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ON MARTINDALE'S LEMMA FOR NONASSOCIATIVE ALGEBRAS*

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Abstract. We give a nonassociative version of Martindale's lemma, and as a consequence, we obtain a semiprime GPI-theorem: if A is a multiplicatively semiprime algebra, $M(A)$ is its multiplication algebra and C is its extended centroid, then the following are equavalent: (1) $CM(A)$ has a finite rank operator over C; (2) $M(A)$ is GPI; (3) there are $F_i, G_i, H_j, K_j \in CM(A)$ and $p_i, q_j \in A$ with $F_iXG_iY(p_i) \neq 0$ for some i, and such that $\sum_{i=1}^{n} F_iXG_iY(p_i) = \sum_{j=1}^{m} H_jYK_jX(q_j)$ (for all $X, Y \in M(A)$); (4) there exists $F \in M(A)$ and $a \in A$ such that the $FM_C(Q)F(a))$ is C-finitely generated.

Introduction. In this paper, we will deal with semiprimes algebras which are not necessarily associative over a fixed field $\mathbb{K}$ of zero characteristic. Recall that an algebra B is said to be *semiprime* (respectively *prime*) if $I^2 \neq 0$ (resp. $IJ \neq 0$) for every nonzero ideal I (respectively all nonzero ideals I, J)

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of B. Given a semiprime associative algebra $\mathcal{A}$, the so called symmetric algebra of quotients $Q_s(\mathcal{A})$ of $\mathcal{A}$ is probably the most confortable algebra of quotients of $\mathcal{A}$. The centre $C_{\mathcal{A}}$ of $Q_s(\mathcal{A})$ is a unital semiprime commutative associative algebra extension of $\mathbb{K}$, called the *extended centroid* of $\mathcal{A}$, and the $C_{\mathcal{A}}$-subalgebra $Q_{\mathcal{A}}$ of $Q_s(\mathcal{A})$ generated by $\mathcal{A}$ is called the *central closure* of $\mathcal{A}$. Both $C_{\mathcal{A}}$ and $Q_{\mathcal{A}}$ play a fundamental role in GPI-theories. In this framework, the standard definition of a generalized polynomial identity (GPI) requires the introduction of an appropriate generalization of a free algebra, which provides a suitable setting for "generalized polynomial". Roughly speaking, for a multilinear generalized polynomial identity of $\mathcal{A}$, we mean an identity of the form

$$\Phi(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in S_n} \sum_{i=1}^{n_\sigma} a_{0i}^\sigma x_{\sigma(1)} a_{1i}^\sigma x_{\sigma(2)} \cdots a_{(n-1)i}^\sigma x_{\sigma(n)} a_{ni}^\sigma,$$

where a_{mi}^σ are fixed elements in $Q_s(\mathcal{A})$. $\mathcal{A}$ is said to be *GPI* if there is a nonzero GPI Φ such that $\Phi(a_1, a_2, \ldots, a_n) = 0$ for all $a_i \in \mathcal{A}$. For a comprehensive treatment and for references to the extensive literature on $Q_s(\mathcal{A})$ we refer to the books [4] and [3]. W. S. Martindale proved in [20, Theorem 1] (often referred as Martindale 's lemma) that if $\mathcal{A}$ is a prime algebra and $p, q \in Q_s(\mathcal{A})$ satisfy that $L_p R_q = L_q R_p$, then there is $\lambda \in C_{\mathcal{A}}$ such $p = \lambda q$. It was extended to semiprime context in [3, Theorem 2.3.11]. As a result in [3, §6.3] it is obtained a semiprime GPI theorem: If $\mathcal{A}$ is a semiprime associative algebra, then $\mathcal{A}$ is GPI if and only if there is an abelian idempotent E of $C\mathcal{A}$ such that $EC\mathcal{A}E$ is C-finitely generated. These results are a cornerstone of GPI-theory. The goal of the present paper is to give a nonassociative version of Martindale's lemma and, as a consequence, to obtain a semiprime GPI-theorem, which may serve in the construction of a GPI-theory for nonassociative algebras.

In the general nonassociative setting, the absence of algebras of quotients complicates the presentation of the extended centroid and the central closure, which were introduced and developed by T. S. Erickson, W. S. Martindale, and J. M. Osborn [18] in the prime context, and by W. E. Baxter and W. S. Martindale [2] in the semiprime context (see also [16]). Later, another approaches to these concepts have appeared in the literature: see the books by Y. P. Razmyslov [21, §3] and R. Wisbauer [22, §32]. For a recent treatment we refer the reader to [11, §2.1]. As it is made clear below, the multiplicatively semiprime algebras turn out to be the appropriate framework for translating to nonassociative setting the semiprime associative results. Given an algebra B, for $a \in B$, we will denote by L_a and R_a the operators of, respectively, left and right multiplication by a on B. The *multiplication algebra* $M(B)$ of B is defined as the subalgebra of $L(B)$

generated by the identity operator Id_B and the set $\{L_a, R_a \ : \ a \in B\}$. We say that an algebra B is *multiplicatively semiprime* (in short *m.s.p.*) whenever both B and $M(B)$ are semiprime algebras.

The need for such extensions is justified for the breadth of the class of m.s.p. algebras. Of course, associative semiprime algebras are multiplicatively semiprime algebras [15, Section 4]), and a similar result holds for many nearly associative algebras (see the papers [5, 7, 10, 17]). Algebras with a semiprime multiplication algebra were first studied by N. Jacobson [19] and A. A. Albert [1] in a finite dimensional context. Without any restriction on the dimension, the study of m.s.p. algebras was initiated in [5].

We shall assume throughout this paper that A is an m.s.p. algebra and we will denote its central closure Q_A simply by Q and its extended centroid C_A simply by C.

1. Preliminaries. In this section we fix the relevant material on the extended centroid for an m.s.p. algebra. The starting point of this path relies on the possibility of associating an idempotent of the extended centroid to each subset of central closure. This result is well-known in an associative context (cf. [3, Theorem 2.3.9 and Lemma 2.3.10]) and it was established in [9] in a general context.

