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ON $\mathbb{Z}_2$-GRADED POLYNOMIAL IDENTITIES OF $sl_2(F)$ OVER A FINITE FIELD

Luís Felipe Gonçalves Fonseca

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Abstract. Let F be a finite field of $\operatorname{char} F > 3$ and $sl_2(F)$ be the Lie algebra of traceless 2×2 matrices over F. In this paper, we find a basis for the $\mathbb{Z}_2$-graded identities of $sl_2(F)$.

1. Introduction. The well-known Ado-Iwasawa theorem gives that any finite-dimensional Lie algebra over an arbitrary field has a faithful finite-dimensional representation. Briefly, any finite-dimensional Lie algebra can be viewed as a subalgebra of a Lie algebra of square matrices under the commutator brackets. Thus, the study of Lie algebras of matrices is of considerable interest.

A task in PI-theory is to describe the identities of $sl_2(F)$, the Lie algebra of traceless 2×2 matrices over a field F of $\operatorname{char} F \neq 2$. The first breakthrough in this area was made by Razmyslov [12], who described a basis for the identities of $sl_2(F)$ when $\operatorname{char} F = 0$. Vasilovsky [16] found a single identity for the identities of $sl_2(F)$ when F is an infinite field of $\operatorname{char} F \neq 2$, and Semenov [13] described a basis (with two identities) for the identities of $sl_2(F)$ when F is a finite field of $\operatorname{char} F > 3$.

The Lie algebra $sl_2(F)$ can be naturally graded by $\mathbb{Z}_2$ as follows:

$$sl_2(F) = (sl_2(F))_0 \oplus (sl_2(F))_1,$$

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where $(sl_2(F))_0, (sl_2(F))_1$ contain diagonal and off-diagonal matrices, respectively. A recent development in PI-theory is the description of the graded identities of $sl_2(F)$. Using invariant theory techniques, Koshlukov [8] described the $\mathbb{Z}_2$-graded identities for $sl_2(F)$ when F is an infinite field of $\operatorname{char} F \neq 2$. Several further papers on graded identities of $sl_2(F)$ over a field of characteristic zero have appeared in recent years (cf. e.g., [4] and [5]).

Up till now, no basis has been found for the $\mathbb{Z}_2$-graded identities of $sl_2(F)$ when F is a finite field. In the present paper we give such a basis when $\operatorname{char} F > 3$.

2. Preliminaries. Let F be a fixed finite field of $\operatorname{char} F > 3$ and size $|F| = q$, let $\mathbb{N} = \{1, 2, \ldots\}$, let $G = (\mathbb{Z}_2, +)$, and let L be a Lie algebra over F. In this paper (unless otherwise stated), all vector spaces and Lie algebras are considered over F. We denote by $\dot{+}, \oplus, \operatorname{span}_F\{a_1, \ldots, a_n\}, \langle a_1, \ldots, a_n\rangle$ (where $a_1, \ldots, a_n \in L$) the direct sum of Lie algebras, the direct sum of vector spaces, the vector space spanned by $a_1, \ldots, a_n$, and the ideal generated by $a_1, \ldots, a_n$, respectively. The associative product is represented by a dot: "$\cdot$" and the commutator "[,]" denotes the multiplication operation of a Lie algebra. We assume that all commutators are left-normed, i.e., $[x_1, x_2, \ldots, x_n] := [[x_1, x_2, \ldots, x_{n-1}], x_n]$, $n \geq 3$. We use the convention $[x_1, x_2^k] = [x_1, x_2, \ldots, x_2]$, where x_2 appears k times in the expanded commutator.

We denote by $gl_2(F)$ the Lie algebra of 2×2 matrices over F and by $sl_2(F)$ the Lie subalgebra of traceless matrices in $gl_2(F)$. Here, $e_{ij} \in gl_2(F)$ denotes the matrix unit whose elements are 1 in the position (i, j) and 0 otherwise.

The basic concepts of Lie algebras adopted in this paper can be found in [6, Chapters 1 and 2]. We denote the center of L by $Z(L)$. If $x \in L$, then $\operatorname{ad}x$ is the linear map defined by $y \mapsto [x, y]$. The Lie algebra L is said to be metabelian if it is solvable of class at most 2. As it is known, if L (over a finite field of $\operatorname{char} F > 3$) is a three-dimensional simple Lie algebra, then $L \cong sl_2(F)$. The algebra L is called a Lie A-algebra if all of its nilpotent subalgebras are abelian.

