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ON n-HOMOGENEOUS C^*-ALGEBRAS OVER TWO-DIMENSIONAL NON-ORIENTED COMPACT MANIFOLDS

Mikhail Shchukin

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Abstract. We consider algebraic bundles over two-dimensional compact non-oriented connected manifold. Every non-oriented compact manifold can be realized as sphere S^2 with k projective planes on it. Let P^k be the sphere S^2 with k projective planes. Let ζ be algebraic bundle over P^k with fiber $\mathrm{Mat}(n)$. If $n = 2m+1$ then the bundle ζ is trivial. If $n = 2m$ then there are two non-isomorphic algebraic bundles over P^k with fiber $\mathrm{Mat}(n)$. J. Fell, J. Tomiyama, M. Takesaki showed in 1961 the correspondence between the classes of algebraic bundles and n-homogeneous C^*-algebras. Hence we can classify non-isomorphic n-homogeneous C^*-algebras over P^k.

1. Introduction. Let A be a n-homogeneous C^*-algebra. The "n-homogeneous" means that all its irreducible representations are n-dimensional. Suppose that the space of primitive ideals of the algebra A be a two-dimensional

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compact non-oriented manifold. J. Fell [3], I. Tomiyama and M. Takesaki [7] described any n-homogeneous C^*-algebra as algebra of all continuous sections $\Gamma(\zeta)$ for appropriate algebraic bundle ζ.

Suppose A_1 and A_2 are n-homogeneous C^*-algebras. Let $f : A_1 \to A_2$ be a continuous bijection such that $f(a \cdot b) = f(a) \cdot f(b)$ and $f(a^*) = f(a)^*$. In this case, two algebras A_1 and A_2 are called isomorphic.

F. Krauss and T. Lawson [4] described the class of algebraic bundles over the toruses T^2 and T^3.

In present work we described the classes of algebraic bundles over the two-dimensional non-oriented manifolds in the hull-kernel topology.

It is well known that every compact non-oriented two-dimensional connected manifold is homeomorphic to the connected sum P^l of l projective planes [5]. In particular, the projective plane P is two-dimensional non-oriented manifold.

A triple $(E; B; p)$ is called bundle, where E and B are topological spaces, $p : E \to B$ is a continuous surjection. The surjection p is called projection. The set $F_x = p^{-1}(x)$ is called fiber over the point $x \in B$. We may assume that the fiber F_x is homeomorphic to F_y for any $x, y \in B$. Note that a triple $(B \times F, B, p)$ is the bundle, where B and F are topological spaces, $p : (x, y) \to x$, where $x \in B, y \in F$.

The bundle (E, B, p) is called locally trivial with a fiber F, if each point of B has a neighbourhood U such that the bundle E over U is trivial. This means that there exists a homeomorphism $\phi : p^{-1}(U) \to U \times F$, which commute with the projections. In this case, each fiber is homeomorphic to the model fiber F.

Let U_j be an open covering of B such that the restrictions of the bundle E on U_j are trivial bundles and let ϕ_j be the corresponding homeomorphisms. Then the mapping $\phi_{j,i} = \phi_j \circ (\phi_i)^{-1}$ is well defined on $(U_j \cap U_i) \times F$ and $\phi_{j,i}(x, y) = (\alpha(x), g_x(y)), y \in F$, where g_x are the homeomorphisms of the fiber F, $\alpha(x) \in U_j \cap U_i$. Suppose the homeomorphisms g_x are belong to a topological subgroup G of the group of all homeomorphisms of the space F. The group G is called the structure group of the bundle. A locally trivial G-bundle (E, B, p) is called algebraic bundle, if the fiber $F = \mathrm{Mat}(n)$ and the structure group $G = \mathrm{Aut}(n)$. Here $\mathrm{Mat}(n)$ is the algebra of square matrices of order n over the complex field C. $\mathrm{Aut}(n)$ denotes the group of automorphisms for the algebra. Two algebraic bundles $\zeta_1 = (E_1, B_1, p_1)$ and $\zeta_2 = (E_2, B_2, p_2)$ are called isomorphic if there is a homeomorphism $\gamma : E_1 \to E_2$ such that $\gamma(F_x) = F_{\alpha(x)}$, $\gamma(B_1) = B_2$. Here $\alpha : B_1 \to B_2$ is a homeomorphism; F_x denotes the fiber over the point $x \in B_1$; $F_{\alpha(x)}$ denotes the fiber over the point $\alpha(x) \in B_2$.