1.1. Notation. First of all, we establish the notation used. Let B be an algebra. For any subspaces S of B and $\mathcal{N}$ of $M(B)$, the subsets S^{ann} of $\mathcal{M}(B)$ and $\mathcal{N}_{\mathrm{ann}}$ of B are defined by

$$S^{\mathrm{ann}} = \{F \in \mathcal{M}(B) \ : \ F(S) = 0\} \text{ and } \mathcal{N}_{\mathrm{ann_B}} = \{a \in B \ : \ \mathcal{N}(a) = 0\}.$$

It is well-known that an ideal of $M(B)$ is essential if and only if $\mathcal{P}_{\mathrm{ann}} = 0$. The set $(S^{\mathrm{ann}})_{\mathrm{ann_B}}$ is called the *ε-closure* of the subspace S of B, and will be denoted by either $\widehat{S}^B$ or $S^{\wedge_B}$. A subspace S of B is said to be a *dense subspace* of B if $B = \widehat{S}^B$, that is to say whenever $S^{\mathrm{ann}} = 0$. The ε-closure enjoys a relevant property, namely the so-called property of continuity [5, Proposition 1.8]: *If $F \in M(B)$, and if S is a subspace of B, then $F(S^{\wedge_B}) \subseteq F(S)^{\wedge_B}$.*

Note that B has a natural structure of left $M(B)$-module for the valuation action. In fact, if B is a dense subalgebra of an algebra Q, then Q also has a natural structure of left $M(B)$-module. Indeed, for each $F \in M(B)$, there exists a unique $F' \in M(Q)$ such that $F'(a) = F(a)$ for every $a \in B$. Moreover, the map $F \mapsto F'$ becomes a canonical algebra embedding $M(B) \hookrightarrow M(Q)$. Thus, Q has a natural structure of left $M(B)$-module given by

$$F \cdot q := F'(q) \quad \text{for all } F \in M(B) \text{ and } q \in Q.$$

By abuse of notation, we will write $F(q)$ instead of $F \cdot q$.

Let us introduce the concepts of extended centroid and central closure.

If C is a semiprime commutative associative unital algebra and Q is a C-algebra then, for a subset S of Q, we will denote by CS the C-subspace of Q generated by S. Of course, if $Q = CB$ then B is dense in Q.

Between the different approaches to the concepts of extended centroid and central closure for a semiprime algebra we prefer to take advantage of the characterization given in [11, Theorem 2.4]: The extended centroid C_B and the central closure Q_B of B are determined by the following properties:

(P1) C_B is a unital semiprime commutative associative algebra, Q_B is an algebra extension of B, and Q_B is generated by B as a C_B-algebra.

(P2) For each $q \in Q_B$, there exists an essential ideal D of B (that is, $D \cap I \neq 0$ for any nonzero ideal I of B) such that $DM(B)(q) + M(B)(q)D \subseteq B$.

(P3) If $q \in Q_B$ satisfies either $DM(B)(q) = 0$ or $M(B)(q)D = 0$ for some essential ideal D of B, then $q = 0$.

(P4) For each essentially defined centralizer $f : D \longrightarrow B$, there exists a unique element $\lambda \in C_B$ such that $f(x) = \lambda x$ for every $x \in D$.

Obviously $Q_B = CB$ and it is easy to prove that C_B is von Neumann regular (c.f. [3, Theorem 2.3.9.(iii)]) and that Q_B is semiprime (see [11, Proposition 2.1]). B is said to be a *centrally closed* algebra whenever $B = Q_B$. Of course, Q_B is a centrally closed algebra (see [2, Theorem 2.15.(c)]).

Recall that the set $\mathcal{I}_{C_B}$ of all idempotents in C_B has a partial order given by $e \leq f$ if and only if $e = ef$. Moreover, $\mathcal{I}_{C_B}$ is a Boolean algebra for the operations

$$e \wedge f = ef, \qquad e \vee f = e + f - ef, \qquad \text{and} \qquad e^* = 1 - e.$$

Proposition 1.1 ([9], Proposition 1.6)**.** *Let B be a semiprime algebra and let S be a nonempty subset of Q_B. Then*

(1) *There exists a unique $e_{[S]}$ in $\mathcal{I}_{C_B}$ such that*

$$\{\lambda \in C_B \, : \lambda S = 0\} = (1 - e_{[S]})C_B;$$

(2) $e_{[S]}p = p$ *for every $p \in S$ and for any $e \in \mathcal{I}_{C_B}$,* $e_{[eS]} = ee_{[S]}$.

On the other hand, it is well known that $Q_s(C_B) = C_B$, and so

$$C_{C_B} = C_B. \tag{1}$$

In particular, by [3, Theorem 2.3.9], given a nonempty subset $\mathcal{S}$ of C_B:

(1) There exists a unique $e_{[\mathcal{S}]}$ in $\mathcal{I}_{C_B}$ such that

$$\{\lambda \in C;\ \lambda\mathcal{S} = 0\} = (1 - e_{[\mathcal{S}]})C_B;$$

(2) $e_{[\mathcal{S}]}\lambda = \lambda$ for every $\lambda \in \mathcal{S}$ and for any $e \in \mathcal{I}_{C_B}$, $e_{[e\mathcal{S}]} = ee_{[\mathcal{S}]}$.

In this paper, frequently use is made of these properties, often without explicit mention. It is obvious that $S = \{0\}$ if and only if $e_{[S]} = 0$. For each element x in Q or C, we will denote by $e_{[x]}$ the idempotent associated to $\{x\}$.

Next let us prove an elemental fact.

Corollary 1.2. *Let B be a semiprime algebra and let λ be in C_B. Then, λ is invertible if and only if $e_{[\lambda]} = 1$. As a consequence, $\lambda + (1 - e_{[\lambda]})$ is invertible for all $\lambda \in C$.*

Proof. Suppose that there exists $\mu \in C$ such that $\mu\lambda = 1$. It is clear that $1 = e_{[\mu\lambda]} \leq e_{[\lambda]}$. For the converse, suppose that $e_{[\lambda]} = 1$. Since C is von Neumann regular, there exists $\mu \in C$ such that $\mu\lambda$ is an idempotent and $\mu\lambda\lambda = \lambda$. Hence $\mu\lambda = \mu\lambda e_{[\lambda]} = e_{[\mu\lambda\lambda]} = e_{[\lambda]} = 1$. Finally, fix $\lambda \in C$ and consider $\mu = \lambda + (1 - e_{[\lambda]})$. It is clear that $\mu e_{[\lambda]} = \lambda$ and $\mu(1 - e_{[\lambda]}) = (1 - e_{[\lambda]})$. In particular $e_{[\mu]}(1 - e_{[\lambda]}) = (1 - e_{[\lambda]})$ and $e_{[\mu]}e_{[\lambda]} = e_{[\lambda]}$, and therefore $e_{[\mu]} = e_{[\mu]}e_{[\lambda]} + e_{[\mu]}(1 - e_{[\lambda]}) = 1$, as required. $\square$

Let B be a semiprime algebra. The algebra $M_C(Q_B)$ of Q_B over C_B is defined as the subalgebra of the algebra $L_{C_B}(Q_B)$ (operators on Q_B) generated by the identity operator Id_{Q_B} and the set $\{L_q, R_q\ :\ q \in Q_B\}$. It is clear that $M_C(Q_B) = M(Q_B) + C_B Id_{Q_B}$.

A second result allows us to relate the idempotent of an element and the idempotent of the ideal generated by itself.