For an additively written group G, a Lie algebra L is said to be G-graded (a graded Lie algebra or graded by G) when there exist subspaces $\{L_g\}_{g\in G} \subset L$ such that $L = \bigoplus_{g\in G} L_g$, and $[L_{g_1}, L_{g_2}] \subset L_{g_1+g_2}$ for any $g_1, g_2 \in G$. G-graded associative algebras are defined in the same way. In that context, $\{L_g\}_{g\in G}$ is said to be a grading for L. An element a is called homogeneous when $a \in \bigcup_{g\in G} L_g$ and $a \neq 0$ is homogeneous of G-degree g when $a \in L_g$. A Lie algebra homomorphism $\phi : L_1 \to L_2$ of two G-graded Lie algebras L_1 and L_2 is G-graded if $\phi(L_{1g}) \subset L_{2g}$ for all $g \in G$. Two gradings on L $\{L_g\}_{g\in G}$ and $\{L'_g\}_{g\in G}$ on L are called isomorphic when there exists a G-graded isomorphism $\phi : L \to L$ such that $\phi(L_g) = L'_g$ for all $g \in G$. An ideal $I \subset L$ is graded when $I = \bigoplus_{g\in G}(I \cap L_g)$ and we define graded Lie subalgebras similarly. Likewise, if I is a graded ideal of L, then $C_L(I) = \{a \in L | [a, I] = \{0\}\}$ is also a graded ideal of L. Furthermore, $Z(L)$, the n-th term of descending central series L^n, and the n-th term of derived series $L^{(n)}$ are graded ideals of L. We use the convention that $L^1 = L$ and $L^{(1)} = [L, L]$.

Let L be a finite-dimensional Lie algebra. Denote by $\mathrm{Nil}(L)$ the greatest nilpotent ideal of L and by $\mathrm{Rad}(L)$ the greatest solvable ideal of L. Clearly, when L is a Lie A-algebra, $Nil(L)$ is the unique maximal abelian ideal of L. Furthermore, when L is a Lie A-algebra, tehn every subalgebra and every factor algebra of L are also A-algebras (see [15, Lemma 2.1] and [10, Lemma 1]).

The next theorem is a structural result on solvable Lie A-algebras.

Theorem 2.1 (Towers, [15, Theorem 3.5]). *Let L be a (finite-dimensional) solvable Lie A-algebra (over an arbitrary field F) of derived length $n+1$ with nilradical* $\mathrm{Nil}(L)$*. Moreover, let K be an ideal of L and B a minimal ideal of L. Then we have the following:*

- $K = (K \cap A_n) \oplus (K \cap A_{n-1}) \oplus \cdots \oplus (K \cap A_0)$;
- $\mathrm{Nil}(L) = A_n \dot{+} (A_{n-1} \cap Nil(L)) \dot{+} \cdots \dot{+} (A_0 \cap \mathrm{Nil}(L))$;
- $Z(L^{(i)}) = \mathrm{Nil}(L) \cap A_i$ *for each* $0 \le i \le n$;
- $B \subseteqq \mathrm{Nil}(L) \cap A_i$ *for some* $0 \le i \le n$,

where $A_n = L^{(n)}$, $A_{n-1}, \ldots, A_0$ are abelian subalgebras of L defined in the proof of [15, *Corollary 3.2*].

Remark 2.2. In vertue of Theorem 2.1, we can prove that, if $L = \bigoplus_{g \in G} L_g$ is a (finite-dimensional) solvable graded Lie A-algebra (over an arbitrary field F) of derived length $n+1$ with nilradical $\mathrm{Nil}(L)$, then $\mathrm{Nil}(L)$ is a graded ideal. Moreover, if L is a finite-dimensional metabelian Lie A-algebra (over an arbitrary field), then $\mathrm{Nil}(L) = [L, L] \dot{+} Z(L)$.

A finite-dimensional Lie algebra L is called semisimple if $\mathrm{Rad}(L) = \{0\}$. Recall that L (finite-dimensional and nonsolvable) has a Levi decomposition when there exists a semisimple subalgebra $S \neq \{0\}$ (called a Levi subalgebra) such that L is a semidirect product of S and $\mathrm{Rad}(L)$. We now present a result.

Proposition 2.3 (Premet and Semenov, [10, Proposition 2] , adapted). *Let L be a finite-dimensional Lie A-algebra over a finite field F of* $\mathrm{char} F > 3$. *Then,*

- $[L, L] \cap Z(L) = \{0\}$;

- *L has a Levi decomposition. Moreover, each Levi subalgebra S is represented as a direct sum of F-simple ideals in S, each one of which splits over some finite extension of the ground field into a direct sum of the ideals isomorphic to $sl_2(F)$.*

A Lie algebra L is said to be G-simple if $[L, L] \neq \{0\}$, and L does not have any proper nontrivial graded ideals.

By mimicking the arguments of Pagon, Repovš, and Zaicev in [11, Lemma 2.1; Section 3; Proposition 3.1, items (i) and (ii)], we have the following.

