarrow [0,1]$ be measurable and $C_\varphi : M^1_\Lambda \longrightarrow L^1$, then C_φ is bounded as soon as C_φ is defined.*

Proof. We shall prove that the graph of C_φ is closed.
Let (f,h) belonging to the closure of the graph of C_φ. There exists a sequence $(f_j)_j \subset M^1_\Lambda$ such that $(f_j)_j$ converges to f and $(C_\varphi(f_j))_j$ converges to h. According to the Corollary 2.6, $(f_j)_j$ converges uniformly to f on every compact subset of $[0,1[$, which implies that $(C_\varphi(f_j)(x))_j$ converges to $C_\varphi(f)(x)$ for every $x \in \varphi^{-1}([0,1[)$. From the above (Proposition 3.3), $m(\varphi^{-1}\{1\}) = 0$, hence $(f_j \circ \varphi)_j$ converges to $f \circ \varphi$ almost everywhere on $[0,1]$. Therefore $h = f \circ \varphi$ (in the space L^1) and the graph of C_φ is closed. $\square$

Lemma 3.5. *Let $\varphi : [0,1] \longrightarrow [0,1]$ be measurable.*

Let us assume that $\|\varphi\|_\infty < 1$. Then $C_\varphi : M^1_\Lambda \longrightarrow L^\infty([0,1]) \subset L^1([0,1])$ is nuclear.

Proof. The crucial point is the following. Thanks to the Clarkson-Erdös theorem, every function $f \in M^1_\Lambda$ admits a Taylor expansion

$$f(x) = \sum_{n\geq 0} \alpha_n(f) x^{\lambda_n}$$

where $x \in [0,1)$ and $\alpha_n(f)$ is uniquely defined.

Let us fix $n \in \mathbb{N}$. The functional $\alpha_n : f \in M^1_\Lambda \longrightarrow \alpha_n(f) \in \mathbb{C}$ is bounded: for instance, thanks to the heart of the proof of the Clarkson-Erdös theorem, for each $t \in [0,1)$, there exists some $C_t > 0$ such that $|\alpha_n(f)| t^{\lambda_n} \leq C_t \|f\|_1$ for every $n \in \mathbb{N}$ and every $f \in M^1_\Lambda$. In particular, fixing $t \in \big(\|\varphi\|_\infty, 1\big)$, there exists $C > 0$ such that for every $n \in \mathbb{N}$: $|\alpha_n(f)| \leq C \|f\|_1 t^{-\lambda_n}$.

Now, we can write: $C_\varphi(f) = \sum_{n\geq 0} \alpha_n(f) \varphi^{\lambda_n}$ with

$$\sum_{n\geq 0} \|\alpha_n\|_{(M^1_\Lambda)^*} \|\varphi^{\lambda_n}\|_\infty \leq \sum_{n\geq 0} C \left(\frac{\|\varphi\|_\infty}{t}\right)^{\lambda_n} < \infty. \qquad \square$$

Lemma 3.6. *Let $\varphi : [0,1] \longrightarrow [0,1]$ such that the composition operator C_φ maps M^1_Λ to L^1. Assume that $\varphi(\alpha) = 1$ where $\alpha \in [0,1]$.*

a. If $\alpha \in [0,1)$ then $\displaystyle\limsup_{\substack{t\to\alpha \\ t>\alpha}} \frac{1-\varphi(t)}{t-\alpha} > 0.$

b. If $\alpha \in (0,1]$ *then* $\limsup\limits_{\substack{t\to\alpha\\ t<\alpha}} \dfrac{1-\varphi(t)}{\alpha - t} > 0$.

c. In particular φ *is differentiable at no point of* $\varphi^{-1}(\{1\}) \cap (0,1)$.

Proof. We only have to prove the first item (the second is similar and the last one easily follows from *a.* and *b.*).

Assume that the conclusion does not hold: for every $\varepsilon \in (0,1)$ there exists $a \in (\alpha, 1)$ such that

$$\forall t \in (\alpha, a) \qquad 1 - \varphi(t) \le \varepsilon(t-\alpha).$$

Then for every integer $j \ge 1$, we have

$$\begin{aligned}\|C_\varphi\| &\ge \int_0^1 (\lambda_j+1)|\varphi(x)|^{\lambda_j}\,dx \ge \int_\alpha^a (\lambda_j+1)\big(1-\varepsilon(x-\alpha)\big)^{\lambda_j}\,dx\\ &= \frac{1}{\varepsilon}\Big(1-\big(1-\varepsilon(a-\alpha)\big)^{\lambda_j+1}\Big).\end{aligned}$$

For j large enough we get $\|C_\varphi\| \ge \dfrac{1}{2\varepsilon}$ which contradicts the hypothesis of boundedness of C_φ. $\square$

Example 3.7. *The following remark shows that some simple smooth maps* φ *do not necessarily define a bounded operator* C_φ *on* M^1_Λ. *For instance, the operator associated to the symbol* $\varphi(x) = 1-(1-x)^2$ *does not induce a bounded operator:* φ *"touches" the "delicate" end point* 1 *only when* $x = 1$ *(but too smoothly since* $\varphi'(1) = 0$*).*

Remark 3.8. Let $\varphi : [0,1] \longrightarrow [0,1]$ be a differentiable function. Assume that $C_\varphi : M^1_\Lambda \longrightarrow L^1$ is bounded. Then

$$\varphi^{-1}(\{1\}) \subset \{0,1\} \ \text{ with } \ [\varphi(x_0) = 1 \Longrightarrow \varphi'(x_0) \ne 0].$$

Indeed, since C_φ is a bounded operator and 1 belongs to the range of φ, the first conclusion follows from Lemma 3.6.

