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TYPE II FAMILY OF BIVARIATE INFLATED-PARAMETER GENERALIZED POWER SERIES DISTRIBUTIONS*

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Abstract. The family of Inflated–parameter Generalized Power Series distributions (IGPSD) was introduced by Minkova in 2002 as a compound Generalized Power Series distributions (GPSD) with geometric compounding distribution. In this paper we introduce a family of compound GPSDs with bivariate geometric compounding distribution. The probability mass function, recursion formulas, conditional distributions and some properties are given. A member of this family is a Type II bivariate Pólya-Aeppli distribution, introduced by Minkova and Balakrishanan [7]. In this paper the particular cases of bivariate compound binomial, negative binomial and logarithmic series distributions are analyzed in detail and compared by the bivariate Fisher index of dispersion.

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1. Introduction. The probability generating function (PGF) of the Generalized Power Series Distributions (GPSD) with a parameter $\theta > 0$, is given by the following relation

$$\psi(s) = \frac{g(\theta s)}{g(\theta)}, \tag{1}$$

where $g(\theta)$ is a finite, differentiable function with positive derivatives in the form

$$g(\theta) = \sum_{m=0}^{\infty} a(m)\theta^m, \tag{2}$$

where

$$a(m) = \frac{g^{(m)}(\theta)}{m!}\,|_{\theta=0}$$

The binomial, negative binomial (NB), logarithmic series and Poisson distributions belong to this class, see Patil [10]. In the binomial and NB cases, the corresponding additional parameters n and r are given positive integer valued.

For a given parameter $\pi \in (0,1)$, the particular cases of the functions $a(m)$ and $g(\theta)$ and the parameter θ, are given in the following table.

$X \sim Bi(n,\theta)$:	$a(m) = \binom{n}{m}$	$g(\theta) = (1+\theta)^n$	$\theta = \frac{\pi}{1-\pi}$
$X \sim NB(r,\theta)$:	$a(m) = \binom{m+r-1}{m}$	$g(\theta) = (1-\theta)^{-r}$	$\theta = 1-\pi;$
$X \sim Po(\theta)$:	$a(m) = \frac{1}{m!}$	$g(\theta) = e^{\theta}$	$\theta = \lambda;$
$X \sim LS(\theta)$:	$a(m) = \frac{1}{m}$	$g(\theta) = -ln(1-\theta)$	$\theta = 1-\pi$

The Inflated-parameter generalized power series distributions (IGPSD) was introduced in Minkova [8] as a compound GPSDs. The random variables have the form $N = X_1 + \cdots + X_Z$, where Z has the GPSD, and the compounding random variables $X_1, X_2, \dots$ are distributed geometrically as the r.v. X with PMF

$$P(X = i) = (1-\rho)\rho^{i-1}, \quad i = 1, 2, \dots \tag{3}$$

and PGF

$$\psi_1(s) = Es^X = \frac{(1-\rho)s}{1-\rho s}. \tag{4}$$

The PGF of the IGPSDs is given by

$$\psi(s) = \frac{g(\theta\psi_1(s))}{g(\theta)}, \tag{5}$$

where $g(\theta)$ is the series function, with particular cases given in the table, and $\psi_1(s)$ is given in (4). The lack-of-memory property of the geometric distribution leads to some nice properties of the distributions and closed form of the PMFs; see Minkova [8].

In the case of $g(\theta) = e^\theta$ and $\theta = \lambda$, the compound Poisson distribution with geometric compounding distribution coincides with the Pólya-Aeppli distribution; see Johnson et al. [2]. The Pólya-Aeppli distribution was applied as a counting distribution in a risk model by Minkova [9]. The corresponding Pólya-Aeppli process was characterized by Chukova and Minkova [1]. The process is very closed to the Poisson process. This motivated us to define two types of bivariate Pólya-Aeppli distributions; see Minkova and Balakrishnan [6] and [7].

In this paper we define a family of bivariate IGPSDs, which member is the Type II bivariate Pólya-Aeppli distribution, given by Minkova and Balakrishnan [7]. The definition of the family with joint probability mass function is given in the next Section 2. In Section 3 we derive the conditional distributions and conditional moments. As examples, in Section 4 we consider additionally three members of the defined family of distributions: bivariate Inflated-parameter binomial distribution, bivariate Inflated-parameter negative binomial distribution and bivariate Inflated-parameter logarithmic series distribution.