Let $\beta : B \to E$ be a continuous mapping such that $p \circ \beta = \text{Id}$. In this case, the mapping β is called continuous section of the bundle (E, B, p). Let $\Gamma(E)$ be the algebra of all continuous sections of the bundle. Every n-homogeneous C^*-algebra is isomorphic to the algebra $\Gamma(E)$ for the appropriate algebraic bundle (E, B, p) ([3], [7]).

2. Algebraic bundles over two-dimensional non-oriented compact connected maniolds. Suppose S^2 be the 2-sphere. Let us identify the opposite points of the sphere S^2. Suppose P be the corresponded factor-space. We say that the space P is the real projective plane. Suppose the space B is homeomorphic to the space P. In this case, we say that B is the projective plane too.

Further, let us construct a connected sum of two surfaces. Suppose S_1 and S_2 be two surfaces such that $S_1 \cap S_2$ is empty. Let us cut out two small open disks D_1 and D_2 from S_1 and S_2 correspondingly. Gluing together the borders $\delta(D_1)$ with $\delta(D_2)$ from the surfaces $S_1 \backslash D_1$ and $S_2 \backslash D_2$, we obtain the surface $S_1 \sharp S_2$. The surface $S_1 \sharp S_2$ is called the connected sum of the surfaces S_1 and S_2.

Proposition 2.1 ([5]). *Let L denotes a non-oriented connected compact 2-dimensional manifold. The manifold L is homeomorphic to a connected sum of projective planes.*

Let P^l be a sphere S^2 with l projective planes. The set P^0 is the sphere S^2. Therefore we suppose that $l \geq 1$. First, consider an algebraic bundle $\zeta = (E, P^l, p)$. Suppose the fiber $F = \text{Mat}(n)$. Let $D \subset P^l$ is homeomorphic to the open unit disk D_1. Cut out the set D from the set P^l. The set $P^1 \backslash D$ is homeomorphic to the cross cap M. The cross cap has homotopic class of a connected sum of two circles with one common point. This connected sum of two circles could be considered as a subset of the plane R^2. Therefore every algebraic bundle over M is trivial [1]. Further, represent the algebraic bundle ζ as a cluing of two bundles $\zeta/_{P^1\backslash D} \underset{\nu_{12}}{\cup} \zeta/_{\bar{D}}$. Here $\zeta/_{P^1\backslash D}$ denotes the restriction of the bundle ζ to the set $P^1\backslash D$; $\zeta/_{\bar{D}}$ denotes the restriction of the bundle ζ to the set $\bar{D}$; $\nu_{12} = \nu_2^{-1}\nu_1$ is the function of cluing generated by a mapping $\gamma_{12}^1 \in C(S^1; \text{Aut}(n))$. The restriction $\zeta/_{P^1\backslash D}$ of the bundle ζ to the set $P^1\backslash D$ is trivial; the mapping $\nu_1 : \zeta/_{P^1\backslash D} \to (P^1\backslash D) \times \text{Mat}(n)$ is a trivialization. Similarly, the restriction $\zeta/_{\bar{D}}$ is trivial; the mapping $\nu_2 : \zeta_{\bar{D}} \to \bar{D} \times \text{Mat}(n)$ is corresponded trivialization.

By construction, let the cross cap M is realized as the unit square I^2 with conditions on its border: $u(0; y) = u(1; 1 - y)$.

Suppose the mapping $\theta : \pi_1(\mathrm{Aut}(n)) \to Z/nZ$ be the isomorphism of groups. Now we need the next lemma.