In the sequel, we concentrate our attention on weighted composition operators with a specific condition which shall ensure the boundedness of the associated composition operators.

Let us precise our framework. In the sequel, for convenience, we recall that we denote $\varphi^{-1}(\{1\})$ by E_φ. The following condition on φ is a smoothness condition.

Definition 3.9. *Let $\varphi : [0,1] \longrightarrow [0,1]$ be a measurable function. We say that φ satisfies condition $(\mathcal{R})$, if*

- *For every $x \in \varphi^{-1}(\{1\})$, the (restricted) functions $\varphi_{|[x,1)}$ and $\varphi_{|(0,x]}$ are C^1 at the point x, with $\varphi'_l(x) > 0$ and $\varphi'_r(x) < 0$*
- $\sup \varphi(K) < 1$ *for every closed subset K of $[0,1] \setminus E_\varphi$, where $K = [0,1] \setminus \Omega$, with $\Omega = \bigcup_{\varphi(x)=1} (x - \varepsilon_x, x + \varepsilon_x)$,*

where $\varphi'_r(x)$ and $\varphi'_l(x)$ stand for the right and left derivative of φ at the point x.

The first condition implies that the set E_φ is a discrete subset of the compact $[0,1]$, hence is at most countable (finite when φ is continuous). A fortiori, it has zero Lebesgue measure, which is a necessary condition to ensure boundedness (recall Proposition 3.3).

On the other hand, when φ is continuous the second condition is clearly irrelevant.

The next Theorem gives a characterization (under condition $(\mathcal{R})$) to obtain a bounded weighted composition operators on Müntz subspaces of L^1. We shall use the following function, associated to φ, verifying

$$L(x) = \begin{cases} \dfrac{1}{\varphi'_l(x)} + \dfrac{1}{|\varphi'_r(x)|} & \text{if } x \in E_\varphi \cap]0,1[\\[2ex] \dfrac{1}{\varphi'_l(1)} & \text{if } x = 1 \in E_\varphi \\[2ex] \dfrac{1}{|\varphi'_r(0)|} & \text{if } x = 0 \in E_\varphi. \end{cases}$$

Theorem 3.10. *Let $\varphi : [0,1] \longrightarrow [0,1]$ satisfying condition $(\mathcal{R})$ and $\psi \in L^\infty$ which is continuous at each point of E_φ. Then*

$$\mathcal{T}_\psi \circ C_\varphi : M^1_\Lambda \longrightarrow L^1 \text{ is bounded if and only if } \sum_{x \in E_\varphi} |\psi(x)| L(x) \text{ converges.}$$

Proof. Let us first assume the boundedness of $\mathcal{T}_\psi \circ C_\varphi$. The sequence $((\lambda_n + 1)x^{\lambda_n})_n$ belongs to the unit ball of M^1_Λ. For each $x \in E_\varphi \cap (0,1)$, according to condition $(\mathcal{R})$, and the continuity of ψ at x, there exists $\varepsilon_x > 0$ such that

$0 < \varphi'(t) < 2\varphi_l'(x)$ for every $t \in [x-\varepsilon_x, x]$; $0 < |\varphi'(t)| < 2|\varphi_r'(x)|$ for every $t \in [x, x+\varepsilon_x]$ and $|\psi(t)| \geq \dfrac{1}{2}|\psi(x)|$ for every $t \in [x-\varepsilon_x, x+\varepsilon_x]$. Moreover, we can assume that the intervals $[x-\varepsilon_x, x+\varepsilon_x]$ are (pairewise) disjoint.