2. Bivariate inflated-parameter generalized power series distributions. Let us consider the sequence $(X_i, Y_i), i = 1, 2, \dots$ of independent and identically distributed random variables, distributed as (X, Y). Define

$$N_1 = X_1 + \cdots + X_Z \quad \text{and} \quad N_2 = Y_1 + \cdots + Y_Z,$$

where Z is independent of (X, Y) and has a GPSD with PGF, given by (1). Suppose that (X, Y) has a bivariate geometric distribution with PGF

$$\psi_1(s_1, s_2) = \frac{\gamma}{1 - \alpha s_1 - \beta s_2}, \tag{6}$$

where $0 < \alpha, \beta < 1$ and $\gamma = 1 - \alpha - \beta \neq 0$; see Kocherlakota and Kocherlakota [4]. Then, the joint PGF of (N_1, N_2) is given by

$$\Psi(s_1, s_2) = \frac{g(\theta\psi_1(s_1, s_2))}{g(\theta)}, \tag{7}$$

where $\psi_1(s_1, s_2)$ is the PGF of the compounding distribution in (6).

Definition 2.1. *The probability distribution of* (N_1, N_2), *corresponding to* (7), *is referred to as a Type II family of Bivariate Inflated-parameter generalized power series distributions* $(BIGPSD_{II})$.

The marginal compounding distributions are geometric with PGFs

$$\psi_1(s_1) = \psi_1(s_1, 1) = \frac{\gamma}{1 - \beta - \alpha s_1} \quad \text{and} \quad \psi_1(s_2) = \psi_1(1, s_2) = \frac{\gamma}{1 - \alpha - \beta s_2}. \tag{8}$$

The random variable X has a geometric distribution starting at zero, with probability of success $\frac{\gamma}{1-\beta}$, i.e, $X \sim Ge_0(\frac{\gamma}{1-\beta})$. We define the parameter ρ_1 of the random variable X as $\rho_1 = \frac{\alpha}{1-\beta}$. Analogously, Y has a geometric distribution with parameter $\frac{\gamma}{1-\alpha}$ and $Y \sim Ge_0(\frac{\gamma}{1-\alpha})$. Denote by $\rho_2 = \frac{\beta}{1-\alpha}$ the parameter of Y. In the terms of ρ_1 and ρ_2, the marginal PGFs in (8) have the form

$$\psi_1(s_1) = \frac{1 - \rho_1}{1 - \rho_1 s_1} \quad \text{and} \quad \psi_1(s_2) = \frac{1 - \rho_2}{1 - \rho_2 s_2}. \tag{9}$$

Then, from (7) we obtain the corresponding marginal PGFs of N_1 and N_2

$$\Psi_{N_1}(s_1) = \Psi(s_1, 1) = \frac{g(\theta\psi_1(s_1))}{g(\theta)} \text{ and } \Psi_{N_2}(s_2) = \Psi(1, s_2) = \frac{g(\theta\psi_1(s_2))}{g(\theta)},$$

from which it follows that N_1 and N_2 belong to the family of univariate IGPSDs.

From (2), it follows that

$$g(\theta\psi_1(s_1, s_2)) = \sum_{m=0}^{\infty} a(m)\theta^m[\psi_1(s_1, s_2)]^m.$$

Then for the PGF we obtain

$$\Psi(s_1, s_2) = \frac{1}{g(\theta)} \sum_{m=0}^{\infty} a(m)\theta^m \frac{\gamma^m}{(1 - \alpha s_1 - \beta s_2)^m}. \tag{10}$$

Differentiation in (6) leads to the following derivatives

$$\psi_1^{(i,j)}(s_1, s_2) = \gamma \frac{(i+j)!\alpha^i \beta^j}{(1 - \alpha s_1 - \beta s_2)^{i+j+1}},$$

where $\psi_1^{(i,j)}(s_1, s_2) = \dfrac{\partial^{i+j}\psi_1(s_1, s_2)}{\partial s_1^i \partial s_2^j}$, for $i, j = 0, 1, \ldots$. Then, for the corresponding derivatives of $\Psi(s_1, s_2)$ in (10) we obtain

$$\Psi^{(i,j)}(s_1, s_2) = \frac{(i+j)!}{g(\theta)} \sum_{m=1}^{\infty} a(m)(\theta\gamma)^m \binom{m+i+j-1}{m-1} \frac{\alpha^i \beta^j}{(1 - \alpha s_1 - \beta s_2)^{m+i+j}}. \tag{11}$$

Upon setting $s_1 = s_2 = 1$ in (11), we obtain the (i, j) factorial moments of (N_1, N_2)

$$E[N_1(N_1 - 1) \ldots (N_1 - i + 1)N_2(N_2 - 1) \ldots (N_2 - j + 1) \\ = \frac{(i+j)!}{g(\theta)} \sum_{m=1}^{\infty} a(m)\theta^m \binom{m+i+j-1}{m-1} \frac{\alpha^i \beta^j}{\gamma^{i+j}}.$$