Proposition 2.4. *Let L be a finite-dimensional graded Lie algebra. The ideal $\mathrm{Rad}(L)$ is a graded ideal. If L is G-simple, then L is a direct sum of simple Lie algebras. If L is a direct sum of simple Lie algebras, then L is a direct sum of G-simple Lie algebras.*

3. Graded identities and varieties of graded Lie algebras. Let $X = \{X_g = \{x_1^g, \ldots, x_n^g, \ldots\} \mid g \in G\}$ be a family of pairwise-disjoint enumerable sets, where X_g denotes the variables of G-degree g. Let $F\langle X\rangle$ be the free associative unital algebra and let $L(X)$ be the Lie subalgebra of $F\langle X\rangle$ generated by X. It is known that $L(X)$ is isomorphic to the free Lie algebra with a set of free generators X. The algebras $L(X)$ and $F\langle X\rangle$ have a natural G-grading. A graded ideal $I \subset L(X)$ invariant under all graded endomorphisms is called a graded verbal ideal. Let $S \subset L(X)$ be a nonempty set. The graded verbal ideal generated by S, $\langle S\rangle_T$, is defined as the intersection of all verbal ideals containing S. A polynomial $f \in L(X)$ is called a consequence of $h \in L(X)$ when $f \in \langle h\rangle_T$, and it is called a graded polynomial identity for a graded Lie algebra L if f vanishes on L whenever the variables from X_g are substituted by elements of L_g for all $g \in G$. We denote by $\mathrm{Id}_G(L)$ the set of all graded identities of L. The variety determined by $S \subset L(X)$ is denoted by

$$\mathcal{V}(S) = \{A \text{ is a } G\text{-graded Lie algebra} \mid \mathrm{Id}_G(A) \supset \langle S\rangle_T\}.$$

The variety generated by a graded Lie algebra L is denoted by

$$\mathrm{var}_G(L) = \{A \text{ is a } G\text{-graded Lie algebra} \mid \mathrm{Id}_G(L) \subset \mathrm{Id}_G(A)\}.$$

We say that a class of graded Lie algebras $\{L_i\}_{i\in\Gamma}$, where Γ is an index set, generates $\mathcal{V}(S)$ when $\langle S\rangle_T = \bigcap_{i\in\Gamma} \mathrm{Id}_G(L_i)$.

We denote by $\mathrm{Id}(L)$ the set of all ordinary polynomial identities of a Lie algebra L, and by $\mathrm{var}(L)$ the variety generated by L. The variety of metabelian Lie algebras over F is denoted by $\mathcal{A}^2$. A set $S \subset L(X)$ of ordinary polynomials (respectively graded polynomials) is called a basis for the ordinary identities (respectively graded identities) of a Lie algebra (respectively a graded Lie algebra) A when S generates $\mathrm{Id}(A)$ as a verbal ideal (respectively $\mathrm{Id}_G(A)$ as a graded verbal ideal).

Example 3.1. In 1990's, Semenov ([13, Proposition 1]) proved that

$$\begin{aligned}
\mathrm{Sem}_1(x_1,x_2) = \;& (x_1)f(\mathrm{ad}(x_2)), \quad f(t) = t^{q^2+2} - t^3,\\
\mathrm{Sem}_2(x_1,x_2) = \;& [x_1,x_2] - [x_1,x_2,x_1^{q^2-1}] - [x_1,x_2^q]\\
& +[x_1,x_2,x_1^{q^2-1},x_2^{q-1}] + [x_1,x_2,x_1^{q^2}-x_1,[x_1,x_2]^{q-2},x_2^{q^2}-x_2]\\
& -[x_2,([x_1^{q^2}-x_1,x_2])^q,x_2^{q^2-2}-x_2^{q-2}].
\end{aligned}$$

are polynomial identities of $sl_2(F)$.

A finite-dimensional ordinary (respectively graded) Lie algebra L is critical if $\mathrm{var}(L)$ (respectively $\mathrm{var}_G(L)$) is not generated by all proper subquotients of L. It is monolithic if it contains a single ordinary (respectively graded) minimal ideal. This single ideal is called the monolith of L. It is known that if L is an ordinary (respectively graded) critical Lie algebra, then L is monolithic. Notice that if $L = \oplus_{g\in G} L_g$ is a critical ordinary Lie algebra, then L is critical also as a G-graded Lie algebra.

Example 3.2. If L is a critical abelian (ordinary or graded) Lie algebra, then $\dim L = 1$. If L is a two-dimensional (nonabelian) metabelian Lie algebra, then L is critical. Furthermore, $sl_2(F)$ is a critical Lie algebra.

Proposition 3.3. *Let $L = \bigoplus_{g\in G} L_g$ be a finite-dimensional (nonabelian) metabelian graded Lie A-algebra over an arbitrary field F. If L is monolithic, then* $\mathrm{Nil}(L) = [L,L]$.