Fix a finite subset E of $E_\varphi \cap (0,1)$. Then we have

$$\begin{aligned}
\|(\lambda_n+1)\psi.\varphi^{\lambda_n}\|_1 &\geq \sum_{x\in E}\int_{x-\varepsilon_x}^{x+\varepsilon_x}(\lambda_n+1)\varphi(t)^{\lambda_n}|\psi(t)|\,dt \\
&\geq \sum_{x\in E}\frac{1}{2}|\psi(x)|\int_{x-\varepsilon_x}^{x+\varepsilon_x}(\lambda_n+1)\varphi(t)^{\lambda_n}\,dt \\
&\geq \sum_{x\in E}|\psi(x)|\Bigg(\frac{1}{4\varphi_l'(x)}\int_{x-\varepsilon_x}^{x}(\lambda_n+1)\varphi(t)^{\lambda_n}\varphi'(t)dt \\
&\qquad + \frac{1}{4|\varphi_r'(x)|}\int_{x}^{x+\varepsilon_x}(\lambda_n+1)\varphi(t)^{\lambda_n}\varphi'(t)dt\Bigg) \\
&\geq \sum_{x\in E}|\psi(x)|\Bigg(\frac{1}{4\varphi_l'(x)}\Big(1-\varphi(x-\varepsilon_x)^{\lambda_n+1}\Big) \\
&\qquad + \frac{1}{4|\varphi_r'(x)|}\Big(1-\varphi(x+\varepsilon_x)^{\lambda_n+1}\Big)\Bigg).
\end{aligned}$$

Letting n going to the infinity, we obtain

$$\|\mathcal{T}_\psi \circ C_\varphi\| \geq \sum_{x\in E}|\psi(x)|\Bigg(\frac{1}{4\varphi_l'(x)} + \frac{1}{4|\varphi_r'(x)|}\Bigg).$$

Since E is arbitrary, we have the conclusion.

Now, we suppose that $\sum_{x\in E_\varphi}|\psi(x)|L(x)$ converges. By assumption, for every $x \in E_\varphi$, there exists $\varepsilon_x \in (0,1/2)$ such that

$$\forall t \in (x-\varepsilon_x, x)\cap[0,1]\,, \quad \frac{\varphi_l'(x)}{2} \leq \varphi'(t) \leq 2\varphi_l'(x)$$

and

$$\forall t \in (x, x+\varepsilon_x)\cap[0,1]\,, \quad \frac{\varphi_r'(x)}{2} \geq \varphi'(t) \geq 2\varphi_r'(x).$$

We fix a summable (countable) family of positive numbers u_x, indexed by E_φ and write $A_x = u_x(\varphi_l'(x) + |\varphi_r'(x)|) > 0$.

Let $\Omega = \bigcup_{x\in E_\varphi} (x-\varepsilon_x, x+\varepsilon_x)$. We can choose ε_x small enough to ensure that this is a union of disjoint subsets and that the oscillation of ψ on $(x-\varepsilon_x, x+\varepsilon_x)$ is less than A_x. Now, $K = [0,1]\setminus\Omega$ is a closed set disjoint from E_φ, hence $M = \max\varphi(K) < 1$. Using Proposition 2.3, we obtain

$$\begin{aligned}\|\psi.C_\varphi(f)\|_1 &\le \|\psi\|_\infty.\|f\|_{[0,M]} + \sum_{x\in E_\varphi}\left(\int_{x-\varepsilon_x}^{x+\varepsilon_x} |\psi(t)|.|f(\varphi(t))|\,dt\right)\\ &\le \gamma(M)\|\psi\|_\infty\,\|f\|_1 + \sum_{x\in E_\varphi}\left(\frac{2(|\psi(x)|+A_x)}{\varphi_l'(x)}\int_{\varphi(x-\varepsilon_x)}^1 |f(u)|\,du\right.\\ &\qquad\left. + \frac{2(|\psi(x)|+A_x)}{|\varphi_r'(x)|}\int_{\varphi(x+\varepsilon_x)}^1 |f(u)|\,du\right).\end{aligned}$$

Thus $\mathcal{T}_\psi\circ C_\varphi$ is bounded and

$$\|\mathcal{T}_\psi\circ C_\varphi\| \le \gamma(M)\|\psi\|_\infty + 2\sum_{x\in E_\varphi}|\psi(x)|\left(\frac{1}{\varphi_l'(x)} + \frac{1}{|\varphi_r'(x)|}\right) + 2\sum_{x\in E_\varphi} u_x. \qquad \square$$

Let us mention that, without any assumption of continuity on ψ, a sufficient condition for the boundedness of the weighted composition operator is that ψ lies in L^∞ and the boundedness of C_φ (see the condition below).

Corollary 3.11. *Let $\varphi : [0,1] \longrightarrow [0,1]$ satisfying condition $(\mathcal{R})$. Then*

$$C_\varphi : M_\Lambda^1 \longrightarrow L^1 \text{ is bounded if and only if } \sum_{x\in E_\varphi}\left(\frac{1}{\varphi_l'(x)} - \frac{1}{\varphi_r'(x)}\right) \text{ converges.}$$

Moreover, in this case,

(i) If $\varphi^{-1}(\{1\})$ is not empty then there exists some constants k_1 and $k_2 > 0$ such that $k_1\|f\|_1 \le \|C_\varphi(f)\|_1 \le k_2\|f\|_1$.

(ii) C_φ is compact if and only if C_φ is nuclear if and only if $\|\varphi\|_\infty < 1$.

Proof. The first part obviously follows from Theorem 3.10.