Upon setting $i = 1,\ j = 0$ and $i = 0,\ j = 1$ in (11) we obtain

$$E(N_1) = \frac{\alpha}{\gamma g(\theta)} \sum_{m=1}^{\infty} m a(m)\theta^m \quad \text{and} \quad E(N_2) = \frac{\beta}{\gamma g(\theta)} \sum_{m=1}^{\infty} m a(m)\theta^m.$$

2.1. Joint probability mass function. The probability mass function of the joint distribution of (N_1, N_2) is given by expanding the PGF $\Psi(s_1, s_2)$ in powers of s_1 and s_2. Denote by $f(i, j) = P(N_1 = i, N_2 = j),\ i, j = 0, 1, 2, \ldots$, the joint probability mass function of (N_1, N_2). On the other hand, from Johnson et al. [3], it is known that

$$f(i, j) = \frac{\Psi^{(i,j)}(s_1, s_2)}{i!j!}\Bigg|_{s_1 = s_2 = 0}. \tag{12}$$

Now we use the PGF of (10) and the derivatives in (11) and have the following theorem.

Theorem 2.1. *The probability mass function of $BIGPS_{II}$ distributions is given by*

$$f(0,0) = \frac{g(\theta\gamma)}{g(\theta)},$$

$$f(i,j) = \frac{\binom{i+j}{i}\alpha^i\beta^j}{g(\theta)} \sum_{m=1}^{\infty} a(m)\binom{i+j+m-1}{m-1}(\theta\gamma)^m, \quad i,j = 0,1,\ldots,\ (i,j) \neq (0,0). \tag{13}$$

Proof. The initial value follows from the PGF $\Psi(0,0) = f(0,0)$. Next, the expression (13) follows from (11) and (12). □

3. Conditional distributions and properties. Let $\Psi_{N_2|(N_1=k)}(s_2)$, $k = 0,1,\ldots$, be the conditional PGF of N_2, given N_1. From Johnson et al. [3], it is known that

$$\Psi_{N_2|(N_1=k)}(s_2) = \frac{\Psi^{(k,0)}(0,s_2)}{\Psi^{(k,0)}(0,1)}, \tag{14}$$

where $\Psi^{(k,0)}(s_1,s_2) = \dfrac{\partial^k \Psi(s_1,s_2)}{\partial s_1^k}$. This leads to the following theorem.

Theorem 3.1. *The PGF of N_2, conditioned on N_1, is given by*

$$\Psi_{N_2|(N_1=0)}(s_2) = \frac{1}{g(\frac{\theta\gamma}{1-\beta})} \sum_{m=0}^{\infty} a(m)(\theta\gamma)^m \frac{1}{(1-\beta s_2)^m} \tag{15}$$

and for $k = 1,2,\ldots$, we have

$$\Psi_{N_2|(N_1=k)}(s_2) = \frac{\sum_{m=1}^{\infty} a(m)(\theta\gamma)^m \binom{m+k-1}{m-1} \frac{1}{(1-\beta s_2)^{m+k}}}{\sum_{m=1}^{\infty} a(m)(\theta\gamma)^m \binom{m+k-1}{m-1} \frac{1}{(1-\beta)^{m+k}}}. \tag{16}$$

Proof. For the initial value, corresponding to $k = 0$, we get

$$\Psi_{N_2|(N_1=0)}(s_2) = \frac{\Psi(0,s_2)}{\Psi(0,1)} = \frac{1}{g(\frac{\theta\gamma}{1-\beta})} \sum_{m=0}^{\infty} a(m)(\theta\gamma)^m \frac{1}{(1-\beta s_2)^m},$$

as given in (15). Upon substituting $(i,j) = (k,0)$, $k = 1,2,\ldots$, and $s_1 = 0$ in (11), we obtain

$$\Psi^{(k,0)}(0,s_2) = \frac{k!}{g(\theta)} \sum_{m=1}^{\infty} a(m)(\theta\gamma)^m \binom{m+k-1}{m-1} \frac{\alpha^k}{(1-\beta s_2)^{m+k}}, \quad k = 1,2,\ldots. \tag{17}$$