P r o o f. According to Theorem 2.1, $\mathrm{Nil}(L) = [L,L]\dot{+}Z(L)$. By hypothesis, L is monolithic. Thus, $Z(L) = \{0\}$ and $\mathrm{Nil}(L) = [L,L]$. □

The next theorem describes the relationship between critical metabelian Lie A-algebras and monolithic Lie A-algebras.

Theorem 3.4 (Sheina, [14, Theorem 1]). *A finite-dimensional monolithic Lie A-algebra L over an arbitrary finite field is critical if and only if its derived algebra cannot be represented as a sum of two ideals strictly contained within it.*

A locally finite Lie algebra is a Lie algebra for which every finitely generated subalgebra is finite. A variety of Lie algebras (respectively graded Lie algebras) is said to be locally finite when every finitely generated Lie algebra (respectively graded Lie algebra) has finite cardinality. It is known that a variety generated by a finite Lie algebra (respectively a graded finite Lie algebra) is locally finite. As in the ordinary case, if a variety of graded Lie algebras is locally finite, then it is generated by its critical algebras. For more details about varieties of Lie algebras, see [1, Chapters 4 and 7].

The next result will be useful for our purposes.

Theorem 3.5 (Semenov, [13, Proposition 2]). *Let $\mathcal{B}$ be a variety of ordinary Lie algebras over a finite field F. If there exists a polynomial*

$$f(t) = a_1 t + \cdots + a_n t^n \in F[t]$$

such that

$$yf(\mathrm{ad}x) := a_1[y, x] + \cdots + a_n[y, x^n] \in \mathrm{Id}(\mathcal{B}),$$

then $\mathcal{B}$ is a locally finite variety.

Let L_1 and L_2 be two graded (finite-dimensional) Lie algebras, and $I_1 \subset L_1$ and $I_2 \subset L_2$ be graded ideals. We say that I_1 (in L_1) is similar to I_2 (in L_2) ($I_1 \trianglelefteq A_1 \sim I_2 \trianglelefteq A_2$) if there exist isomorphisms $\alpha_1 : I_1 \to I_2$ and $\alpha_2 : L_1/C_{L_1}(I_1) \to L_2/C_{L_2}(I_2)$ such that for all $a \in I_1$ and $b + C_{L_1}(I_1) \in L_1/C_{L_1}(I_1)$:

$$\alpha_1([a, c]) = [\alpha_1(a), d],$$

where $c + C_{L_1}(I_1) = b + C_{L_1}(I_1)$ and $d + C_{L_2}(I_2) = \alpha_2(b + C_{L_1}(I_1))$.

By proceeding as in [9, cf. pages 162 to 166], we have the following.

Proposition 3.6. *If two critical graded Lie algebras L_1 and L_2 generate the same variety, then their monoliths are similar.*

4. $\mathbb{Z}_2$-graded identities of $sl_2(F)$.

From now on, we denote by $Y = \{y_1, y_2, \ldots\}$ the even variables and by $Z = \{z_1, , z_2, \ldots\}$ the odd variables.

Lemma 4.1. *Let $sl_2(F)$ be the Lie algebra of traceless 2×2 matrices over F endowed with the natural $\mathbb{Z}_2$-grading. The following polynomials are graded identities of $sl_2(F)$*

$$[y_1, y_2], \quad [z_1, y_1^q] - [z_1, y_1].$$

P r o o f. It is clear that $[y_1, y_2] \in \mathrm{Id}_G(sl_2(F))$, because the diagonal is commutative. Choose $a_i = \lambda_{11,i} e_{11} - \lambda_{11,i} e_{22}$ and $b_j = \lambda_{12,j} e_{12} + \lambda_{21,j} e_{21}$. Then

$$[b_j, a_i^q] = \lambda_{11,i}^q [b_j, h^q] = \lambda_{11,i}^q ((-2)^q \lambda_{12,j} e_{12} + 2^q \lambda_{21,j} e_{21})$$

$$= \lambda_{11,i}(-2\lambda_{12,j} e_{12} + 2\lambda_{21,j} e_{21}) = [b_j, a_i].$$

Thus, $[z_1, y_1^q] - [z_1, y_1] \in \mathrm{Id}_G(sl_2(F))$. The proof is complete. $\square$

We now cite two papers. First we present a corollary of Bahturin, Kochetov, and Montgomery [2].

Proposition 4.2 ([2, Corollary 1]). *Let* $R = M_n(F)$, $\mathrm{char}F = p > 0$, $p \neq 2$. *Let* G *be an elementary abelian* p*-group. Suppose that* $R = \bigoplus_{g\in G} R_g$ *is a grading on* $R^{(-)}$. *Then* $R = \bigoplus_{g\in G} R_g$ *is a grading on* R *if and only if* $1 \in R_0$.

Remark 4.3. Here, 1 denotes the identity matrix of $M_n(F)$, and 0 denotes the neutral element of G (recall that G is written additively).