Corollary 1.3. *Let B a semiprime algebra and let S be a subset of Q_B. Then $e_{M_C(Q_B)(S)} = e_{[S]}$.*

Proof. Since $S \subseteq M_C(Q_B)(S)$ we have $e_{[S]} \leq e_{[M_C(Q_B)(S)]}$. On the other hand, $e_{[S]}M_C(Q_B)(S) = M_C(Q)(e_{[S]}S) = M_C(Q_B)(S)$, so,

$$e_{[M_C(Q_B)(S)]} = e_{[e_{[S]}M_C(Q_B)(S)]} = e_{[S]}e_{[M_C(Q_B)(S)]},$$

thus $e_{[M_C(Q_B)(S)]} \leq e_{[S]}$. $\square$

Given a C_B-submodule N of Q_B, we will say that N is *C_B-finitely generated* if there exist $q_1, q_2, \ldots, q_n \in Q_B$ such that $N \subseteq \sum_{i=1}^{n} C_B q_i$. Note that if $p, q \in Q_B$ then it may happen that $p \in C_B q$ but $q \notin C_B p$. Borrowing the definition given in [13], we will say that n nonzero elements $q_1, q_2, \ldots, q_n$ of Q_B are *linearly C-independent* (or that the set $S := \{q_1, q_2, \ldots, q_n\}$ is *linearly C_B-independent*) if, for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in C_B$, $\sum \lambda_i q_i = 0$ implies $\lambda_i q_i = 0$ for all $i \in \{1, \ldots, n\}$, or equivalently, if the C-linear envelope N of the subset S satisfies that: $N = \bigoplus_{i=1}^{n} C_B q_i$.

A careful reading of the proof of [13, Corollary 1.3 and Corollary 1.4] allows us to assure that the next result remains true for semiprime nonassociative algebras.

Corollary 1.4. *Let B a semiprime algebra and let M be a C_B-finitely generated C_B-submodule of B. If $N \subsetneqq M$ is a C_B-submodule of B, then there are $p_1, p_2, \ldots, p_m \in Q_B$ such that $N = \bigoplus_{i=1}^{k} C_B p_i$ and $M = N \oplus \bigoplus_{i=k+1}^{m} C_B p_i$.*

Given a nonzero finitely generated C-submodule M, we will say that $dim_{\mathcal{I}_{C_B}}(M) = n$ whenever

$$n = \min\left\{ k \in \mathbb{N} : \ \exists p_i, p_2, \ldots, p_k \in Q_B \backslash \{0\} \text{ such that } M \subseteq \sum_{i=1}^{k} C_B p_i \right\}.$$

1.2. Semiprime associative algebras. In this subsection we give a slight extension of Martindale's lemma (see [3, Theorem 2.3.11]). This one is essentially known (see [3, Sections 2.3 and 6.3], but we will include its proof by the difficulty of giving a specific reference and by highlight the kinds of ideas that are handled.

Lemma 1.5. *Let $\mathcal{A}$ be a semiprime associative algebra and let $p_1, \ldots, p_n, q_1, \ldots, q_n \in Q_s(\mathcal{A})$. Assume that $\{p_1, \ldots, p_n\}$ or $\{q_1, q_2, \ldots, q_n\}$ are linearly C-independent sets. Then $\sum_{i=1}^{n} p_i x q_i = 0$ for every $x \in \mathcal{A}$ if and only if $e_{[p_i]} e_{[q_i]} = 0$ for all $i = 1, 2, \ldots, n$. As a consequence, given $r_1, \ldots, r_n \in Q_{\mathcal{A}}$, then $\sum_{i=1}^{n} p_i x q_i y r_i = 0$ for all $x, y \in \mathcal{A}$ if and only if $e_{[p_i]} e_{[q_i]} e_{[r_i]} = 0$ (or equivalently, $e_{[q_i]} e_{[r_i]} p_i = 0$, or $e_{[p_i]} e_{[q_i]} r_i = 0$) for all i.*

Proof. Assume that $q_1, q_2, \dots, q_n \in Q$ are linearly C-independent. It is clear that if $e_{[p_i]}e_{[q_i]} = 0$ for all $i \in \{1, \dots, n\}$, then $\sum_{i=1}^{n} p_i x q_i = 0$ for all $x \in \mathcal{A}$.

In order to prove the converse, assume that $\sum_{i=1}^{n} p_i x q_i = 0$ for all $x \in \mathcal{A}$ and there is j such that $e_{[p_j]}e_{[q_j]} \neq 0$. For simplicity we can suppose that $e_{[p_1]}e_{[q_1]} \neq 0$. Therefore $e_{[p_1]}q_1, q_2, \dots, q_n$ are linearly C-independent. By [3, Theorem 2.3.3] there exist $s_j, t_j \in \mathcal{A}$ such that $G \in M(\mathcal{A})$ defined by $G(x) = \sum_{j=1}^{m} s_j x t_j$ verifies that $G(e_{[p_1]}q_1) \neq 0$ and $G(q_i) = 0$ for all $i \in \{2, \dots, n\}$. Put $q_1' = G(e_{[p_1]}q_1) \neq 0$, and note that, for every $x \in \mathcal{A}$, we have:

$$0 = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i x s_j q_i \right) t_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i x s_j q_i t_j \right) = \sum_{i=1}^{n} p_i x G(e_{[p_i]} q_i) = p_1 x q_1'.$$

That is, $0 = p_1 x q_1' = p_1 x G(q_1)$ for all $x \in \mathcal{A}$, which is a contradiction because $G(e_{[p_1]}q_1) \neq 0$ (see [3, Corollary 2.3.10]).

Finally, if we assume that $\sum_{i=1}^{n} p_i x q_i y r_i = 0$ for all $x, y \in \mathcal{A}$ then by the first assertion $e_{[p_i x q_i]}e_{[r_i]} = 0$, for all i and for every $x \in \mathcal{A}$. Therefore, again by [3, Corollary 2.3.10] we deduce that $e_{[p_i]}e_{[q_i]}e_{[r_i]} = 0$. The converse is also obvious. □

Thus, we obtain the first extension:

Proposition 1.6. *Let $\mathcal{A}$ be a semiprime associative algebra and let p_i, q_i, c_j, $d_j \in Q_s(\mathcal{A})$ be such that for all $x \in \mathcal{A}$*

$$\sum_{i=1}^{n} p_i x q_i = \sum_{j=1}^{m} c_j x d_j. \tag{2}$$

If $p_1, p_2, \dots, p_n$ are linearly C-independent, then each $e_{[p_i]}q_i$ is a C-linear combination of $d_1, d_2, \dots, d_m$. Similarly, if $q_1, q_2, \dots, q_n$ are linearly C-independent, then each $e_{[q_i]}p_i$ is a C-linear combination of $c_1, c_2, \dots, c_m$.