Now, (16) is obtained from (14) and (17). □

Differentiation of (15) and (16) leads to the conditional mean

$$E[N_2|N_1=0] = \frac{\beta}{(1-\beta)g(\frac{\theta\gamma}{1-\beta})}\sum_{m=0}^{\infty} a(m)m\left(\frac{\theta\gamma}{1-\beta}\right)^m$$

and for $k = 1, 2, \dots,$

$$E[N_2|N_1=k] = \frac{(k+1)\beta}{1-\beta}\frac{\sum_{m=1}^{\infty} a(m)\binom{m+k}{m-1}\left(\frac{\theta\gamma}{1-\beta}\right)^m}{\sum_{m=1}^{\infty} a(m)\binom{m+k-1}{m-1}\left(\frac{\theta\gamma}{1-\beta}\right)^m}.$$

The PGF of N_1+N_2 is obtained from $\Psi(s_1, s_2)$ in (10), when $s_1 = s_2 = s$ and is given by

$$\Psi(s) = \Psi(s,s) = \frac{g\left(\frac{\theta\gamma}{1-(\alpha+\beta)s}\right)}{g(\theta)}. \tag{18}$$

Since $\psi(s) = \dfrac{\gamma}{1-(\alpha+\beta)s}$ is a PGF of geometric distribution with parameter γ, from (18) it follows that N_1+N_2 is a compound GPSD with compounding random variable $X+Y \sim Ge(\gamma)$.

4. Examples.

As a first example we have to mention the Type II Bivariate Pólya-Aeppli distribution, introduced by Minkova and Balakrishnan (2014), [7]. It is obtained in the case of $g(\theta) = e^\theta$ and $\theta = \lambda$. Then $a(m) = \dfrac{1}{m!}$. In the same paper, the following definition is given:

Definition 4.1. *We define*

$$FI_2(N_1,N_2) = \left[\frac{Var(N_1)}{E(N_1)} + \frac{Var(N_2)}{E(N_2)} - 2R\frac{Cov(N_1,N_2)}{\sqrt{E(N_1)}\sqrt{E(N_2)}}\right](1-R^2)^{-1}, \tag{19}$$

where $R = Corr(N_1,N_2)$, as a bivariate Fisher index of dispersion.

If (Y_1, Y_2) is a bivariate Poisson random vector, then for the bivariate Fisher index of dispersion we have $FI_2(Y_1,Y_2) = 2$. For the Type II bivariate Pólya-Aeppli distribution, in the case of $\rho_1 = \rho_2 = \rho$, Fisher index of dispersion is equal to

$$FI_2(N_1,N_2) = 2\frac{1+\rho}{1-\rho}\,\frac{(1+\rho)^2-4\rho^2}{(1+\rho)^2-4\rho^2} = 2\frac{1+\rho}{1-\rho}.$$

Then, we can say that the bivariate distribution is over-dispersed (equi-dispersed, under-dispersed) if $FI_2 > 2$ ($FI_2 = 2, FI_2 < 2$).

Related to this measure of variability, the Type II bivariate Pólya-Aeppli distribution is over-dispersed with respect to bivariate Poisson distribution. The distributions of the family of $BIGPS_{II}$ can be compare with the bivariate Pólya-Aeppli distributions.

According to (19), if the marginal distributions are over-dispersed related to Poisson distribution, i.e. $\frac{Var(N_1)}{E(N_1)} > 1$ and $\frac{Var(N_2)}{E(N_2)} > 1$, then $FI_2(N_1, N_2) > 2$. In this case (N_1, N_2) is over-dispersed related to bivariate Poisson distribution. In the general case, if

$$\frac{Var(N_1)}{E(N_1)}\frac{Var(N_2)}{E(N_2)} > 1, \tag{20}$$

then $FI_2(N_1, N_2) > 2$. For comparing with bivariate Pólya-Aeppli distribution we have to check $\frac{Var(N_1)}{E(N_1)} > \frac{1+\rho}{1-\rho}$ and $\frac{Var(N_2)}{E(N_2)} > \frac{1+\rho}{1-\rho}$. In this case $FI_2(N_1, N_2) > 2\frac{1+\rho}{1-\rho}$ and (N_1, N_2) is overdispersed related to bivariate Pólya-Aeppli distribution. In the general case, if $\frac{Var(N_1)}{E(N_1)}\frac{Var(N_2)}{E(N_2)} > \left(\frac{1+\rho}{1-\rho}\right)^2$, then $FI_2(N_1, N_2) > 2\frac{1+\rho}{1-\rho}$.