In this paper, F is a finite field of $charF = p > 3$ and size q. So, there exists $b \in F - \{0\}$ which is not a perfect square.

Notice that if $sl_2(F) = (sl_2(F))_0 \oplus (sl_2(F))_1$ is a $\mathbb{Z}_2$-grading on $sl_2(F)$, then $((sl_2(F))_0 \oplus F(e_{11} + e_{22})) \oplus (sl_2(F))_1$ is a $\mathbb{Z}_2$-grading on $gl_2(F)$. By Proposition 4.2, $((sl_2(F))_0 \oplus F(e_{11} + e_{22})) \oplus (sl_2(F))_1$ is a $\mathbb{Z}_2$-grading on $M_2(F)$. The next proposition describes the $\mathbb{Z}_2$-gradings on $M_2(F)$.

Proposition 4.4 (Khazal, Boboc, and Dăscălescu, [7, Theorem 1.1], adapted). *Let* F *be a field of* $\mathrm{char}F \neq 2$. *Then any* $\mathbb{Z}_2$*-grading of* $M_2(F)$ *is isomorphic to one of the following:*

- $(M_2(F)_0, M_2(F)_1) = (M_2(F), 0)$;
- $(M_2(F)_0, M_2(F)_1) = (Fe_{11} \oplus Fe_{22}, Fe_{12} \oplus Fe_{21})$;
- $(M_2(F)_0, M_2(F)_1)$
  $= (F(e_{11} + e_{22}) \oplus F(e_{12} + be_{21}), F(e_{11} - e_{22}) \oplus F(e_{12} - be_{21}))$, *where* $b \in F - F^2$.

Remark 4.5. It is well known that $(F - \{0\}, \cdot)$ is a cyclic group of order $q - 1$. By elementary theory of groups, for every divisor d of $q - 1$, there exists a unique subgroup H' of $(F - \{0\}, \cdot)$ of order d. Let H be the subgroup of order $\frac{q-1}{2}$. It is easy to see that there exists $b' \in (F - F^2) \cap (F - H)$. Finally, note that

$$[(e_{11} - e_{22}), (e_{12} + b'e_{21})] \neq [(e_{11} - e_{22}), (e_{12} + b'e_{21}), \dots, (e_{12} + b'e_{21})],$$

where $(e_{12} + b'e_{21})$ appears q times in the expanded commutator.

Proposition 4.6. *Let* $sl_2(F) = (sl_2(F))_0 \oplus (sl_2(F))_1$ *be a* $\mathbb{Z}_2$*-grading on* $sl_2(F)$ *having the following characteristics:*

- $\dim(sl_2(F))_0 = 1$,

- $[a, c^q] = [a, c]$ *for all* $a \in (sl_2(F))_1$ *and* $c \in F(e_{11} + e_{22}) \oplus (sl_2(F))_0$.

Then the $\mathbb{Z}_2$*-gradings* $((sl_2(F))_0, (sl_2(F))_1)$ *and* $(F(e_{11} - e_{22}), Fe_{12} \oplus Fe_{21})$ *are isomorphic.*

Proof. First, note that $(((sl_2(F))_0 \oplus F(e_{11} + e_{22})) \oplus (sl_2(F))_1)$ is a $\mathbb{Z}_2$-grading on $gl_2(F)$. According to Proposition 4.2, $(((sl_2(F))_0 \oplus F(e_{11} + e_{22})) \oplus (sl_2(F))_1)$ is a $\mathbb{Z}_2$-grading on $M_2(F)$. It is clear that this grading on $M_2(F)$ is not isomorphic to the first grading presented in Proposition 4.4. Notice also that $(F(e_{11} + e_{22}) \oplus (sl_2(F))_0, (sl_2(F))_1)$ cannot be isomorphic to the third grading presented in Proposition 4.4, because $[z_1, y_1] = [z_1, y_1^q]$ is not a polynomial identity for $M_2(F)$ endowed with the third grading (Remark 4.5). According to Proposition 4.4, there exists a $\mathbb{Z}_2$-graded isomorphism $\phi : M_2(F) \to M_2(F)$ such that

$$\phi((sl_2(F))_0 \oplus F(e_{11} + e_{22})) = Fe_{11} \oplus Fe_{22} \text{ and } \phi((sl_2(F))_1) = Fe_{12} \oplus Fe_{21}.$$

Note that $\phi : gl_2(F) \to gl_2(F)$ is an isomorphism of Lie algebras and $\phi(sl_2(F)) = sl_2(F)$. Thus, $((sl_2(F))_0, (sl_2(F))_1)$ and $(F(e_{11} - e_{22}), Fe_{12} \oplus Fe_{21})$ are isomorphic. The proof is complete. □