Proof. Assume that $p_1, p_2, \dots, p_n$ are linearly C-independent and put $S = \{p_1, \dots, p_n, q_1, \dots, q_m\}$. By suitably reordering of the summands appearing in the right hand side of (2) we may assume, by Corollary 1.4, the existence of

$r \in \{1, \ldots, m\}$ such that $\bigoplus_{i=1}^{n} Cp_i + \sum_{j=1}^{m} Cc_j = \bigoplus_{i=1}^{n} Cp_i \oplus \bigoplus_{k=1}^{r} Cc'_k$ for convenient c'_k. For each $j \in \{1, \ldots, m\}$ we write

$$c_j = \sum_{i=1}^{n} \alpha_i^j p_i + \sum_{k=1}^{r} \beta_k^j c'_k \text{ for suitable } \alpha_i^j, \beta_k^j \in C.$$

Then, for each $x \in \mathcal{A}$ we have

$$\sum_{i=1}^{n} p_i x q_i = \sum_{j=1}^{m} c_j x d_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \alpha_i^j p_i + \sum_{k=1}^{r} \beta_k^j c'_k \right) x d_j,$$

and hence

$$\sum_{i=1}^{n} p_i x \left[q_i - \left(\sum_{j=1}^{m} \alpha_i^j d_j \right) \right] = \sum_{k=1}^{r} c'_k x \left(\sum_{j=1}^{m} \beta_k^j d_j \right).$$

Therefore, by Lemma 1.5, $e_{[q_i - \sum_{j=1}^{m} \alpha_i^j d_j]} e_{[p_i]} = 0$, for all i. As a consequence, $e_{[p_i]} q_i = \sum_{j=1}^{m} e_{[p_i]} \alpha_i^j d_j$ for all i. $\square$

1.3. M.s.p. algebras. We begin this subsection with an essentially known result.

Proposition 1.7. *Let B be a semiprime algebra. Then B is an m.s.p. algebra if and only if Q_B is.*

P r o o f. Suppose that B is an m.s.p. algebra. Combining [12, Corollary 4.4] and [8, Proposition 4.4], we obtain that Q_B is also m.s.p. Now suppose that Q_B is m.s.p. The conclusion is a consequence of [14, Proposition 2.2] since B is dense in Q_B. $\square$

Recall that, in what follows, A will be an m.s.p. algebra and we will denote its central closure Q_A simply by Q and its extended centroid C_A simply by C.

Corollary 1.8. *Let $q \in Q$. Then*

$$Q = (M_C(Q)(q) + (1 - e_{[q]})Q \cap Q)^{\wedge_Q}. \tag{3}$$

P r o o f. Set $J := M_C(Q)(q)$. It is clear that J is an ideal of Q. Taking in mind Proposition 1.7 and [9, Theorem 1.8] (c.f. [12, Corollary 1.6]), by [5, Theorem 2.6], we have that $Q = (J + (1 - e_{[J]})Q \cap Q)^{\wedge_Q}$. Therefore, to conclude it is enough to apply that $e_{[J]} = e_{[q]}$ in virtue of Corollary 1.3. $\square$

Recall that the extended centroids of A and $M(A)$ are isomorphic.

Theorem 1.9 ([11], Theorem 4.3)**.**

$$Q_{M(A)} = M_C(Q) \quad \textit{and} \;\; C_{M(A)} = C,$$

and, as a consequence,

$$\mathcal{I}_C = \mathcal{I}_{C_{M(A)}}.$$

Regarding $M(A)$ as a subalgebra of $M(Q)$, and so, of $M_C(Q)$, it is immediate to verify that

$$M_C(Q) = CM(A). \tag{4}$$

As a consequence we have a nonassociative version of [3, Theorem 2.3.3].

Corollary 1.10. *Let $q_1, q_2, \ldots, q_n \in Q$ and suppose that $q_1 \notin \sum_{i=2}^{n} Cq_i$. Then there exists $F \in M(A)$ such that $F(q_1) \neq 0$ and $F(q_i) = 0$, for all $i = 2, \ldots, n$.*

Proof. A careful reading of the proof of [18, Theorem 3.1] shows that the primeness of B is not essential, and so, we can assert that there is $G \in M(Q)$ such that $G(q_1) \neq 0$ and $G(q_i) = 0$, for all $i = 2, \ldots, n$. Taking in mind Theorem 1.9, by property (P2), there exists an essential ideal $\mathcal{P}$ of $M(A)$ such that $\mathcal{P}M(M(A))(G) \subseteq M(A))$. In particular $\mathcal{P}_{\text{ann}} = 0$ (see [5, Theorem 2.4]) and $\mathcal{P}G \subseteq M(A)$. If $\mathcal{P}G(q_1) = 0$, then $G(q_1) \in \mathcal{P}_{\text{ann}} = 0$, which is a contradiction. Hence, there exists $T \in \mathcal{P}$ such that $TG(q_1) \neq 0$. Therefore $F := TG \in M(A)$ satisfies $F(q_2) = \cdots = F(q_n) = 0$ and $F(q_1) \neq 0$. $\square$

The above statement allows us to prove the following

Corollary 1.11. *If S is a finitely generated C-submodule of Q contained in A, then S is an ε-closed subspace of A.*

Proof. By Corollary 1.4, there are $\{q_1, q_2, \ldots, q_n\}$ such that $S = \bigoplus_{i=1}^{n} Cq_i$. Take $a \in A \setminus S$. By Corollary 1.10 there exists $F \in M(A)$ such tat $F(q_1) = \cdots = F(q_n) = 0$ (so $F \in S^{\text{ann}}$) and $F(a) \neq 0$. Therefore, $a \notin \widehat{S}^A$, that is, $\widehat{S}^A = S$. $\square$

2. Martindale's lemma for nonassociative algebras. In this section we will present an extension of Martindale's lemma for nonassociative algebras.

Recall that if I is an ideal of A we denote by $[I : A]$ the ideal of $M(A)$ defined by

$$[I : A] := \{F \in M(A) \, : \, F(A) \subseteq I\}.$$

Lemma 2.1. *Let I be an ideal of A. Then $e_I = e_{L_I} = e_{[I:A]}$.*

Proof. Fix $x \in I$ and $a \in A$, and let us see that $e_{L_I} xa = e_{L_I} L_x(a) = L_x(a) = xa$, and so $(e_{L_I} x - x)a = 0$. As a consequence, we have $(e_{L_I} x - x)A = 0$. Since A is dense in Q, we deduce that $e_{L_I} x = x$. Taking in mind the arbitrariness of x, we have $e_{L_I} I = I$. Thus $e_I = e_{e_{L_I}} = e_{L_I} e_I$, and hence $e_I \leq e_{L_I}$. On the other hand, since $L_I \subseteq [I : A]$, we have $e_{L_I} \leq e_{[I:A]}$. Moreover, for each $F \in [I : A]$ and for each $a \in A$, we have $F(a) \subseteq I$, and so $e_I F(a) = F(a)$ or equivalently $(1 - e_I)F(a) = 0$. In particular, again since A is dense in Q, we deduce that $(1 - e_I)[I : A] = 0$. Therefore, $0 = e_{(1-e_I)[I:A]} = (1 - e_I)e_{[I:A]}$, thus $e_{[I:A]} \leq e_I$. $\square$

The net result can be seen as a nonassociative version of [3, Lemma 2.3.10].