4.1. Bivariate inflated-parameter binomial distribution. In the case of $g(\theta) = (1+\theta)^n$, for $n > 0$, $\theta = \frac{\pi}{1-\pi}$ and $a(m) = \binom{n}{m}$, we have the bivariate Inflated-parameter binomial distribution with parameters n, α, β and π, ($BIBi_{II}(n, \alpha, \beta, \pi)$) and PGF given by

$$\Psi(s_1, s_2) = (1-\pi)^n \left(1 + \frac{\pi\gamma}{(1-\pi)(1-\alpha s_1 - \beta s_2)}\right)^n. \tag{21}$$

In this case we have the following corollary of the Theorem 2.1

Corollary 4.1. *The PMF of the $BIBi_{II}(n, \alpha, \beta, \pi)$ distribution is given by*

$$f(i,j) = \binom{i+j}{i}\alpha^i\beta^j \sum_{m=1}^{n} \binom{n}{m}\binom{m+i+j-1}{m-1}(\pi\gamma)^m(1-\pi)^{n-m}, \tag{22}$$

for $i, j = 0, 1, \ldots,$ $(i,j) \neq (0,0)$ and $f(0,0) = [1-\pi(1-\gamma)]^n$, where $\gamma = 1-\alpha-\beta$.

The following proposition gives a recursion formulas for the PMF.

Proposition 4.1. *The PMF of the $BIBi_{II}(n,\alpha,\beta,\pi))$ distribution satisfies the following recursions:*

$$(1+\theta\gamma)f(i,0)=\alpha\left[2+\theta\gamma+\frac{(n-1)\theta\gamma-2}{i}\right]f(i-1,0)$$
$$-\alpha^2\left(1-\frac{2}{i}\right)f(i-2,0),\quad i=1,2,\dots,$$

$$(1+\theta\gamma)f(0,j)=\beta\left[2+\theta\gamma+\frac{(n-1)\theta\gamma-2}{j}\right]f(0,j-1)$$
$$-\beta^2\left(1-\frac{2}{j}\right)f(0,j-2),\quad j=1,2,\dots,$$

with $f(-1,0)=f(0,-1)=0$. In addition

$$(1+\theta\gamma)[f(i+1,j)-\beta f(i+1,j-1)]=\beta[f(i+1,j-1)-\beta f(i+1,j-2)]$$
$$+2\alpha\left(1-\frac{1}{i+1}\right)[f(i,j)-\beta f(i,j-1)]+\alpha\theta\gamma\left(1+\frac{n-1}{i+1}\right)f(i,j)$$
$$-\alpha^2\left(1-\frac{2}{i+1}\right)f(i-1,j),\quad i,j=1,2,\dots$$

$$(1+\theta\gamma)[f(i,j+1)-\alpha f(i-1,j+1)]=\alpha[f(i-1,j+1)-\alpha f(i-2,j+1)]$$
$$+2\beta\left(1-\frac{1}{j+1}\right)[f(i,j)-\alpha f(i-1,j)]+\beta\theta\gamma\left[1+\frac{n-1}{j+1}\right]f(i,j)$$
$$-\beta^2\left(1-\frac{2}{j+1}\right)f(i,j-1),\quad i,j=1,2,\dots$$

P r o o f. Differentiation of (21) with respect to s_1 and s_2 yields to

$$(1+\theta\gamma-\alpha s_1-\beta s_2)(1-\alpha s_1-\beta s_2)\frac{\partial\Psi(s_1,s_2)}{\partial s_1}=n\alpha\theta\gamma\Psi(s_1,s_2)\tag{23}$$

and

$$(1+\theta\gamma-\alpha s_1-\beta s_2)(1-\alpha s_1-\beta s_2)\frac{\partial\Psi(s_1,s_2)}{\partial s_2}=n\beta\theta\gamma\Psi(s_1,s_2),\tag{24}$$

where $\Psi(s_1,s_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i,j)s_1^i s_2^j$, $\dfrac{\partial\Psi(s_1,s_2)}{\partial s_1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)f(i+1,j)s_1^i s_2^j$ and $\dfrac{\partial\Psi(s_1,s_2)}{\partial s_2} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)f(i,j+1)s_1^i s_2^j$.

The required recursions are obtained by equating the coefficients of $s_1^i s_2^j$ on both sides of (23) and (24), for fixed $i,j = 0,1,\ldots$. □

For the moments we have

$$E(N_1) = n\pi\frac{\alpha}{\gamma} \quad \text{and} \quad E(N_2) = n\pi\frac{\beta}{\gamma},$$

and for the second moments

$$Var(N_1) = n\pi\frac{\alpha}{\gamma}\left[1+\frac{\alpha}{\gamma}(2-\pi)\right], \quad Var(N_2) = n\pi\frac{\beta}{\gamma}\left[1+\frac{\beta}{\gamma}(2-\pi)\right]$$

and

$$Cov(N_1,N_2) = n\frac{\alpha\beta}{\gamma^2}\pi(2-\pi).$$

In the terms of ρ_1 and ρ_2

$$FI(N_1) = \frac{Var(N_1)}{E(N_1)} = 1+\frac{\rho_1}{1-\rho_1}(2-\pi) = \frac{1+\rho_1}{1-\rho_1} - \pi\frac{\rho_1}{1-\rho_1}$$

and

$$FI(N_2) = \frac{Var(N_2)}{E(N_2)} = 1+\frac{\rho_2}{1-\rho_2}(2-\pi) = \frac{1+\rho_2}{1-\rho_2} - \pi\frac{\rho_2}{1-\rho_2}.$$