Lemma 2.2. *The mapping $f \in C(\delta M, \mathrm{Aut}(n))$ has a continuous extension $f_1 \in C(M, \mathrm{Aut}(n))$ if and only if $\theta([f]) = 2s, s \in Z$.*

P r o o f. Let $f : \delta M \to \mathrm{Aut}(n)$ be a mapping such that $\theta([f]) = 2s, s \in Z$. Let x_0 be a point of I such that $\theta(f/_{[0,x_0]\times 1}) = s$. Let $\alpha : M \to M$ be a homeomorphism such that $\alpha(x_0; 1) = (1; 1)$. We may assume that the mapping α is already used for the cross cap M. This yields that $\theta([f/_{I\times 1}]) = \theta([f/_{I\times 0}]) = s$.

Let $f_2(1; 1)$ be a representative of the class $f(1; 1) \in \mathrm{Aut}(n)$. Let $a \in C\backslash 0$ be the determinant of $f_2(1; 1)$. Let $f^1(x, y)$ be a mapping $f(x; y)\cdot f_2^{-1}(0; 1)$. Hence $f^1(0; 1) = f(1; 0) = I$. Therefore it can be assumed that $f(0; 1) = f(1; 0) = I$.

Let $S_1(x; y)$ be a function such that $S_1(x, y) = 1 + x(a - 1), if a > 0; S_1(x, y) = 1 + x(|a| - 1)\exp(\pi ix(6 - 6y)), if a < 0; (x \in I, y \in [5/6; 1])$. Therefore $S_1(0; y) = 1, y \in [5/6; 1]$ and $S_1(1; 5/6) = a, S_1(x; y) \neq 0, x \in [0; 1], y \in [5/6; 1]$. Let $f_1(x; y)$ be a mapping $f(x; y)/S_1(x; y), x \in [0; 1], y \in [5/6; 1]$. Thus $f_1(0; 5/6) = I$. Let $det f_1(1; 5/6)$ be a determinant for a representative of the class $f_1(1; 5/6) \in \mathrm{Aut}(n)$. Therefore $det f_1(1; 5/6) = 1$. Let $f_{1,2}(1; 5/6)$ be a representative for the class $f_1(1; 5/6) \in \mathrm{Aut}(n)$. Let $p(t)$ be a path in $SL_n(C)$ such that $p(0) = I, p(1) = f_{1,2}(1; 5/6)$.

To each point

$$(x; y) \in [[0, 1 - \sqrt{1/36 - (y - 2/3)^2}]; [2/3, 5/6]]$$

assign $f_1(x; y) = f_1(x/(1 - \sqrt{1/36 - (y - 2/3)^2}); y)$. To each point $(x; y)$ such that $\sqrt{(x - 1)^2 + (y - 2/3)^2} \leq 1/6$ assign $f_1(x; y) = p^*(6\sqrt{(x - 1)^2 + (y - 2/3)^2})$. Therefore $f_1(0; 2/3) = I = f_1(1; 2/3)$. Let $T(\exp(2\pi isx)I)$ be the appropriate class in the group $\mathrm{Aut}(n)$ for each $x \in [0; 1]$. Let $\Gamma(t)$ be a homotopy between $f_1/_{(I\times 2/3)}$ and $T(\exp(2\pi isx)I), x \in [0; 1]$. Notice that $[f_1/_{(I\times 2/3)}] = [\exp(2\pi isx)] = \theta^{-1}(s)$. This means that

$$\Gamma(0) = f_1/_{(I\times 2/3)},\ \Gamma(1) = T(\exp(2\pi isx)I),\ \Gamma(t) \in \mathrm{Aut}(n), t \in [0; 1].$$

For each point $(x; y) \in I \times [1/2; 2/3]$ assign $f_1(x; y) = \Gamma(-6y + 4)$. Therefore $f_1/_{(I\times 1/2)} = T(\exp(2\pi isx)I)$. In the same way, we extend f_1 to the lower half of the square.

