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NOTE ON A PAPER OF ELABBASY AND HASSAN

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ABSTRACT. We give a correction to the proof of Theorem 2.3 in the paper of E. M. Elabbasy and T. S. Hassan, Serdica Math. J. **34** (2008), 531–542.

Introduction. In the paper [1], the authors established some new sufficient conditions for oscillation of all solutions of nonlinear neutral delay differential equations. The results extended and improved some of the well known results in the literature. They considered the first order neutral delay differential equation

$$(E) \qquad [x(t) - q(t)x(t - \sigma(t))]' + f(t, x(t - \tau(t))) = 0, \quad \text{for} \quad t \geq t_0 > 0,$$

subject to the conditions

$$(1.5) \qquad \begin{aligned} &\sigma, \tau \in C\left([t_0, \infty), \mathbb{R}^+\right), \ \mathbb{R}^+ = (0, \infty), \\ &\lim_{t\to\infty} (t - \sigma(t)) = \infty, \ \lim_{t\to\infty} (t - \tau(t)) = \infty, \end{aligned}$$

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$$q \in C\left([t_0,\infty),\mathbb{R}^+\right),\ 0 \le q_1 \le q(t) \le q_2 \le 1, \tag{1.6}$$

and

$$f \in C[t_0,\infty) \times \mathbb{R},\mathbb{R}),\ \mathbb{R} = (-\infty,\infty),\ uf(u) \ge 0. \tag{1.7}$$

Moreover, there exist

$$p \in C\left([t_0,\infty),\mathbb{R}^+\right), \quad \text{and} \quad g \in C(\mathbb{R},\mathbb{R}), \tag{1.8}$$

such that

$$g'(u) \ge 0,\ f(t,u) \ge p(t)g(u), \quad \text{and} \quad |g(u) - u| \le M|u|^{1+r}, \tag{1.9}$$

for $u \in (-\epsilon,\epsilon)$, $\epsilon > 0$, $M \ge 0$, $r > 0$, $ug(u) > 0$ for $u \ne 0$.

E. M. Elabbasy et al. [1] proved that if $\tau(t) = \tau > 0$,

$$0 < \frac{1}{a} \le \int_{t-\tau}^{t} p(s)ds,\ for\ t \ge t_0,$$

and

$$\int_{t_0}^{\infty} p(t)\exp\left(a \int_{t-\sigma+\tau}^{t} p(s)ds\right) dt = \infty, \tag{2.10}$$

then every solution of equation (E) oscillates.

Remark 1.1. In page 538, line 5 of paper [1], the authors assumed that the limits of integration are T and N, where

$$T \le N \text{ such that } 0 < N - \tau < T.$$

Unfortunately, τ takes very small value $(\tau \to 0)$. So, if N tends to infinity, also T will tend to infinity, and the limits of integration become incorrect (please, see the $R.\ H.\ S$ of the inequality in page 539, line 3, in [1], and (2.10)). Therefore, the proof of Theorem 2.3, in [1] is incorrect.

We present the correct proof in the section below.

2. The correct proof. In this section, we prove Theorem 2.3 in the paper of E. M. Elabbasy and T. S. Hassan [1].

P r o o f. Proceeding as in the proof of Theorem 2.3 in [1], one obtains the following estimate

$$\lambda(t)A(t) - \frac{1}{2}p(t)\int_{t-\tau}^{t} \lambda(s)ds \geq \frac{1}{2}p(t)A(t),$$

where

$$A(t) = \exp\left(a\int_{t-\tau}^{t} p(s)ds\right).$$

Integrating from T to N where $N > T + \tau$ yields

$$\int_{T}^{N} \lambda(t)A(t)dt - \frac{1}{2}\int_{T}^{N}\left(p(t)\int_{t-\tau}^{t} \lambda(s)ds\right)dt \geq \frac{1}{2}\int_{T}^{N} p(t)A(t)dt.$$