Henceforth we consider only $sl_2(F)$ and $Fe_{11} \oplus Fe_{12}$ endowed with the natural grading by $(\mathbb{Z}_2, +)$. Recall that $\mathrm{Sem}_1(x_1, x_2), \mathrm{Sem}_2(x_1, x_2) \in \mathrm{Id}(sl_2(F))$. We denote by S the set of the following polynomials

$$\mathrm{Sem}_1(y_1 + z_1, y_2 + z_2), \mathrm{Sem}_2(y_1 + z_1, y_2 + z_2), [y_1, y_2], \text{ and } [z_1, y_1^q] = [z_1, y_1].$$

Corollary 4.7. *The variety* $\mathcal{V}(S)$ *is locally finite.*

Proof. Let $L = L_0 \oplus L_1 \in \mathcal{V}(S)$ be a finitely generated algebra. By the definition of S, $\mathrm{Sem}_1(y_1 + z_1, y_2 + z_2) \in \mathrm{Id}_G(L)$. Hence, $\mathrm{Sem}_1(x_1, x_2) \in \mathrm{Id}(L)$. So, by Theorem 3.5, it follows that L is a finite Lie algebra. □

Corollary 4.8. *Let* $L \in \mathcal{V}(S)$ *be a finite-dimensional Lie algebra. Then every nilpotent subalgebra of* L *is abelian.*

Proof. From the definition of S, it follows that $\mathrm{Sem}_2(x_1, x_2) \in \mathrm{Id}(L)$. Let $M \neq \{0\}$ be a nilpotent (unnecessarily graded) subalgebra of L. If $M^t = \{0\}$ for a positive integer $t \leq q + 1$, it is clear that M is abelian. If the index of nilpotency is equal to $q + 2$, then $M/Z(M)$ is abelian. Consequently, M is abelian. Induction on the index of nilpotency will give the desired result. □

It is well known that a verbal ideal (and, respectively, a graded verbal ideal) over an infinite field is multihomogeneous. This fact can be weakened, as stated in the next lemma.

Lemma 4.9. *Let I be a graded verbal ideal over a field of size q. If $f(x_1,\dots,x_n)\in I$ and $0\le deg_{x_1}f,\dots,deg_{x_n}f<q$, then each multihomogeneous component of f belongs to I as well.*

Lemma 4.10. *If $L=\operatorname{span}_F\{e_{11},e_{12}\}\subset gl_2(F)$, then the $\mathbb{Z}_2$-graded identities of L follow from*

$$[y_1,y_2],[z_1,z_2] \text{ and } [z_1,y_1^q]-[z_1,y_1].$$

P r o o f. It is clear that L satisfies the identities $[y_1,y_2],[z_1,z_2]$ and $[z_1,y_1^q]-[z_1,y_1]$. We shall prove that the reverse inclusion holds true. Let f be a polynomial identity of L. We may write

$$f=g+h,$$

where $h\in\langle[y_1,y_2],[z_1,z_2],[z_1,y_1^q]-[z_1,y_1]\rangle$ and $g(x_1,\dots,x_n)\in \mathrm{Id}_G(L)$, with $0\le deg_{x_1}g,\dots,deg_{x_n}g<q$. In this way, we may suppose that g is a multihomogeneous polynomial. If $g(y_1)=\alpha_1y_1$ or $g(z_1)=\alpha_2z_1$ we can easily see that $\alpha_1=\alpha_2=0$. In the other case, we may assume that

$$g(z_1,y_1,\dots,y_l)=\alpha_3[z_1,y_1^{a_1},\dots,y_l^{a_l}],\quad 1\le a_1,\dots,a_l<q.$$

However, $g(e_{12},e_{11},\dots,e_{11})$ is a nonzero multiple scalar of e_{12}, and consequently, $\alpha_3=0$. Hence, $f=h$ and the proof is complete. □

Lemma 4.11. *Let $L=L_0\oplus L_1\in\mathcal{A}^2\cap\mathcal{V}(S)$ be a critical Lie A-algebra. Then*

$$L\in \operatorname{var}_G(\operatorname{span}_F\{e_{11},e_{12}\}).$$

P r o o f. According to Lemma 4.10, it is sufficient to prove that L satisfies the identity $[z_1,z_2]$.

By assumption, L is critical and therefore L is monolithic. If L is abelian, then $\dim L=1$. In this case $L=L_0\cong\operatorname{span}_F\{e_{11}\}$ or $L=L_1\cong\operatorname{span}_F\{e_{12}\}$.