Conversely, let we have a mapping $f_1 \in C(M, \mathrm{Aut}(n))$. It can be shown in the usual way that the middle line of a cross cap is the circle S^1 but with two movings on it. Let us construct a homotopy $\Gamma(t)$ between $f_1/\delta M$ and the restriction of f_1 to the middle line on the cross cap. For each point $(x; y) \in I \times [1/2; 1]$ assign $\Gamma(t) = f_1(x; 1 - t/2)$. In the same way, for each point $(x; y) \in I \times [0; 1/2]$ assign $\Gamma(t) = f_1(x; t/2)$. Let $S = \theta([f_1/_{(I\times 1/2)}])$. Hence $\theta([f_1/_{\delta M}]) =$

$2\theta([f_1/_{(I\times 1/2)}]) = 2s, (f_1(0;1/2) = f_1(1;1/2))$. This concludes the proof of the lemma. □

Let P_k be a sphere S^2 with k handles. Hence P_k is a oriented two-dimensional manifold.

Lemma 2.3. *Let S^1 be a border $\delta(P_k \backslash D)$ for the set $P_k\backslash D)$. Let $f : S^1 \to \mathrm{Aut}(n)$ be a continuous mapping. Let $[f]$ be a class of f in $\pi_1(\mathrm{Aut}(n))$. There is a extension for f to a continuous mapping $f^* : P_k\backslash D \to \mathrm{Aut}(n)$ if and only if $\theta([f]) = 0$.*

Lemma 2.4. *A mapping $f \in C(\delta(P^k\backslash M), \mathrm{Aut}(n))(k \geq 2)$ has an extension $f_1 \in C(P^k\backslash M, \mathrm{Aut}(n))$ if and only if $\theta([f]) = 2s, s \in Z$. If $k = 1$ then the equality $\theta([f]) = 0$ should has a place*

P r o o f. The proof is by induction on k.

1. Let k be 1. In this case, $P^1\backslash M$ is homeomorphic to $P^1\backslash D$. By using the Lemma 2.3, we obtain the statement.

Let k be 2. The space $P^2\backslash M = M_1$ is homeomorphic to the cross cap M. By using the Lemma 2.2, we obtain that the mapping $f \in C(\delta(P^2\backslash M), \mathrm{Aut}(n))$ has an extension $f \in C(M_1, \mathrm{Aut}(n))$.

The induction hypothesis. Suppose the lemma is true for all $k \leq m$.

The step of induction. Let ζ denote an algebraic bundle over P^{m+1}. Let M_1 be a projective plane on $P^{m+1}\backslash M$. Denote by L_1 the set $\delta(M_1)\bigcap(P^{m+1}\backslash M)$. The set L_1 is homeomorphic to the unit interval I. Therefore $P^{m+1}\backslash M = (P^m\backslash M_1)\bigcup M_2$, where M_2 is homeomorphic to the cross cap and $(P^m\backslash M_1)\bigcap M_2 = L_1$.

Let $f \in C(\delta M \to \mathrm{Aut}(n))(\delta M = S^1)$ is a continuous mapping. Suppose f has an extension to $f_1 \in C((P^{m+1}\backslash M), \mathrm{Aut}(n))$. Therefore $\theta([f_1/_{(S_1\cup L_1)}]) = 2s, \theta([f_1/_{(S_2\cup L_1)}]) = 2r$ and $\theta([f/_{\delta M}]) = \theta(f_1/_{(S_1\cup L_1)}) + \theta([f_1/_{(S_2\cup L_1)}]) = 2s + 2r = 2(s + r)$.

Conversely, suppose $\theta([f/_{\delta M}]) = 2s$. Suppose $L_1(t), S_1(t)$ be parametrizations for L_1 and S_1 for $t \in [0;1]$. Define $f_1(L_1(t)) = f(S_1(t))$ for each $t \in [0;1]$. Hence $\theta(f_1/_{(S_1\cup L_1)}) = 0, \theta(f/_{(S_1\cup L_1)}) = 2s$. It follows that there is an extension for f_1 to $P^{m+1}\backslash(M \cup M_1)$ and M_1 by Lemma 2.2 and induction hypothesis. This completes the proof. □

Theorem 2.5. *Let ζ_1 and ζ_2 are algebraic bundles over $P^k(k \geq 1)$. The bundles ζ_1 and ζ_2 are isomorphic if and only if $\theta([f_1/_{\delta M}]) \pm \theta([f_2/_{\delta M}]) = 2s$, $s \in Z$.*