Now, let us prove (i). Let us assume that $x_0 \in \varphi^{-1}(\{1\})$ with $x_0 > 0$ (else it is easy to adapt the argument). There exists $\delta, \varepsilon_0 > 0$ such that $0 < \varphi'(x) < \delta$

for every $t \in (x_0 - \varepsilon_0, x_0)$. We have

$$\begin{aligned}\|C_\varphi(f)\|_1 &\geq \int_{x_0-\varepsilon_0}^{x_0} |f(\varphi(t))|dt \geq \frac{1}{\delta}\int_{x_0-\varepsilon_0}^{x_0} |f(\varphi(t))|\varphi'(t)\,dt \\ &= \frac{1}{\delta}\int_{\varphi(x_0-\varepsilon_0)}^{1} |f(y)|\,dy \geq \frac{1}{\delta.\gamma(\varphi(x_0-\varepsilon_0))}\|f\|_1\end{aligned}$$

thanks to Proposition 2.3.

The assertion (ii) is clear: if $\|\varphi\|_\infty < 1$, then C_φ is bounded and nuclear, thanks to Lemma 3.4. If C_φ is nuclear, it is compact. Finally, if C_φ is (bounded and) compact, it clearly follows from (i) that $\varphi^{-1}(\{1\})$ must be empty, hence $\varphi([0,1]) \subset [0,1[$, so $\varphi([0,1]) \subset [0,\alpha]$ where $\alpha < 1$. $\square$

In the rest of the paper, our functions φ and ψ shall satisfy the following condition, which ensures boundedness of the associated weighted composition operator (thanks to Theorem 3.10):

Definition 3.12. *Let $\varphi : [0,1] \longrightarrow [0,1]$ be a measurable function. We say that (φ,ψ) satisfies condition $(\mathcal{B})$, if*

- *φ satisfies condition $(\mathcal{R})$.*
- *$\psi \in L^\infty$ and is continuous at each point of E_φ.*
- *$\sum_{x\in E_\varphi} |\psi(x)|L(x)$ converges.*

In the sequel, we shall simply say that φ satisfies condition $(\mathcal{B})$ when $(\varphi, 1\!\!1)$ satisfies condition $(\mathcal{B})$ (i.e. $\psi = 1\!\!1$).

4. Compact operators.

We characterize the compactness of weighted composition operators $\mathcal{T}_\psi \circ C_\varphi$ whose associated symbols satisfy condition $(\mathcal{B})$.

Theorem 4.1. *Let (φ,ψ) satisfies condition $(\mathcal{B})$.*
Then the operator $\mathcal{T}_\psi \circ C_\varphi : M^1_\Lambda \longrightarrow L^1$ is compact if and only if $\psi_{|E_\varphi} = 0$.

By convention $\psi_{|\{\emptyset\}} = 0$.

Proof. Assume that $\psi_{|\,E_\varphi} = 0$.

Let $(f_n)_n \subset M^1_\Lambda$ such that $\|f_n\|_1 \leq 1$. By Corollary 2.5, there is a subsequence $(f_{n_k})_k$ that converges to f uniformly on every compact subset of

$[0,1)$, where f belongs to the unit ball of M^1_Λ. Then $f\circ\varphi$ is defined almost everywhere on $[0,1]$, because $m(\varphi^{-1}(\{1\}))=0$ and f is defined almost everywhere on $[0,1]$.

Let $h=(f\circ\varphi)\cdot\psi$. The function h is a well defined measurable function on $[0,1]$. We claim that $\|\mathcal{T}_\psi\circ C_\varphi(f_{n_k})-h\|_1\to 0$ when $k\to+\infty$, so that $h=\lim\limits_{k\to\infty}\mathcal{T}_\psi\circ C_\varphi(f_{n_k})$ belongs to L^1 and hence $\mathcal{T}_\psi\circ C_\varphi$ is compact.

Indeed, let $\varepsilon>0$. We can find a compact subset K of $[0,1]\setminus E_\varphi$ such that $|\psi_{|K^c}|<\varepsilon$. Then, writing $A=\sup\varphi(K)<1$:

$$\begin{aligned}\|\mathcal{T}_\psi\circ C_\varphi(f_{n_k})-h\|_1 &= \int_0^1 |f_{n_k}(\varphi(x))\psi(x)-h(x)|dx\\ &\le \|f_{n_k}-f\|_{[0,A]}\cdot\|\psi\|_\infty+2\|C_\varphi\|\cdot\sup_{x\in K^c}|\psi(x)|\end{aligned}$$

which implies that there exists some $k_0\in\mathbb{N}$ such that

$$\|\mathcal{T}_\psi\circ C_\varphi(f_{n_k})-h\|_1\le(1+2\|C_\varphi\|)\varepsilon\quad\text{for every } k\ge k_0$$

and thus $\|\mathcal{T}_\psi\circ C_\varphi(f_{n_k})-h\|_1\to 0$ when $k\to+\infty$.

Conversely, assume now that $\mathcal{T}_\psi\circ C_\varphi$ is compact.