Lemma 2.2. *Let $\mathcal{T}$ be a subset of $M_C(Q)$ and let S be a subset of A. Then the following assertion are equivalent:*

(1) $\mathcal{T}\mathcal{P}(S) = 0$ *for some essential ideal $\mathcal{P}$ of $M(A)$;*

(2) $[M(A)(S) : A]M_C(Q)\mathcal{T} = 0$*;*

(3) $e_S \mathcal{T} = 0$*;*

(4) $e_S e_{\mathcal{T}} = 0$*;*

(5) $e_{\mathcal{T}} S = 0$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{T}\mathcal{P}(S) = 0$, in particular $\mathcal{T}\mathcal{P}M(A)(S) = 0$, thus $\mathcal{T}\mathcal{P}[M(A)(S) : A] = 0$. Hence the ideal $\mathcal{V} = [M(A)(S) : A]M_C(Q)\mathcal{T}\mathcal{P} \cap M(A)$ of $M(A)$ satisfies $\mathcal{V}^2 = 0$. Since $M(A)$ is semiprime, we have $[M(A)(S) : A]M_C(Q)\mathcal{T}\mathcal{P} \cap M(A) = 0$. By [11, Proposition 2.1] we know that $[M(A)(S) : A]M_C(Q)\mathcal{T}\mathcal{P} = 0$. By [6, Proposition 3.4], $\mathcal{P}(A)$ is a dense ideal of A, and so, by [12, Proposition 2.3], we obtain that $[M(A)(S) : A]M_C(Q)\mathcal{T} = 0$.

(2) $\Rightarrow$ (3). Taking in mind [9, Theorem 1.8], $\mathcal{T} \subseteq (1 - e_{[M(A)(S):A]})M_C(Q) \cap M_C(Q)$. It follows that $e_{[M(A)(S):A]}\mathcal{T} = 0$, and since $S \subseteq M(A)(S) \subseteq M_C(Q)(S)$, by Lemma 2.1 and Lemma 1.3, $e_{[M(A)(S):A]} = e_S$, thus we have $e_S \mathcal{T} = 0$.

(3) $\Rightarrow$ (4). $0 = e_{e_S \mathcal{T}} = e_S e_{\mathcal{T}} = 0$.
(4) $\Rightarrow$ (5). $e_{\mathcal{T}} S = e_{\mathcal{T}} e_S S = 0$.
(5) $\Rightarrow$ (1). $\mathcal{T}\mathcal{P}(S) = e_{\mathcal{T}} \mathcal{T}\mathcal{P}(S) = \mathcal{T}\mathcal{P}(e_{\mathcal{T}} S) = 0$. □

Next, we present an operator that performs a similar role to the operator $M_{a,b} = L_a R_b$ in the associative context. For $F \in M_C(Q)$ and $q \in Q$ we denote by $W_{F,q}$ the linear operator from $M_C(Q)$ in Q given by

$$W_{F\cdot q}(S) = FS(q) \quad \text{for all} \quad S \in M_C(Q).$$

Moreover, for each subset $\mathcal{S}$ of $M_C(Q)$, we will denote by $W^{\mathcal{S}}_{F\cdot q}$ the restriction of $W_{F,q}$ to $\mathcal{S}$. We can rewrite Lemma 2.2 as follows.

Proposition 2.3. *Let $\mathcal{P}$ be an essential ideal of $M(A)$. Let $F \in M_C(Q)$ and let $q \in Q$. Then the following assertions are equivalent:*

(1) $W_{F\cdot q} = 0$; (2) $W^{\mathcal{P}}_{F\cdot q} = 0$; (3) $e_{[F]} q = 0$; (4) $e_{[q]} F = 0$; (5) $e_{[F]} e_{[q]} = 0$.

As a first consequence we get two curious results.

Corollary 2.4. *Let $p \in Q$ and $\lambda \in C$. Then $e_{[\lambda p]} = e_{[\lambda]} e_{[p]}$.*

Proof. First, suppose that A is associative. It is obvious that $\lambda q(1 - e_{\lambda p})p = 0$ for all $q \in Q$. Therefore, by [3, Lemma 2.3.10], $e_{[\lambda]}(1 - e_{[\lambda p]})e_{[p]} = 0$. Hence $e_{[\lambda]} e_{[p]} \leq e_{[\lambda p]}$. On the other hand, it is clear that $e_{[\lambda]} e_{[p]} e_{[\lambda p]} = e_{[\lambda p]}$, thus $e_{[\lambda p]} = e_{[\lambda]} e_{[p]}$. Suppose now that A is nonassociative. It is clear that

$$W_{\lambda Id_Q \cdot (1 - e_{[\lambda p]})p} = W_{Id_Q \cdot (1 - e_{[\lambda p]})\lambda p} = W_{Id_Q \cdot \lambda p} - W_{Id_Q \cdot e_{[\lambda p]} \lambda p} = 0.$$

Since $M(Q)$ is an associative algebra, by the first assertion, we deduce that $e_{[\lambda Id_Q]} = e_{[\lambda]}$. Hence, by Proposition 2.3, $e_{[\lambda]} e_{[p]}(1 - e_{[\lambda p]}) = 0$, thus $e_{[\lambda} e_{[p]} \leq e_{[\lambda p]}$. The other inequality is obvious as we have seen. □

Corollary 2.5. *Let $\mathcal{P}$ be an essential ideal of $M(A)$, $i = 1, 2, \ldots, n$, $F_i \in M_C(Q)$ and $q_i \in Q$ with some $e_{[F_i]} e_{[q_i]} \neq 0$. Then there exist $F \in M(A)$, $a \in A$, and $e \in \mathcal{I}_C$ such that*

$$0 \neq F\mathcal{P}(a) \subseteq e \sum_{i=1}^{n} F_i \mathcal{P}(q_i).$$

Proof. Suppose for simplicity that $e_{[F_1]} e_{[q_1]} \neq 0$. By Corollary 1.10, there exists $G \in M(A)$ such that $G(e_{[F_1]} q_1) \neq 0$ and $G(q_i) = 0$ for all $i \in$

$\{2,\ldots,n\}$. Put $q_1' = G(e_{F_1}q_1)$ and note that for each $T \in \mathcal{P}$, we have

$$\sum_{i=1}^{n} F_i T G(q_i) = \sum_{i=1}^{n} F_i T G(e_{[F_i]}q_i) = F_1 T(q_1').$$

Thus

$$F_1\mathcal{P}q_1' \subseteq \sum_{i=1}^{n} F_i\mathcal{P}(q_i).$$

On the other hand, since $M(A)$ is an associative algebra and by Theorem 1.9 $G \in M_C(Q) = Q(M(A))$, there exists an essential ideal $\mathcal{U}$ of $M(A)$ such that $\mathcal{U} \subseteq \mathcal{P}$ and $F_1\mathcal{U} \subseteq M(A)$. Take $S \in \mathcal{U}$ such that $F_1 S \neq 0$ and $e = e_{[F_1 S(q_1')]}$. In particular $eq_1' \neq 0$ (in other case, $0 = ee_{[q_1']} = e_{F_1 S(q_1')}e_{[q_1']} = e_{[F_1 S(q_1')}]$, which is a contradiction).