P r o o f. Let $\gamma : \zeta_1 \to \zeta_2$ be isomorphism. It generates an homeomorphism $\alpha : P^k \to P^k$ for the bases of bundles ζ_1 and ζ_2. Cut out the cross cap M from P^k. Let $\nu_{12} = u_2^{-1}u_1$ be a function of cluing for the bundle ζ_1 over $(P^k\backslash M) \cup \overline{M}$. Let $\mu_{12} = v_2^{-1}v_1$ be a function of cluing for the bundle ζ_2 over $(P^k\backslash\alpha(M)) \cup \alpha(\overline{M}), u_1 : \zeta_1/(P^k\backslash M) \to (P^k\backslash M) \times \mathrm{Mat}(n), u_2 : \zeta_1/(\overline{M}) \to (\overline{M}) \times \mathrm{Mat}(n)$, $v_1 : \zeta_2/(P^k\backslash\alpha(M)) \to (P^k\backslash\alpha(M)) \times \mathrm{Mat}(n), v_2 : \zeta_2/(\alpha(\overline{M})) \to (\alpha(\overline{M})) \times \mathrm{Mat}(n)$.

Let $\beta : P^k \to P^k$ be a homeomorphism such that $\alpha(\overline{M}) = \overline{M}$ and the orientation of $\alpha(\delta(\overline{M})) \cong S^1$ is not changed. We have $\beta(\overline{M}) = \overline{M}$ and $\beta(P^k\backslash M) = P^k\backslash M$.

Denote by β_1 the extension of β to an isomorphism of trivial bundles $\beta_1 : \zeta_2/_{(P^k\backslash\alpha(\overline{M}))} \to \zeta_2/_{(P^k\backslash\overline{M})}$. Define by β_2 the extension of β to an isomorphism $\beta_2 : \zeta_2/_{\alpha(\overline{M})} \to \zeta_2/_{\overline{M}}$ of trivial bundles.

Define the mapping $\mu_{12}* : \zeta_2/_{\delta(P^k\backslash M)} \to \zeta_2/_{\delta(\overline{M})}$ such that the next diagram is commutative: $\begin{pmatrix} \zeta_2/_{(P^k\backslash\alpha(M))} & \to^{\mu_{12}} & \zeta_2/_{(\alpha(\overline{M}))} \\ \downarrow\beta_1 & & \downarrow\beta_2 \\ \zeta_2/_{(P^k\backslash M)} & \to^{\mu_{12}*} & \zeta_2/_{\overline{M}} \end{pmatrix}$

It follows that the bundle $\beta_1(\zeta_2/_{(P^k\backslash\alpha(M))}) \underset{\mu_{12}*}{\cup} \beta_2(\zeta_2/_{\alpha(\overline{M})})$ is isomorphic to the bundle ζ_2. Denote by β_3 this isomorphism.

The isomorphism $\beta_3 \circ \gamma : \zeta_1 \to \zeta_2$ generates an homeomorphism $\beta \circ \alpha : P^k \to P^k$ such that $\beta \circ \alpha(\delta(\overline{M})) = \delta(\overline{M})$. The restriction of the bundle ζ_1 to the set $P^k\backslash M$ is trivial. Hence the isomorphism $\beta_3 \circ \gamma$ generates a mapping $\beta_5 \in C(P^k\backslash M, \mathrm{Aut}(n))$. For each fiber F_x we have $\beta_5(x)(F_x) = \beta_3 \circ \gamma(F_x), x \in P^k\backslash M$. For each $x \in \overline{M}$ the isomorphism $\beta_3 \circ \gamma$ generates a mapping $\beta_6 \in C(\overline{M}, \mathrm{Aut}(n))$. For each fiber F_x we have $\beta_6(x)(F_x) = \beta_3 \circ \gamma(F_x), x \in \overline{M}$. Therefore $\gamma_4(\beta \circ \alpha(x))\beta_5(x) = \beta_6(x) \cdot \gamma(x)$. It follows that

$$\theta([\gamma_4(\beta \circ \alpha(x))]) + \theta([\beta_5(x)]) = \theta([\beta_6(x)]) + \theta([\gamma_3(x)]) \tag{2.1}$$