In the case of $\rho_1 = \rho_2 = \rho$ it is easy to check that $2 < FI_2(N_1,N_2) < 2\dfrac{1+\rho}{1-\rho}$. This means that the $BIBi_{II}$ distribution is under-dispersed related to Type II bivaraite Pólya-Aeppli distribution and over-dispersed related to bivariate Poisson distribution.

4.2. Bivariate inflated-parameter negative binomial distribution. In the case of $g(\theta) = (1-\theta)^{-r}$, for a given $r > 0$, $\theta = 1-\pi$ and $a(m) = \binom{r+m-1}{m}$, we have the bivariate Inflated-parameter negative binomial distribution with parameters r,α,β and π, $(BINB_{II}(r,\alpha,\beta,\pi))$ and PGF

$$\Psi(s_1,s_2) = \left(\frac{\pi(1-\alpha s_1-\beta s_2)}{1-(1-\pi)\gamma-\alpha s_1-\beta s_2}\right)^r. \tag{25}$$

In this case we have following corollary of the Theorem 2.1

Corollary 4.2. *The PMF of the $BINB_{II}(r, \alpha, \beta, \pi))$ distribution is given by*

$$f(i,j) = \pi^r \binom{i+j}{i} \alpha^i \beta^j \sum_{m=1}^{\infty} \binom{m+r-1}{m} \binom{m+i+j-1}{m-1} ((1-\pi)\gamma)^m, \tag{26}$$

for $i, j = 0, 1, \ldots,\ (i,j) \neq (0,0)$ *and* $f(0,0) = \left[\dfrac{\pi}{1-(1-\pi)\gamma}\right]^r$.

Proposition 4.2. *The PMF of the $BINB_{II}(r, \alpha, \beta, \pi))$ distribution satisfies the following recursions:*

$$\begin{aligned}(1-\theta\gamma)f(i,0) = {} & \alpha\left[2-\theta\gamma+\frac{(r+1)\theta\gamma-2}{i}\right]f(i-1,0)\\ & -\alpha^2\left(1-\frac{2}{i}\right)f(i-2,0), \quad i = 1, 2, \ldots,\end{aligned}$$

$$\begin{aligned}(1-\theta\gamma)f(0,j) = {} & \beta\left[2-\theta\gamma+\frac{(r+1)\theta\gamma-2}{j}\right]f(0,j-1)\\ & -\beta^2\left(1-\frac{2}{j}\right)f(0,j-2), \quad j = 1, 2, \ldots,\end{aligned}$$

with $f(-1,0) = f(0,-1) = 0$. *In addition*

$$\begin{aligned}&(1-\theta\gamma)[f(i+1,j)-\beta f(i+1,j-1)] = \beta[f(i+1,j-1)-\beta f(i+1,j-2)]\\ &+2\alpha\left(1-\frac{1}{i+1}\right)[f(i,j)-\beta f(i,j-1)] - \alpha\theta\gamma\left[1-\frac{r+1}{i+1}\right]f(i,j)\\ &-\alpha^2\left(1-\frac{2}{i+1}\right)f(i-1,j), \quad i,j = 1,2,\ldots\\ &(1-\theta\gamma)[f(i,j+1)-\alpha f(i-1,j+1)] = \alpha[f(i-1,j+1)-\alpha f(i-2,j+1)]\\ &+2\beta\left(1-\frac{1}{j+1}\right)[f(i,j)-\alpha f(i-1,j)] - \beta\theta\gamma\left[1-\frac{r+1}{j+1}\right)f(i,j)\\ &-\beta^2\left(1-\frac{2}{j+1}\right)f(i,j-1), \quad i,j = 1,2,\ldots\end{aligned}$$