By properties (P2) and (P3) there exists an essential ideal D of A, such that $0 \neq DM(A)(q_1') \subseteq A$. Setting $F = F_1 S$ and $0 \neq a \in DM(A)(eq_1') = eDM(A)(q_1')$, we have

$$F\mathcal{P}(a) = F_1 S\mathcal{P}(a) \subseteq F_1 S\mathcal{P}DM(A)(eq_1') \subseteq F_1\mathcal{P}(eq_1') \subseteq e\sum_{i=1}^{n} F_i\mathcal{P}(p_i).$$

Finally, let us see that $e_{[F]}a = a$. Indeed, there exist $x \in D$ and $H \in M(A)$ such that $a = xH(eq_1')$ and since $e_{[F]}e = e$, we have $e_{[F]}a = a$. In particular $e_{[F]}e_{[a]} = e_{[a]} \neq 0$. Thus, by Proposition 2.3, $0 \neq F\mathcal{P}(a)$. □

For an arbitrary sum we have

Corollary 2.6. *Let $q_1,\ldots,q_n \in Q$ and $F_1,\ldots,F_n \in M_C(Q)$ and let $\mathcal{P}$ be an essential ideal of $M(A)$. If $q_1,\ldots,q_n$ or $F_1,\ldots,F_n$ are linearly C-independent, then $\sum_{i=1}^{n} W^{\mathcal{P}}_{F_i,q_i} = 0$ if and only if $e_{[F_i]}e_{[q_i]} = 0$ for all $i \in \{1,\ldots,n\}$.*

Proof. Suppose that $\sum_{i=1}^{n} W_{F_i,q_i} = 0$. Assume that $q_1,\ldots,q_n$ are linearly C-independent. In order to obtain a contradiction, we assume that there exists i such that $e_{[F_i]}e_{[q_i]} \neq 0$. For simplicity we can suppose that $e_{[F_1]}e_{[q_1]} \neq 0$. By Corollary 1.10, there exists $G \in M(A)$ such that $G(e_{[F_1]}q_1) \neq 0$ and $G(q_i) = 0$ for all $i \in \{2,\ldots,n\}$. Put $q_1' = G(e_{F_1}q_1)$ and note that for each $T \in M_C(Q)$ we have

$$0 = \sum_{i=1}^{n} F_i T G(q_i) = F_1 T G(e_{[F_1]}q_1) = F_1 T(q_1').$$

Thus, by Proposition 2.3 we have $0 = e_{[F_1]}q_1' = q_1'$, which is a contradiction.

If we assume that $F_1, \ldots, F_n$ are linearly C-independent then, according to [3, Theorem 2.3.3], we can follows a similar argument: Take $H_j, K_j \in M(A)$ such that $\mathcal{H} \in M(M(A))$ defined by $\mathcal{H}(H) = \sum_{j=1}^{m} H_j H K_j$ verifies that $\mathcal{H}(e_{[q_1]}F_1) \neq 0$ and $\mathcal{H}(F_i) = 0$ for all $i \in \{2, \ldots, n\}$. Put $F_1' = \mathcal{H}(e_{[p_1]}q_1) \neq 0$, and note that, for every $H \in M(A)$, we have:

$$0 = \sum_{j=1}^{m} H_j \left(\sum_{i=1}^{n} F_i K_j H \right) (q_i) = \sum_{i=1}^{n} \mathcal{H}(F_i) H(q_i) = F_i' H(q_1).$$

Thus, by Proposition 2.3 we have $0 = e_{[q_1]}F_1'$, which is a contradiction.

In both cases, the converse is obvious since

$$\sum_{i=1}^{n} W_{F_i, e_{[F_i]} e_{[q_i]} q_i} = \sum_{i=1}^{n} W_{F_i, q_i}. \qquad \square$$

Remark 2.7. Taking in mind Corollary 1.4, it is easy to prove that for every $F_1, \ldots, F_n \in M_C(Q)$, $q_1, \ldots, q_n \in Q$, and for every essential ideal $\mathcal{P}$ of $M(A)$:

$$\sum_{i=1}^{n} W_{F_i, q_i}^{\mathcal{P}} = 0 \quad \text{if and only if} \quad \sum_{i=1}^{n} W_{F_i, q_i} = 0.$$

Our next result is an m.s.p. version of Proposition 1.6, which can be considered as an m.s.p. version of Matindale's lemma.

Theorem 2.8. *Let $p_i, q_j \in Q$ and $F_i, G_j \in M_C(Q)$ $(1 \leq i \leq n, 1 \leq j \leq m)$, and let $\mathcal{P}$ be an essential ideal of $M(A)$ satisfying*

$$\sum_{i=1}^{n} F_i X(p_i) = \sum_{j=1}^{m} G_j X(q_j) \quad \textit{for every} \quad X \in \mathcal{P}. \tag{5}$$

If $p_1, \ldots, p_n$ are linearly C-independent, then each $e_{p_i}F_i$ is a C-linear combination of $G_1, \ldots, G_m$. If $F_1, \ldots, F_n$ are linearly C-independent, then each $e_{F_i}p_i$ is a C-linear combination of $q_1, \ldots, q_m$.

P r o o f. Assume that $p_1, p_2, \ldots, p_n$ are linearly C-independent. By a similar argument to that used in the proof of Proposition 1.6, we deduce that, for each $X \in \mathcal{P}$,

$$\sum_{i=1}^{n} F_i X(p_i) = \sum_{j=1}^{m} G_j X(q_j) = \sum_{j=1}^{m} G_j X\left(\sum_{i=1}^{n} \alpha_i^j p_i + \sum_{k=1}^{r} \beta_k^j q'_k\right),$$

for suitable $r \in \mathbb{N}$, $\alpha_i^j, \beta_k^j \in C$ and $q'_k \in Q$ such that $p_1, p_2, \cdots, p_n$, $q'_1, q'_2, \cdots, q'_r$ are linearly C-independent. Hence

$$\sum_{i=1}^{n} \left(F_i - \sum_{j=1}^{m} \alpha_i^j G_j\right) X(p_i) = \sum_{k=1}^{r} \left(\sum_{j=1}^{m} \beta_k^j G_j\right) X(q'_k).$$

Therefore, by Corollary 2.6, $e_{[F_i - \sum_{j=1}^{m} \alpha_i^j G_j]} e_{[p_i]} = 0$, for all i. As a consequence, $e_{[p_i]} F_i = \sum_{j=1}^{m} e_{[p_i]} \alpha_i^j G_j$.

Assume that $F_1, F_2, \ldots, F_n$ are linearly C-independent. Taking in mind Theorem 1.9 and Corollary 1.4, by a similar argument, we can write, for each $j \in \{1, \ldots, m\}$

$$\sum_{i=1}^{n} F_i X\left(p_i - \sum_{j=1}^{m} \alpha_i^j q_j\right) = \sum_{k=1}^{r} G'_j X\left(\sum_{j=1}^{m} \beta_k^j q_j\right),$$

for suitable $r \in \mathbb{N}$, $\alpha_i^j, \beta_k^j \in C$ and $G'_k \in M_C(Q)$ such that $F_1, F_2, \ldots, F_n$, $G'_1, G'_2, \ldots, G'_r$ are linearly C-independent. Therefore, again by Corollary 2.6, $e_{[p_i - \sum_{j=1}^{m} \alpha_i^j q_j]} e_{[F_i]} = 0$, for all i. As a consequence, $e_{[F_i]} p_i = \sum_{j=1}^{m} e_{[F_i]} \alpha_i^j q_j$. $\square$

From Theorem 2.8 we find an m.s.p.-version of [3, Theorem 2.3.11].