The mappings $\beta_5(x)$ and $\beta_6(x)$ are well-defined on the sets $P^k\backslash M$ and $\overline{M}$. Therefore $\theta([\beta_5(x)]) = 2r$ ($r = 0$ for $k = 1$ by Lemma 2.3), by Lemma 2.4. We get $\theta([\beta_6(x)]) = 2m (m \in Z)$. The equality 2.1 is equivalent to the next equality:

$$\theta([\gamma_4(\beta \circ \alpha(x))]) = \theta([\gamma_3(x)]) + 2(m - r) \tag{2.2}$$

Further, suppose the homeomorphism $\beta \circ \alpha$ changed the orientation for the circle $\delta(\overline{M})$. Therefore $\theta([\gamma_4(\beta \circ \alpha(x))]) = -\theta([\gamma_4(x)])$.

Suppose the homeomorphism $\beta \circ \alpha$ doesn't changed the orientation of the circle $\delta(\overline{M})$. This yields that $\theta([\gamma_4]) = \pm\theta([\gamma_3]) + 2(m - r)$. In the converse case, suppose that $\theta([\gamma_4]) = \pm\theta([\gamma_3]) + 2s, s \in Z$. First let $\theta([\gamma_4]) = -\theta([\gamma_3]) + 2s$. Let $\alpha : P^k \to P^k$ be a homeomorphism such that $\alpha(\overline{M}) = \overline{M}$. Suppose that α does not changes the orientation of the circle $S^1 = \delta(\overline{M})$. Let $\nu_1 : \zeta_2/_{(P^k\backslash M)} \to (P^k\backslash M) \times \mathrm{Mat}(n)$ be an isomorphism of bundles that generates the homeomorphism α on $P^k\backslash M$. Let $\gamma_1 = v_1^{-1} \circ u_1$ is an isomorphism of bundles $\zeta_2/_{P^k\backslash M} \to (P^k\backslash M) \times \mathrm{Mat}(n)$. The isomorphism $u_1 : \zeta_1/_{(P^k\backslash M)} \to (P^k\backslash M) \times \mathrm{Mat}(n)$ produces the identity homeomorphism I for the bases of the bundles. Let the isomorphism γ_1 produces a mapping $\gamma_5 \in C(P^k\backslash M, \mathrm{Aut}(n))$. Therefore the mapping $\nu_{12} \circ \gamma_1 \circ \mu_{12}^{-1} : \zeta_1/_{\delta\overline{M}} \to \zeta_2/_{\delta\overline{M}}$ is isomorphism for the trivial bundles. The homeomorphism α changed the orientation of the circle $S^1 = \delta M$. Hence $\theta([\gamma_4(\alpha x)]) = -\theta([\gamma_4(x)])$. Therefore $\theta([\gamma_4(\alpha x)\cdot\gamma_5(x)\cdot(\gamma_3(x))^{-1}]) = \theta([\gamma_4(\alpha x)]) + \theta([\gamma_5(x)]) + \theta([\gamma_3(x)^{-1}]) = -\theta([\gamma_4]) - \theta([\gamma_3]) + 2r = 2m + 2l + 2r$. We get $\theta([\gamma_5(x)]) = 2r, r \in Z$ by Lemma 2.4. In addition, $\theta([\gamma_5(x)^{-1}]) = -\theta([\gamma_5(x)])$. Therefore there is an extension of $\gamma_4(\alpha x)\cdot\gamma_5(x)\cdot(\gamma_3(x))^{-1}$ to $\gamma_7 \in C(\overline{M}, \mathrm{Aut}(n))$ by Lemma 2.4.