P r o o f. Differentiation of (25) with respect to s_1 and s_2 leads to

$$(1-\theta\gamma-\alpha s_1-\beta s_2)(1-\alpha s_1-\beta s_2)\frac{\partial\Psi(s_1,s_2)}{\partial s_1} = r\alpha\theta\gamma\Psi(s_1,s_2) \tag{27}$$

and

$$(1-\theta\gamma-\alpha s_1-\beta s_2)(1-\alpha s_1-\beta s_2)\frac{\partial\Psi(s_1,s_2)}{\partial s_2}=r\beta\theta\gamma\Psi(s_1,s_2), \tag{28}$$

where $\Psi(s_1,s_2)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}f(i,j)s_1^is_2^j$, $\frac{\partial\Psi(s_1,s_2)}{\partial s_1}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)f(i+1,j)s_1^is_2^j$ and $\frac{\partial\Psi(s_1,s_2)}{\partial s_2}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)f(i,j+1)s_1^is_2^j$. The required recursions are obtained by equating the coefficients of $s_1^is_2^j$ on both sides of (27) and (28), for fixed $i,j=0,1,\dots$. $\square$

For the moments we have

$$E(N_1)=r\frac{\alpha(1-\pi)}{\gamma\pi}\quad\text{and}\quad E(N_2)=r\frac{\beta(1-\pi)}{\gamma\pi},$$

and for the second moments

$$Var(N_1)=r\frac{\alpha(1-\pi)}{\gamma\pi}\left[1+\frac{\alpha(1+\pi)}{\gamma\pi}\right],\quad Var(N_2)=r\frac{\beta(1-\pi)}{\gamma\pi}\left[1+\frac{\beta(1+\pi)}{\gamma\pi}\right],$$

and

$$Cov(N_1,N_2)=r\frac{\alpha\beta(1-\pi)}{\gamma\pi^2}.$$

Again, in terms of ρ_1 and ρ_2 we obtain the marginal Fisher indexes

$$FI(N_1)=\frac{1+\rho_1}{1-\rho_1}+\frac{\rho_1(1-\pi)}{(1-\rho_1)\pi}$$

and

$$FI(N_2)=\frac{1+\rho_2}{1-\rho_2}+\frac{\rho_2(1-\pi)}{(1-\rho_2)\pi}.$$

Now, in the case of $\rho_1=\rho_2=\rho$, for the bivariate Fisher index of dispersion we obtain $FI_2(N_1,N_2)>2\frac{1+\rho}{1-\rho}$, i.e., the $BINB_{II}$ distribution is over-dispersed related to Type II bivariate Pólya-Aeppli distribution.

4.3. Bivariate inflated-parameter logarithmic series distribution. In the case of $g(\theta)=-\log(1-\theta)$, $\theta=1-\pi$ and $a(m)=\frac{1}{m}$, we have the Bivariate Inflated-parameter logarithmic series distribution with parameters α,β and π, $(BILS_{II}(\alpha,\beta,\pi))$ and PGF

$$\Psi(s_1,s_2)=\frac{1}{\ln(\pi)}\ln\left(1-\frac{(1-\pi)\gamma}{1-\alpha s_1-\beta s_2}\right). \tag{29}$$

In this case we have the following corollary of the Theorem 2.1

Corollary 4.3. *The PMF of the $BILS_{II}(\alpha,\beta,\pi))$ distribution is given by*

$$f(i,j) = -\frac{\alpha^i\beta^j}{\log(\pi)}\binom{i+j}{i}\sum_{m=1}^{\infty}\binom{m+i+j-1}{m-1}\frac{((1-\pi)\gamma)^m}{m}, \tag{30}$$

for $i,j=0,1,\ldots,\ (i,j)\neq(0,0)$ *and* $f(0,0)=\dfrac{\log(1-(1-\pi)\gamma)}{\log(\pi)}$.

Proposition 4.3. *The PMF of the $BILS_{II}(\alpha,\beta,\pi)$ distribution satisfies the following recursions:*

$$\begin{aligned}
&(1-\theta\gamma)[f(i+1,j)-\beta f(i+1,j-1)] = \beta[f(i+1,j-1)-\beta f(i+1,j-2)]\\
&\quad +2\alpha\left(1-\frac{1}{i+1}\right)[f(i,j)-\beta f(i,j-1)] - \alpha\theta\gamma\left(1-\frac{1}{i+1}\right)f(i,j)\\
&\quad -\alpha^2\left(1-\frac{2}{i+1}\right)f(i-1,j), \quad i,j=1,2,\ldots
\end{aligned}$$

and

$$\begin{aligned}
&(1-\theta\gamma)[f(i,j+1)-\alpha f(i-1,j+1)] = \alpha[f(i-1,j+1)-\alpha f(i-2,j+1)]\\
&\quad +2\beta\left(1-\frac{1}{j+1}\right)[f(i,j)-\alpha f(i-1,j)] - \beta\theta\gamma\left(1-\frac{1}{j+1}\right)f(i,j)\\
&\quad -\beta^2\left(1-\frac{2}{j+1}\right)f(i,j-1), \quad i,j=1,2,\ldots
\end{aligned}$$

with initial values

$$(1-\theta\gamma)f(1,0) = -\frac{\alpha\theta\gamma}{\log(\pi)} \quad \textit{and} \quad (1-\theta\gamma)f(0,1) = -\frac{\beta\theta\gamma}{\log(\pi)},$$

and $f(-1,j)=f(i,-1)=0$.