Corollary 2.9. *Let $F, G \in M_C(Q)$ and $p, q \in Q$. Then the following conditions are equivalent:*

(i) $W_{F,p} = W_{G,q}$*;*

(ii) *There exists $\mu \in C$ invertible such that $e_{[p]}(\mu F - e_{[F]} G) = 0$ and $e_{[G]}(\mu q - e_{[q]} p) = 0$;*

(iii) *There exists $\lambda \in C$ such that $e_{[p]}(F - \lambda G) = 0$ and $e_{[G]}(\lambda p - q) = 0$.*

In this case, $e_{[F]} e_{[p]} = e_{[G]} e_{[q]}$.

P r o o f. (i) $\Rightarrow$ (ii). First of all, note that $e_{[F]}e_{[p]} = e_{[G]}e_{[q]}$. In fact

$$W_{F,(1-e_{[G]}e_{[q]})p} = W_{F,p} - e_{[G]}e_{[q]})W_{F,p} = W_{F,p} - e_{[G]}e_{[q]})W_{G,q} = 0.$$

Therefore, by Proposition 2.3 $e_{[F]}(1 - e_{[G]}e_{[q]})e_{[p]} = 0$, that is, $e_{[F]}e_{[p]} \leq e_{[G]}e_{[q]}$. Applying a similar argument for G and q allows us to conclude the equality. Let us call $e = e_{[F]}e_{[p]}$. By Theorem 2.8, there exists $\lambda \in C$ such that $e_{[p]}F = \lambda G$. It is clear that $eF = \lambda eG$, and taking in mind Corollary 2.4,

$$e = e_{[e_{[p]}F]} = e_{[\lambda G]} = e_{[\lambda]}e_{[G]]} = e_{[\lambda]}e_{[G]]}e = e_{[\lambda]}e = e_{[e\lambda]}.$$

Take $\beta = e\lambda + (1 - e)$. By Corollary 1.2, β is invertible and it is clear that $eF = e_{[p]}F = e\lambda G = e\beta G$. Therefore,

$$W_{G,q} = W_{F,p} = W_{eF,p} = W_{\beta eG,p} = W_{G,\beta ep}$$

and by Corollary 2.6, $e_{[G]}(q - \beta p) = 0$. Finally, multiplying by $\mu = \beta^{-1}$ in both equalities we obtain that $0 = e_{[G]}(\mu q - ep) = e_{[G]}(\mu q - e_{[q]}p)$.

(ii) $\Rightarrow$ (iii). Multiply by μ^{-1} in both equalities and take $\lambda = \mu^{-1}e$.

(iii) $\Rightarrow$ (i). Suppose that there exists $\lambda \in C$ such that $e_{[p]}(F - \lambda G) = 0$ and $e_{[G]}(\lambda p - q) = 0$. It is clear that

$$W_{F,p} = W_{e_{[p]}\lambda G,p} = W_{G,\lambda p} = W_{G,e_{[g]}q} = W_{G,q}. \qquad \square$$

3. Semiprime GPI-theorem. In this section we will give a nonassociative version of a semiprime GPI-theorem (see [3, section 3]). To this end, we need to know what happens when the rank of an operator of $M(Q)$ is finite over the extended centroid.

First of all let us see some sufficient condition that ensures the existence of finite rank operators.

Proposition 3.1. *Let $F_i, G_i, H_j, K_j \in M_C(Q)$ and p_i, q_j be in Q such that*

$$\sum_{i=1}^{n} F_i X G_i Y(p_i) = \sum_{j=1}^{m} H_j Y K_j X(q_j).$$

Then there exist $F \in M_C(Q)$ and $q \in Q$ such that the operator $W_{F,q}$ is nonzero and has finite rank, whenever $e_{[F_i]}e_{[G_i]}e_{[p_i]} \neq 0$ for some i.

Proof. By Corollary 1.4 and Corollary 2.6 we can assume without loss of generality that the set $\{F_1, F_2, \ldots, F_n\}$ is linearly C-independent and $e_{[F_i]}e_{[G_i]}e_{[p_i]} \neq 0$ for some i. Fix $Y \in M_C(Q)$. By Theorem 2.8, we have that $e_{[F_i]}G_iY(p_i)$ is a C-linear combination of $\{q_1, q_2, \ldots, q_m\}$. Therefore, take $F = e_{[F_i]}G_i$ and $q = p_i$ to conclude. □

Now, we see some equivalent conditions.

Proposition 3.2. *Then the following assertions are equivalent:*

(i) *There exists F be in $M_C(Q) \setminus \{0\}$ such that $F(Q)$ is C-finitely generated;*

(ii) *There exists G in $M(A)\setminus\{0\}$ such that $GM_C(Q)G(a)$ is C-finitely generated for some $a \in A$ such that $G(a) \neq 0$.*

Proof.

(i) $\Rightarrow$ (ii). Since $F \in M_C(Q) \setminus \{0\}$ there exists $0 \neq p \in Q$ such that $q = F(p) \neq 0$, that is, $0 \neq e_{F(p)} = e_F e_{F(p)} = e_F e_q$, and so, by Proposition 2.3, $e_q F \neq 0$ and, of course, by assumption $FM_C(Q)(q)$ C-finitely generated. By Corollary 2.5 there is $G \in M(A)$ and $b \in A$ such that $0 \neq GM(A)(b) \subseteq FM_C(Q)(q)$. Since $0 \neq GM(A)(b)$, there is $T \in M(A)$ such that $0 \neq GT(b)$. If we take $a = T(b)$, then we have

$$GM_C(Q)G(a) \subseteq GM_C(Q)(b) \subseteq FM_C(Q)(q),$$

and so, $GM_C(Q)G(a)$ is C-finitely generated.

(ii) $\Rightarrow$ (i). Assume that there is $a \in A$ and $G \in M(A)$ satisfying that $GM_C(Q)G(a)$ is C-finitely generated and $G(a) \neq 0$. Take $F = e_{[G(a)]}G$. By Corollary 1.8,

$$F(Q) \subseteq F([M_C(Q)G(a) + (1 - e_{[G(a)]})Q \cap Q)]^{\wedge}),$$

and so, by the continuity of the ε-closure

$$F(Q) \subseteq [F(M_C(Q)G(a) + (1 - e_{[G(a)]})Q \cap Q)]^{\wedge_Q} = [G(M_C(Q)G(a))]^{\wedge_Q}.$$

Thus, by Corollary 1.11, $F(Q) \subseteq G(M_C(Q)G(a)))$, which asserts that $F(Q)$ is C-finitely generated. □

In light of Proposition 3.2, the following result becomes an m.s.p. version of [3, Lemma 6.1.4] (see also [20, Theorem 2]).