The sequence $((\lambda_n+1)x^{\lambda_n})_n$ belongs to the unit ball of M^1_Λ, therefore there exists $h\in L^1$ and a subsequence n_k such that $\lim\limits_{k\to\infty}\|\mathcal{T}_\psi\circ C_\varphi((\lambda_{n_k}+1)x^{\lambda_{n_k}})-h\|_1=0$. Without loss of generality we may assume that $(\lambda_{n_k}+1)\psi\varphi^{\lambda_{n_k}}$ converges to h almost everywhere (a.e.) on $[0,1]$. Now, since $\varphi(x)<1$ a.e. (and ψ is bounded) we infer $h(x)=0$ a.e. and therefore

$$\int_0^1(\lambda_{n_k}+1)\varphi(x)^{\lambda_{n_k}}|\psi(x)|dx\underset{k\to\infty}{\longrightarrow}0.$$

Let $x_0\in E_\varphi$, so according to condition $(\mathcal{B})$, and the continuity of ψ at x_0, there exists $\varepsilon_0>0$ such that $0<\varphi'(x)<2\varphi'_l(x_0)$ and $|\psi(x)|\ge\frac{1}{2}|\psi(x_0)|$ for all $x\in[x_0-\varepsilon_0,x_0]$ (if $x_0=0$ we work on the right of x_0). Then we have

$$\begin{aligned}\int_0^1(\lambda_{n_k}+1)\varphi(x)^{\lambda_{n_k}}|\psi(x)|dx &\ge \frac{1}{2}|\psi(x_0)|\int_{x_0-\varepsilon_0}^{x_0}(\lambda_{n_k}+1)\varphi(x)^{\lambda_{n_k}}dx\\ &\ge\frac{|\psi(x_0)|}{4\varphi'_l(x_0)}\int_{x_0-\varepsilon_0}^{x_0}(\lambda_{n_k}+1)\varphi(x)^{\lambda_{n_k}}\varphi'(x)dx.\end{aligned}$$

So we obtain,

$$\int_0^1 (\lambda_{n_k}+1)\varphi(x)^{\lambda_{n_k}}|\psi(x)|dx \geq \frac{|\psi(x_0)|}{4\varphi_l'(x_0)}\Big(1-\varphi(x_0-\varepsilon_0)^{\lambda_{n_k}+1}\Big)$$

$$\underset{k\to\infty}{\longrightarrow} \frac{|\psi(x_0)|}{4\varphi_l'(x_0)}.$$

This imposes $\psi(x_0)=0$. □

Corollary 4.2. *Let $\varphi : [0,1] \longrightarrow [0,1]$ satisfying the condition $(\mathcal{B})$, and ψ a continuous function. Then we have:*

(1) *C_φ is compact on M^1_Λ if and only if $\|\varphi\|_\infty < 1$.*

(2) *$\mathcal{T}_\psi$ is compact on M^1_Λ if and only $\psi(1)=0$.*

Proof. Applying Theorem 4.1 with $\psi = \mathbb{1}$, we get $\mathcal{T}_\psi \circ C_\varphi = C_\varphi$ and, C_φ is compact if and only if $1_{|E_\varphi} = 0$, equivalently $E_\varphi = \emptyset$ equivalently $\|\varphi\|_\infty < 1$ which gives (1).

We now apply Theorem 4.1 with $\varphi(x) = x$ to get $\mathcal{T}_\psi \circ C_\varphi = \mathcal{T}_\psi$ and then $\mathcal{T}_\psi$ is compact if and only if $\psi_{|E_\varphi} = \psi_{|\{1\}} = \psi(1) = 0$ which proves (2). □

5. Essential norm. Recall that the essential norm of an operator $T : X \to Y$ is

$$\|T\|_e = \inf\big\{\|T-S\| : S \text{ is a compact operator from } X \text{ to } Y\big\}.$$

Clearly an operator is compact if and only if its essential norm vanishes.

The next result gives the exact value of the essential norm of the weighted composition operator $\mathcal{T}_\psi \circ C_\varphi$. The estimation uses the functions $f_n(x) = (\lambda_n+1)x^{\lambda_n}$.

Theorem 5.1. *Let (φ,ψ) satisfies condition $(\mathcal{B})$. Then we have*

$$\|\mathcal{T}_\psi \circ C_\varphi\|_e = \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1 = \sum_{x\in E_\varphi} |\psi(x)|L(x).$$

Proof. Let $\varepsilon > 0$. For every $x \in E_\varphi$, there exists $\varepsilon_x > 0$ such that (of course, if $x=0$ or $x=1$, we have to replace $(x-\varepsilon_x, x+\varepsilon_x)$ by $(0,\varepsilon_0)$ or $(1-\varepsilon_1, 1)$):

(i) For every $t \in (x-\varepsilon_x, x+\varepsilon_x)$, we have $|\psi(x)-\psi(t)| < \varepsilon$.

(ii) For every $t \in (x-\varepsilon_x, x)$, we have $\varphi'(t) > 0$.

(ii) For every $t \in (x, x+\varepsilon_x)$, we have $\varphi'(t) < 0$.