P r o o f. Differentiation of (29) with respect to s_1 and s_2 leads to

$$(1-\theta\gamma-\alpha s_1-\beta s_2)(1-\alpha s_1-\beta s_2)\frac{\partial\Psi(s_1,s_2)}{\partial s_1} = -\frac{\alpha\theta\gamma}{\log(\pi)}. \tag{31}$$

and

$$(1-\theta\gamma-\alpha s_1-\beta s_2)(1-\alpha s_1-\beta s_2)\frac{\partial\Psi(s_1,s_2)}{\partial s_2}=-\frac{\beta\theta\gamma}{\log(\pi)}. \tag{32}$$

Equating the coefficients of $s_1^i s_2^j$ on both sides of (31) and (32) we obtain the required recursions. $\square$

For the moments we have

$$E(N_1)=-\frac{\alpha(1-\pi)}{\gamma\pi\log(\pi)} \quad \text{and} \quad E(N_2)=-\frac{\beta(1-\pi)}{\gamma\pi\log(\pi)},$$

and for the second moments

$$Var(N_1)=-\frac{\alpha(1-\pi)}{\gamma\pi^2\log(\pi)}\left[\pi+\frac{\alpha(1+\pi)}{\gamma}+\frac{\alpha(1-\pi)}{\gamma\log(\pi)}\right]$$

and

$$Var(N_2)=-\frac{\beta(1-\pi)}{\gamma\pi^2\log(\pi)}\left[\pi+\frac{\beta(1+\pi)}{\gamma}+\frac{\beta(1-\pi)}{\gamma\log(\pi)}\right].$$

For the marginal Fisher indexes we obtain

$$FI(N_1)=\frac{Var(N_1)}{E(N_1)}=1+\frac{\alpha}{\gamma\pi}\frac{1-\pi+(1+\pi)\log(\pi)}{\log(\pi)}$$

and

$$FI(N_2)=\frac{Var(N_2)}{E(N_2)}=1+\frac{\beta}{\gamma\pi}\frac{1-\pi+(1+\pi)\log(\pi)}{\log(\pi)}.$$

If the nominator on the right hand side is negative, the marginal variables are over-dispersed and (N_1,N_2) is over-dispersed with respect to bivariate Poisson distribution. In this case $-\log(\pi)>\frac{1-\pi}{1+\pi}$. If $-\log(\pi)<\frac{1-\pi}{1+\pi}$, (N_1,N_2) is under-dispersed with respect to bivariate Poisson distribution. For comparing with bivariate Pólya-Aeppli distribution we rewrite the Fisher indexes in the following way

$$FI(N_1)=\frac{1+\rho_1}{1-\rho_1}+\frac{\rho_1}{1-\rho_1}\left[\frac{1-\pi+(1+\pi)\log(\pi)}{\pi\log(\pi)}-2\right]$$

and

$$FI(N_2)=\frac{1+\rho_2}{1-\rho_2}+\frac{\rho_2}{1-\rho_2}\left[\frac{1-\pi+(1+\pi)\log(\pi)}{\pi\log(\pi)}-2\right].$$

In the case of $\rho_1 = \rho_2 = \rho$, (N_1, N_2) is over-dispersed related to bivariate Pólya-Aeppli distribution if $-\log(\pi) > 1$, and under-dispersed if $-\log(\pi) < 1$.

5. Concluding remarks. In this paper we have defined a new family of bivariate discrete distributions. Similar to the univariate case, the compounding with bivariate geometric distribution results in distributions with closed forms of PMFs and nice properties. The particular cases are the corresponding bivariate extensions of the Inflated-parameter generalized power series distributions.

Possible applications of the defined distributions are in queueing theory, risk theory, reliability. For example, if there are two types of claims to the insurance company, the counting process in the risk model has to be bivariate. Kostadinova and Minkova in [5] have defined a risk model with bivariate counting process, such that the compounding distribution is a bivariate negative binomial distribution. As a future work we are planning at first to define the counting processes with distributions introduced in this paper. Then, the defined bivariate counting processes will be used in analyzing the corresponding risk models with two types of claims.

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