Lemma 3.3. *If there exists $F \in M_C(Q)$ such that $F(Q)$ is a finitely generated C-submodule of Q, then $FM_C(Q)F$ is a finitely generated C-submodule of Q.*

Proof. Let $F \in M_C(Q)$ be such that $F(Q)$ is a finitely generated C-submodule of Q. By [13, Corollary 1.4] there exist $q_1, q_2, \dots, q_n \in Q$ such that $F(Q) = \bigoplus_{i=1}^{n} Cq_i$. Let $p_i \in Q$ be such that $F(p_i) = q_i$. On the other hand, for each i consider the map $\varphi_i : Q \to e_{[q_i]}C$ defined by $F(q) = \sum_{i=1}^{n} \varphi_i(q)q_i$ for all $q \in Q$. Note that by assumption on q_i's, φ_i is unique. Thus, set

$$F = \varphi_1 \otimes q_1 + \dots + \varphi_n \otimes q_n,$$

where $(\varphi_i \otimes q_i)(q) := \varphi_i(q)q_i$, for every $q \in Q$. Then for any $G \in L_C(Q)$, $i, j \in \{1, \dots, n\}$ and $q \in Q$, we have

$$(\varphi_j \otimes q_j)G(\varphi_i \otimes q_i)(q) = (\varphi_j \otimes q_j)G(\varphi_i(q)q_i) = (\varphi_j \otimes q_j)(\varphi_i(q)G(q_i))$$

$$= \varphi_i(q)\varphi_j(G(q_i))q_j = \varphi_j(G(q_i))(\varphi_i \otimes q_j)(q),$$

which proves that $FM_C(Q)F$ is spanned by the rank one operators $\varphi_i \otimes q_j$, for $i, j \in \{1, \dots, n\}$, and therefore is C-finite dimensional. $\square$

Let $\mathcal{A}$ be a semiprime associative algebra. Given a non-empty set $\mathbf{X}$ of variables, we can consider the C-algebra $Q_s(\mathcal{A})\langle \mathbf{X} \rangle_C$ given by the coproduct over C of the C-algebra $Q_s(\mathcal{A})$ and the unital free associative algebra $C\langle \mathbf{X} \rangle_1$. The elements of $Q_s(\mathcal{A})\langle \mathbf{X} \rangle_C$ are called *generalized polinomial identities* (in short GPI's). As usual we write a GPI Φ in the form $\Phi(x_1, \dots, x_n)$ to indicate the variables that Φ involves. Given a map $s : \mathbf{X} \to Q_s(\mathcal{A})$, there exists a unique unital C-algebra homomorphism from $Q_s(\mathcal{A})\langle \mathbf{X} \rangle_C$ to $Q_s(\mathcal{A})$ extending the maps s and $\mathrm{Id}_{Q_s(\mathcal{A})}$. Such a homomorphism is also denoted by s and is called a *substitution*. Given a GPI $\Phi = \Phi(x_1, \dots, x_n)$ and $q_1, \dots, q_n \in Q_s(\mathcal{A})$, we put $\Phi(q_1, \dots, q_n)$ to indicate the value of $s(\Phi)$ for any substitution s such that $s(x_i) = q_i$ $(1 \leq i \leq n)$. The algebra $\mathcal{A}$ is said to be *GPI* whenever $\mathcal{A}$ satisfies a nonzero GPI.

Finally, we obtain the semiprime GPI-theorem in nonassociative context.

Theorem 3.4. *Let A be an m.s.p. algebra. Then the following assertions are equivalent:*

(1) *There exists a nonzero operator $F \in M_C(Q)$ such that $F(Q)$ is C-finitely generated;*

(2) *$M(A)$ is GPI;*

(3) *There are* $F_i, G_i, H_j, K_j \in M_C(Q)$ *and* $p_i, q_j \in Q$ *with* $1 \le i \le n, 1 \le j \le m$ *such that*

$$\sum_{i=1}^{n} F_i X G_i Y(p_i) = \sum_{j=1}^{m} H_j Y K_j X(q_j). \tag{6}$$

for all $X, Y \in M_C(Q)$ *and* $e_{[F_i]} e_{[G_i]} e_{[p_i]} \neq 0$ *for some* i*;*

(4) *There exists* $F \in M(A)$ *and* $a \in A$ *such that the* $0 \neq FM_C(Q)F(a))$ *is* C*-finitely generated.*

Proof. (1) $\Rightarrow$ (2). Suppose that $F \in M_C(Q)$ verifies that $F(Q)$ is C-finitely generated. By Lemma 3.3, $FM_C(Q)F$ is also C-finitely generated. By [13, Theorem 2.1] (see also [3, Remark 6.3.4 and Theorem 6.3.11]) we can assume without loss of generality that F is idempotent. Taking in mind Theorem 1.9, if $\dim_{\mathcal{I}_C}(FM_C(Q)F) = n$ and St_{n+1} is the standard polynomial in $n+1$ variables, then

$$\phi = St_{n+1}(EX_1E, EX_2E, \cdots, EX_{n+1}E)$$

is the required a GPI.

(2) $\Rightarrow$(3). Taking in mind Theorem 1.9, by [3, Theorem 6.3.8] there exists an idempotent $F \in M_C(Q)$ such that $FM_C(Q)F$ is C-finitely generated. Let $n = \dim_{\mathcal{I}_C}(FM_C(Q)F)$. If $n = 1$ then $FXFYF - FYFXF = 0$ for all X and Y, and so the proof is easily derived. Now, suppose that $n > 1$. Without loss of generality, we assume that there are $F'_1, \ldots, F'_n \in M_C(Q)$ such that $St_n(FF'_1F, \ldots, FF'_nF) \neq 0$ and $0 = St_{n+1}(FXF, FYF, FF'_3F, \ldots, FF'_nF, F)$ for all X and Y. Rearranging summands, we have

$$0 = \sum_{i=1}^{n} F_i X G_i Y H_i - \sum_{j=1}^{m} J_j Y K_j X L_j$$

for all $X, Y \in M_C(Q)$ and for convenient $F_i, G_i, H_i, J_j, K_j, L_j \in M_C(Q)$. In fact, by the assumption on F_i''s, we have $e_{[F_i]} e_{[G_i]} e_{[H_i]} \neq 0$ for some i. In particular, by Lemma 1.5 there is $p \in Q$ such that $0 \neq \sum_{i=1}^{n} F_i X G_i Y H_i(p)$, and hence for all $X, Y \in M_C(Q)$

$$\sum_{i=1}^{n} F_i X G_i Y(p_i) = \sum_{j=1}^{m} K_j Y L_j X(q_j),$$

where $p_i = H_i(p)$ and $q_j = M_j(p)$.

(3) ⇒ (4). It follows from Proposition 3.1 and Proposition 3.2.

(4) ⇒ (1). It follows from Proposition 3.2. □

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