(iv) The intervals $(x-\varepsilon_x, x+\varepsilon_x)$ (where x runs over E_φ) are disjoints and included in $[0,1]$.

For any $x_0 \in E_\varphi$, we write $J_{x_0} = (x_0-\varepsilon_{x_0}, x_0+\varepsilon_{x_0}) \cap [0,1]$.

Let $\Omega = \bigcup_{x_0\in E_\varphi} J_{x_0}$. This is an open subset of $[0,1]$. The set $K = [0,1]\setminus\Omega$ is compact and, thanks to condition $(\mathcal{R})$, we have $s = \sup\varphi(K) < 1$.

Step I. We first claim that $\lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1 = \sum_{x_0\in E_\varphi} |\psi(x_0)|L(x_0)$.

Indeed

$$\|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1 = \int_K |f_n(\varphi(t))\psi(t)|\,dt + \int_\Omega |f_n(\varphi(t))\psi(x)|\,dt.$$

On K, some uniform majorizations give:

$$\int_K |f_n(\varphi(t))\psi(t)|\,dt \le \|\psi\|_\infty \sup_{u\le s}|f_n(u)|$$

and the right hand side converges to 0.

Next, we claim that

$$\lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1 = \sum_{x_0\in E_\varphi} \lim_{n\to\infty} \int_{J_{x_0}} f_n(\varphi(t))|\psi(t)|\,dt \tag{1}$$

and that it suffices to show that, for each $x_0 \in E_\varphi$,

$$\lim_{n\to\infty} \int_{J_{x_0}} |f_n(\varphi(t))\psi(t)|\,dt = |\psi(x_0)|L(x_0). \tag{2}$$

We first give now the details for (2). Let $x_0 \in E_\varphi \setminus \{0,1\}$ (and the computation easily adapts when $x_0 = 0$ or 1): for every $t \in J_{x_0}$, we have

$$(1-\varepsilon)|\psi(x_0)| \le |\psi(t)| \le (1+\varepsilon)|\psi(x_0)|,$$

which implies

$$\begin{aligned}(1-\varepsilon)|\psi(x_0)|\int_{J_{x_0}}|f_n(\varphi(t))|\,dt &\leq \int_{J_{x_0}}|f_n(\varphi(t))\psi(t)|\,dt\\ &\leq (1+\varepsilon)|\psi(x_0)|\int_{J_{x_0}}|f_n(\varphi(t))|\,dt.\end{aligned}$$

Making on each (left-right) sub-interval of J_{x_0} the (natural) change of variables, we have

$$\int_{J_{x_0}}|f_n(\varphi(t))|\,dt=\int_{\varphi(x_0-\delta)}^{1}(\lambda_n+1)u^{\lambda_n}\frac{du}{\varphi'(\varphi^{-1}(u))}-\int_{\varphi(x_0+\delta)}^{1}(\lambda_n+1)u^{\lambda_n}\frac{du}{\varphi'(\varphi^{-1}(u))},$$

but $(1-\varepsilon)\varphi_l'(x_0)\leq\varphi'(\varphi^{-1}(u))\leq(1+\varepsilon)\varphi_l'(x_0)$ for every $u\in[\varphi(x_0-\delta),1]$, and $-(1-\varepsilon)\varphi_r'(x_0)\leq-\varphi'(\varphi^{-1}(u))\leq-(1+\varepsilon)\varphi_r'(x_0)$ for every $u\in[\varphi(x_0+\delta),1]$ hence

$$\begin{aligned}\int_{J_{x_0}}|f_n(\varphi(t))|\,dt \geq \;& \frac{1}{1+\varepsilon}\left(\frac{1}{\varphi_l'(x_0)}\int_{\varphi(x_0-\delta)}^{1}(\lambda_n+1)u^{\lambda_n}du\right.\\ &\left.+\frac{1}{|\varphi_r'(x_0)|}\int_{\varphi(x_0+\delta)}^{1}(\lambda_n+1)u^{\lambda_n}du\right)\end{aligned}$$

and

$$\begin{aligned}\int_{J_{x_0}}|f_n(\varphi(t))|\,dt \leq \;& \frac{1}{1-\varepsilon}\left(\frac{1}{\varphi_l'(x_0)}\int_{\varphi(x_0-\delta)}^{1}(\lambda_n+1)u^{\lambda_n}du\right.\\ &\left.+\frac{1}{|\varphi_r'(x_0)|}\int_{\varphi(x_0+\delta)}^{1}(\lambda_n+1)u^{\lambda_n}du\right)\end{aligned}$$

Collecting the quantities and letting $n\to\infty$, we obtain that for n large enough

$$\frac{1-2\varepsilon}{1+\varepsilon}|\psi(x_0)|L(x_0)\leq\int_{J_{x_0}}|f_n(\varphi(t))\psi(t)|\,dt\leq\frac{1+2\varepsilon}{1-\varepsilon}|\psi(x_0)|L(x_0)$$

and, since ε is chosen arbitrarily small, claim (2) is justified.