In the sequel, we suppose that L is nonabelian. From Proposition 3.3, we have $[L,L]=\mathrm{Nil}(L)=[L_1,L_1]\oplus[L_0,L_1]$. From the identity $[z_1,y_1]=[z_1,y_1^q]$, $\{0\}=[L_1,[L_1,L_1]]=-[L_1,L_1,L_1]$. So, by the identity $\mathrm{Sem}_2(y_1+z_1,y_2+z_2)$, we have $[z_1,z_2]\in\mathrm{Id}_G(span_F\{e_{11},e_{12}\})$, as required. The proof is complete. □

Corollary 4.12. *$\mathcal{A}^2\cap\operatorname{var}_G(sl_2(F)),\mathcal{A}^2\cap\mathcal{V}(S)$ and $\operatorname{var}_G(\operatorname{span}_F\{e_{11},e_{12}\})$ coincide.*

P r o o f. First, notice that $\mathcal{A}^2\cap\operatorname{var}_G(sl_2(F))\subset\mathcal{A}^2\cap\mathcal{V}(S)$ which is a locally finite variety. By Lemma 4.11, all critical algebras of $\mathcal{A}^2\cap\mathcal{V}(S)$ belong to

$$\operatorname{var}_G(\operatorname{span}_F\{e_{11},e_{12}\})\subset A^2\cap\operatorname{var}_G(sl_2(F)).$$

Therefore, $\mathcal{A}^2 \cap \mathcal{V}(S) \subset \mathrm{var}_G(\mathrm{span}_F\{e_{11}, e_{12}\})$. Thus, we have

$$\mathcal{A}^2 \cap \mathrm{var}_G(sl_2(F)) = \mathcal{A}^2 \cap \mathcal{V}(S) = \mathrm{var}_G(\mathrm{span}_F\{e_{11}, e_{12}\}). \qquad \square$$

Lemma 4.13. *Let L be a critical solvable Lie A-algebra belonging to $\mathcal{V}(S)$. Then L is metabelian.*

Proof. Let $L \in \mathcal{V}(S)$ be a critical (nonabelian) solvable Lie algebra with monolith W. By Proposition 2.3, we have $[L, L] \cap Z(L) = \{0\}$. Consequently, $Z(L) = \{0\}$. Notice that $Z(C_L(\mathrm{Nil}(L))) = \mathrm{Nil}(L)$. If $\mathrm{Nil}(L)_1 = L_1$, then L is metabelian. Now, we suppose that $\mathrm{Nil}(L)_1 \subsetneq L_1$. We assert that $\mathrm{Nil}(L)_0 = \{0\}$. Suppose, on the contrary, that there exists $a \neq 0 \in \mathrm{Nil}(L)_0$. Hence, there exists $b \in L_1 - \mathrm{Nil}(L)_1$ such that $[b, a] \neq 0$, because $Z(L) = \{0\}$. However, $[b, a] = [b, a^q] = 0$. This is a contradiction. Thus, $[L_1, \mathrm{Nil}(L)] = \{0\}$. Consequently $C_L(\mathrm{Nil}(L)) \supset L_1 \cup [L_1, L_1]$. By Proposition 2.3

$$Z(C_L(\mathrm{Nil}(L))) \cap [C_L(\mathrm{Nil}(L)), C_L(\mathrm{Nil}(L))] = \{0\}.$$

Hence, $[C_L(\mathrm{Nil}(L)), C_L(\mathrm{Nil}(L))] = \{0\}$. So, $L^{(2)} = \{0\}$ and the proof is complete. $\square$

Lemma 4.14. *Let L be a critical nonsolvable Lie A-algebra belonging to $\mathcal{V}(S)$. Then L is G-simple.*

Proof. Let W be the monolith of L. We claim that L is semisimple. Suppose on the contrary that $\mathrm{Rad}(L) \neq \{0\}$. Thus, $W \subset \mathrm{Rad}(L) \cap [L, L]$. The nontrivial subspace W is an abelian ideal and it is contained in $L^{(n)}$, where n is the least nonnegative integer such that $L^{(n)} = L^{(n+1)}$. According to Proposition 2.3, $[L, L] \cap Z(L) = \{0\}$. So $Z(L) = \{0\}$ and $[L, W] = W$. The identities $[y_1, y_2]$ and $[z_1, y_1] = [z_1, y_1^q]$ mean that the subspace $W_0 = \{0\}$. Notice that $[W, [L, L]] = \{0\} = [W, L^{(n)}]$ and $Z(L^{(n)}) \supset W$. By Proposition 2.3, $Z(L^{(n)}) \cap L^{(n)} = \{0\}$. This is a contradiction, so L is semisimple. By Propositions 2.3 and 2.4, L is a direct sum of G-simple Lie algebras. Given that L is monolithic, we conclude that L is G-simple. $\square$

The next lemma was proved by Drensky in [3, Lemma, page 991 of the English translation].

Lemma 4.15. *Let V be a finite-dimensional vector space over F and let A be an abelian Lie algebra of the linear transformations $\phi : V \to V$, where each ϕ satisfies the equality*

$$\phi^q = \phi.$$

Then, every $\phi \in A$ is diagonalizable.