Define an isomorphism $\gamma_2 : \zeta_1/_{\overline{M}} \to \zeta_2/_{\overline{M}}$ by the rule $(x; y) \to (\alpha(x); \gamma_7(x)\cdot y), (x \in \overline{M}, y \in F_x)$. The isomorphism γ_2 is agreed with conditions of cluing for the bundles ζ_1 and ζ_2. Let $\gamma : \zeta_1 \to \zeta_2$ is defined by γ_1 on $\zeta_1/_{P^k\backslash M}$ and by γ_2 on $\zeta_1/_{\overline{M}}$. Hence γ is the necessary isomorphism between the bundles ζ_1 and ζ_2. Secondly let $\theta([\gamma_4]) = \theta([\gamma_3])$. Let α be identity homeomorphism $I : P^k \to P^k$. As before, we construct the mapping $\gamma_4(\alpha x)\cdot\gamma_5(x)\cdot(\gamma_3(x))^{-1} \in C(S^1, \mathrm{Aut}(n))$ such that $\theta([\gamma_4(\alpha x)\cdot\gamma_5(x)\cdot(\gamma_3(x))^{-1}]) = \theta([\gamma_4(\alpha x)]) + \theta([\gamma_5(x)]) + \theta([(\gamma_3(x))^{-1}]) = \theta([\gamma_4]) - \theta([\gamma_3]) + 2r{=}2m + 2l + 2r$. Therefore we can extend the mapping $\gamma_4(\alpha x)\cdot\gamma_5(x)\cdot(\gamma_3(x))^{-1}$ to $\gamma_7 \in C(\overline{M}, \mathrm{Aut}(n))$. The mapping γ_7 generates an isomorphism $\gamma_2 : \zeta_1/_{\overline{M}} \to \zeta_2/_{\overline{M}}$ that is agreed with conditions of cluing for the bundles ζ_1 and ζ_2. Also, γ_2 is agreed with isomorphism γ_1. This means that γ_1 and γ_2 generate an isomorphism $\gamma : \zeta_1 \to \zeta_2$. This completes the proof. □

Theorem 2.6. *Let ζ be algebraic bundle over two-dimensional non-oriented manifold P^k. Let the fiber F_x be* Mat(n). *If $n = 2m$ then there are two non-isomorphic algebraic bundles over P^k. If $n = 2m+1$ then the bundle ζ is trivial. It is isomorphic to the product-bundle $P^k \times$* Mat(n).

Proof. Let $n = 2m$. Consider the members of Z/nZ. Let $\rho : \zeta \to Z/nZ$ be the connection between algebraic bundles and $Z/nZ \cong \pi_1(\mathrm{Aut}(n))$ that was constructed in the proof of the theorem 2.5. Two algebraic bundles ζ_1 and ζ_2 are isomorphic if and only if

$$\rho(\zeta_1) \pm \rho(\zeta_2) = 2l, l \in Z/nZ \tag{2.3}$$

The next elements from Z/nZ satisfy 2.3:

$$(0; 2; 4; 6; \ldots; 2m-2) \tag{2.4}$$

Let

$$(1; 3; 5; \dots; 2m-1) \tag{2.5}$$

be the second class of elements from Z/nZ. Two elements from the set 2.4 are generated by isomorphic algebraic bundles ζ_1 and ζ_2. Two elements from 2.5 are generated by isomorphic bundles. Let ζ_1 and ζ_2 be algebraic bundles over P^k such that $\rho(\zeta_1) \in 2.4$ and $\rho(\zeta_2) \in 2.5$. Hence ζ_1 is not isomorphic to the bundle ζ_2.

Secondly let $n = 2m+1$. Let ζ_1 and ζ_2 be algebraic bundles over P^k with fiber $\mathrm{Mat}(n)$. Suppose $\rho(\zeta_1) = 0, \rho(\zeta_2) = 1$. Then $\rho(\zeta_2) + 2m = 2m + 1 = 0$ in Z/nZ. Therefore ζ_1 is isomorphic to ζ_2, by theorem 2.5.

This shows that every bundle ζ over P^k with fiber $\mathrm{Mat}(n)$ is isomorphic to ζ_1 or ζ_2. Therefore ζ is trivial. This concludes the proof. □

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Chair of Higher Mathematics N1
Belarusian National Technical University
ul. Hmelnizkogo 9, 220045, Minsk, Belarus
e-mail: mvs777777@gmail.com

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