Concerning claim (1), let us first point out that

$$\lim_{n\to\infty}\|\mathcal{T}_\psi\circ C_\varphi(f_n)\|_1=\lim_{n\to\infty}\sum_{x_0\in E_\varphi}\int_{J_{x_0}}f_n(\varphi(t))|\psi(t)|\,dt$$

Now, it suffices to apply the Lebesgue domination theorem (with respect to the counting measure). The domination is justified by the previous estimates. Indeed, for each $x_0 \in E_\varphi$, we have

$$\int_{J_{x_0}} f_n(\varphi(t))|\psi(t)|\,dt \le \frac{1}{1-\varepsilon}\left(\frac{1}{\varphi_l'(x_0)} + \frac{1}{|\varphi_r'(x_0)|}\right)$$

which is summable thanks to condition $(\mathcal{B})$.

Step II. We claim now that $\|\mathcal{T}_\psi \circ C_\varphi\|_e \le \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1$.

There exists a function h, which is continuous at each point $x_0 \in E_\varphi$. and such that the restricted functions satisfy $h_{|\,E_\varphi} = 0$ and $h_{|\,K} = 1$; with h taking its valued in $[0,1]$. Indeed, for instance, define $h(t) = |t - x_0|/\varepsilon_{x_0}$ when t belongs to J_{x_0} and $h = 1$ on K.

Let $\psi_\varepsilon = h \cdot \psi$. We have

$$\begin{aligned}
\|\mathcal{T}_\psi \circ C_\varphi - \mathcal{T}_{\psi_\varepsilon} \circ C_\varphi\| &= \sup_{\|f\|_1 \le 1} \int_\Omega (1-h)|\psi(x)||f(\varphi(x))|dx \\
&\le \sup_{\|f\|_1 \le 1} \sum_{x_0 \in E_\varphi} \int_{J_{x_0}} |\psi(x)||f(\varphi(x))|dx \\
&\le \sum_{x_0 \in E_\varphi} \sup_{\|f\|_1 \le 1} \|\psi\|_{J_{x_0}} \int_{J_{x_0}} |f(\varphi(x))|dx.
\end{aligned}$$

If $x \in J_{x_0}$, then $|x - x_0| < \varepsilon_{x_0}$ so $\|\psi\|_{J_{x_0}} = \sup_{x \in J_{x_0}} |\psi(x)| \le |\psi(x_0)| + \varepsilon$.

On the other hand, using the computation in step I, we get

$$\int_{J_{x_0}} |f(\varphi(x))|dx \le \frac{1}{1-\varepsilon} L(x_0)\|f\|_1.$$

We obtain

$$\|\mathcal{T}_\psi \circ C_\varphi - \mathcal{T}_{\psi_\varepsilon} \circ C_\varphi\| \le \frac{1}{1-\varepsilon} \sum_{x_0 \in E} (|\psi(x_0)| + \varepsilon) L(x_0).$$

Now, since $(\psi_\varepsilon)_{|\,E} = 0$ and is continuous at each point of E_φ, thanks to Theorem 4.1, we know that $\mathcal{T}_{\psi_\varepsilon} \circ C_\varphi$ is compact.

Hence, $\|\mathcal{T}_\psi \circ C_\varphi\|_e = \inf\{\|\mathcal{T}_\psi \circ C_\varphi - S\| : S$ is a compact operator on $M^1_\Lambda\}$

$$
\begin{aligned}
&\leq \|\mathcal{T}_\psi \circ C_\varphi - \mathcal{T}_{\psi_\varepsilon} \circ C_\varphi\| \\
&\leq \frac{1}{1-\varepsilon} \sum_{x_0 \in E} (|\psi(x_0)| + \varepsilon) L(x_0)
\end{aligned}
$$

Since ε is arbitrary we get

$$
\|\mathcal{T}_\psi \circ C_\varphi\|_e \leq \sum_{x_0 \in E} |\psi(x_0)| L(x_0) = \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1.
$$

Step III. It remains to prove $\|\mathcal{T}_\psi \circ C_\varphi\|_e \geq \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1$.

If $E_\varphi = \emptyset$, we have $\|\varphi\|_\infty < 1$ and C_φ is compact, as well as $\mathcal{T}_\psi \circ C_\varphi$, therefore $\|\mathcal{T}_\psi \circ C_\varphi\|_e = 0 = \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1$.

We may now assume that $E_\varphi \neq \emptyset$. Let $S : M^1_\Lambda \longrightarrow L^1$ be a compact operator.

We want to show that $\|\mathcal{T}_\psi \circ C_\varphi - S\| \geq \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1$.

Since S is compact and $\|f_n\|_\infty = 1$, then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ and $f \in L^1$ such that $\lim_{j\to\infty} \|S(f_{n_j}) - f\|_1 = 0$.

We have $\limsup_{j\to\infty} \|(\mathcal{T}_\psi \circ C_\varphi - S)(f_{n_j})\|_1 \geq \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1$.