Interchanging the order of integration, we get

$$\begin{aligned}\int_{T}^{N}\left(p(t)\int_{t-\tau}^{t} \lambda(s)ds\right)dt &\geq \int_{T}^{N-\tau}\left(\lambda(s)\int_{s-\tau}^{s} p(t)dt\right)ds \\ &\geq \int_{T}^{N-\tau}\left(\lambda(t)\int_{s-\tau}^{s} p(s)ds\right)dt.\end{aligned}$$

Thus, we have

$$\int_{T}^{N} \lambda(t)A(t)dt - \frac{1}{2}\int_{T}^{N-\tau}\left(\lambda(t)\int_{s-\tau}^{s} p(s)ds\right)dt \geq \frac{1}{2}\int_{T}^{N} p(t)A(t)dt.$$

Now, we denote

$$B(t) = \frac{1}{2}\int_{t-\tau}^{t} p(s)ds.$$

Then

$$(i) \qquad 0 < \frac{1}{2a} \leq B(t) \leq 1,$$

as

$$\int_{t-\tau}^{t} p(s)ds \leq 2, \quad \text{(see [1], page 537).}$$

Since

$$\begin{aligned} A(t) &= \exp\left(a\int_{t-\tau}^{t} p(s)ds\right) \geq a\int_{t-\tau}^{t} p(s)ds \\ &\geq \frac{1}{2}\int_{t-\tau}^{t} p(s)ds = B(t), \quad \text{where} \ \ a \geq \frac{1}{2}, \end{aligned}$$

then

(ii)
$$A(t) \geq B(t), \quad \text{for} \ \ all \ t \geq T,$$

and since

$$1 \leq a\int_{t-\tau}^{t} p(s)ds \leq 2a,$$

then

(iii)
$$e \leq A(t) = \exp\left(a\int_{t-\tau}^{t} p(s)ds\right) \leq e^{2a} = k.$$

From (i), (ii), (iii) we obtain

(iv)
$$0 < e - 1 = k_2 \leq A(t) - B(t) \leq e^{2a} - \frac{1}{2a} = k_1.$$

Then

$$\begin{aligned} &\int_{T}^{N} \lambda(t)A(t)dt - \int_{T}^{N-\tau} \lambda(t)A(t)dt + \int_{T}^{N-\tau} \lambda(t)A(t)dt \\ &\qquad - \frac{1}{2}\int_{T}^{N-\tau}\left(\lambda(t)\int_{s-\tau}^{s} p(s)ds\right)dt \geq \frac{1}{2}\int_{T}^{N} p(t)A(t)dt. \end{aligned}$$

Hence,

$$\left[\int_T^N \lambda(t)A(t)dt - \int_T^{N-\tau} \lambda(t)A(t)dt\right] + \int_T^{N-\tau} \lambda(t)[A(t)-B(t)]dt \geq \frac{1}{2}\int_T^N p(t)A(t)dt,$$

and therefore

$$\int_{N-\tau}^N \lambda(t)A(t)dt + \int_T^{N-\tau} \lambda(t)[A(t)-B(t)]dt \geq \frac{1}{2}\int_T^N p(t)A(t)dt.$$

From (iii), (iv) we obtain

$$k\int_{N-\tau}^N \lambda(t)dt + k_1\int_T^{N-\tau} \lambda(t)dt \geq \frac{1}{2}\int_T^N p(t)A(t)dt,$$

and

$$\ln\frac{z(N-\tau)}{z(N)} - \frac{k_1}{k}(\ln(z(N-\tau)) - \ln(z(T))) \geq \frac{1}{2k}\int_T^N p(t)\exp\left(a\int_{t-\tau}^t p(s)ds\right)dt.$$

Since, $z(t) > 0$ (as in [1], page 537), we obtain by (2.10) that

$$\lim_{t\to\infty}\frac{z(t-\tau)}{z(t)} = \infty,$$

which contradicts the fact that

$$\lim_{t\to\infty}\frac{z(t-\tau)}{z(t)} < \infty \text{ is bounded for } t > T.$$

This completes the proof of the theorem. □

REFERENCES

[1] E. M. Elabbasy, T. S. Hassan. Oscillation of nonlinear neutral delay differential equations. *Serdica Math. J.* **34**, *3* (2008), 531–542.

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