Indeed,

$$
\|(\mathcal{T}_\psi \circ C_\varphi - S)(f_{n_j})\|_1 \geq \|\mathcal{T}_\psi \circ C_\varphi(f_{n_j}) - f\|_1 - \|S(f_{n_j}) - f\|_1,
$$

which implies that

$$
\limsup_{j\to\infty} \|(\mathcal{T}_\psi \circ C_\varphi - S)(f_{n_j})\|_1 \geq \limsup_{j\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_{n_j}) - f\|_1.
$$

So it suffices to show that $\limsup_{j\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_{n_j}) - f\|_1 \geq \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1$.

Let $\varepsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that $\int_U |f(x)| dx \leq \varepsilon$ where $U = (E_\varphi + (-\delta, \delta)) \cap [0,1]$, thus

$$
\begin{aligned}
\|\mathcal{T}_\psi \circ C_\varphi(f_{n_j}) - f\|_1 &\geq \int_U |\mathcal{T}_\psi \circ C_\varphi(f_{n_j})| dx - \int_U |f(x)| dx \\
&\geq \int_U |\mathcal{T}_\psi \circ C_\varphi(f_{n_j})| dx - \varepsilon \\
&\geq \|\mathcal{T}_\psi \circ C_\varphi(f_{n_j})\|_1 - \|\psi\|_\infty \Big(\sup_{t \in \varphi([0,1]\setminus U)} f_{n_j}(t) \Big) - \varepsilon
\end{aligned}
$$

According to step I, the sequence $(\|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1)_n$ is convergent. On the other hand, (f_n) is uniformly convergent to 0 on compact subsets of $[0,1)$. So letting $j \to \infty$, we get

$$\limsup_{j\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_{n_j}) - f\|_1 \geq \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1 - \varepsilon.$$

Since ε is arbitrary, we deduce that

$$\limsup_{j\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_{n_j}) - f\|_1 \geq \lim_{n\to\infty} \|\mathcal{T}_\psi \circ C_\varphi(f_n)\|_1.$$

which proves the last step and completes the proof of the theorem. □

Corollary 5.2. *Let $\varphi : [0,1] \longrightarrow [0,1]$ be a function satisfying condition $(\mathcal{B})$ and $\psi \in C([0,1])$.*

Then,

- $$\|C_\varphi\|_e = \begin{cases} 0 & \text{if } \|\varphi\|_\infty < 1 \\ \displaystyle\sum_{x\in E_\varphi} L(x) & \text{if } \|\varphi\|_\infty = 1. \end{cases}$$

- $\|\mathcal{T}_\psi\|_e = |\psi(1)|$.

Remark 5.3. If we denote by $\|\cdot\|_e^\infty$ (respectively $\|\cdot\|_e^1$) the essential norm of an operator defined on M_Λ^∞ (respectively on M_Λ^1), we note that $\|\mathcal{T}_\psi\|_e^\infty = \|\mathcal{T}_\psi\|_e^1$, contrariwise (see [2]) we have

$$1 = \|C_\varphi\|_e^\infty \neq \|C_\varphi\|_e^1 = \sum_{x\in E_\varphi} \left(\frac{1}{\varphi_l'(x)} - \frac{1}{\varphi_r'(x)} \right).$$

Corollary 5.4. *Let $\varphi : [0,1] \longrightarrow [0,1]$ be a function satisfying condition $(\mathcal{B})$, such that C_φ is not compact (i.e. $1 \in \operatorname{Im}\varphi$) and $\psi \in C([0,1])$. Let $C_\varphi \circ \mathcal{T}_\psi$ from M_Λ^1 to L^1, then its essential norm is*

$$|\psi(1)| \cdot \sum_{x\in E_\varphi} \left(\frac{1}{\varphi_l'(x)} - \frac{1}{\varphi_r'(x)} \right).$$

P r o o f. Let $f \in M^1_\Lambda$, then $C_\varphi \circ \mathcal{T}_\psi(f) = (f \circ \varphi) \cdot (\psi \circ \varphi) = \mathcal{T}_{\psi\circ\varphi} \circ C_\varphi(f)$. Therefore $C_\varphi \circ \mathcal{T}_\psi = \mathcal{T}_{\psi\circ\varphi} \circ C_\varphi$ and hence by the preceding theorem, we have

$$\|C_\varphi \circ \mathcal{T}_\psi\|_e = \|\mathcal{T}_{\psi\circ\varphi} \circ C_\varphi\|_e = |\psi(1)| \sum_{x \in E_\varphi} \left(\frac{1}{\varphi'_r(x)} - \frac{1}{\varphi'_l(x)} \right). \qquad \square$$

Remark 5.5. It is easy to see that most of the results of this paper are still valid when M^1_Λ is replaced by a Banach space X satisfying : $M^1_\Lambda \subset X \subset L^1$ and each $f \in X$ is continuously differentiable on $[0, 1)$. Nevertheless the natural examples of such spaces seem to be only Müntz spaces (i.e. $X = M^1_{\Lambda'}$ with $\Lambda \subset \Lambda'$).

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