Definition 4.16. *Let L be a finite-dimensional Lie algebra with a diagonalizable operator $T : L \to L$. We denote by $V(T)$ a basis of L formed by the eigenvectors of T. Moreover, we denote $V(T)_\lambda = \{v \in V(T) \mid T(v) = \lambda \cdot v\}$. We denote $EV(w)$ the eigenvalue associated with the eigenvector $w \in V(T)$.*

Let $L \in \mathcal{V}(S)$ be a finite-dimensional Lie algebra. It is not difficult to see that $\mathrm{ad}(L_0) = \{\mathrm{ad}a : L \to L \mid a \in L_0\}$ is an abelian subalgebra of linear transformations of L. Moreover, $(\mathrm{ad}a_0)^p = \mathrm{ad}a_0$ for all $a_0 \in L_0$. By Lemma 4.15, we have the following.

Corollary 4.17. *Let $L \in \mathcal{V}(S)$ be a finite-dimensional Lie algebra. Let $a_0 \in L_0$. Then there exists $V(\mathrm{ad}a_0) \subset L_0 \cup L_1$.*

Proposition 4.18. *Let $L \in \mathcal{V}(S)$ be a finite-dimensional G-simple algebra. Let $a_0 \in L_0$. Then there exists $V(\mathrm{ad}a_0) \subset L_0 \cup L_1$. Moreover, $V(\mathrm{ad}a_0)_0 \cap L_1 = \emptyset$ for any basis $V(\mathrm{ad}a_0) \subset L_0 \cup L_1$.*

Proof. According to Corollary 4.17, there exists $V(\mathrm{ad}a_0) \subset L_0 \cup L_1$.

Let $b_1, b_2 \in V(\mathrm{ad}a_0) \cap L_1$. Notice that if $[b_1, b_2] \neq 0$, then $EV(b_1) = -EV(b_2)$.

It is clear that $\langle a_0 \rangle$ is a graded ideal, and that it is equal to L. Notice also that $L = \mathrm{span}_F\{[a_0, b_1, \dots, b_n] | b_1, \dots, b_n \in V(\mathrm{ad}a_0), n \geq 1\}$.

If there was a nonzero element $b \in V(\mathrm{ad}a_0)_0 \cap L_1$, we could easily check that $[a_0, b_1, b] = 0$ for any $b_1 \in V(\mathrm{ad}(a_0))$. More generally, by an inductive argument and routine calculations, we would have $[a_0, b_1, \dots, b_n, b] = 0$ for any $n \geq 1$ and $b_1, \dots, b_n \in V(\mathrm{ad}a_0)$. However, an element such as b cannot be, because $Z(L) = \{0\}$. So, $V(\mathrm{ad}a_0)_0 \cap L_1 = \emptyset$. □

Lemma 4.19. *Let $L \in \mathcal{V}(S)$ be a critical nonsolvable algebra, then $L \cong sl_2(F)$.*

Proof. First of all, notice that $\dim L_0 \geq 1$ and $\dim L_1 \geq 2$. According to Lemma 4.14 L is G-simple. Let $a_0 \in L_0$. By Proposition 4.18, there exists $V(\mathrm{ad}a_0) = \{b_1, \dots, b_n\} \subset L_0 \cup L_1$. Moreover, $V(\mathrm{ad}a_0)_0 \cap L_1 = \emptyset$.

Let $-\lambda_1 \leq \cdots \leq -\lambda_m < 0 < \lambda_m \leq \cdots \leq \lambda_1$ be the eigenvalues associated with the eigenvectors of $V(\mathrm{ad}a_0)$. Notice that

$$L_0 = \sum_{i=1}^{m} \mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_i}, V(\mathrm{ad}a_0)_{-\lambda_i}]\}.$$

Without loss of generality, suppose that $\mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_1}, V(\mathrm{ad}a_0)_{-\lambda_1}]\} \neq \{0\}$.

We assert that $\mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_1}, V(\mathrm{ad}a_0)_{-\lambda_1}]\} \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$ is a subalgebra of L.

In fact, let $a \in V(\mathrm{ad}a_0)_{\lambda_1}$ and $b \in \mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_1}, V(\mathrm{ad}a_0)_{-\lambda_1}]\}$. Consider $[a,b] = \sum_{i=1}^{n} \alpha_i b_i$. So,

$$[a,b,a_0] = -\sum_{i=1}^{n} \alpha_i \cdot EV(b_i) b_i.$$

On the other hand,

$$[a,b,a_0] = -\lambda_1 [a,b] = -\lambda_1 (\sum_{i=1}^{n} \alpha_i b_i).$$

Hence

$$(-EV(b_j)\cdot \alpha_j + \lambda_1 \cdot \alpha_j) b_j = 0.$$

Consequently, if $\alpha_j \neq 0$, then $\lambda_1 = EV(b_j)$.

Similarly, the subspace

$$\mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_1}, V(\mathrm{ad}a_0)_{-\lambda_1}]\} \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{-\lambda_1}\}$$

is a subalgebra. Notice that

$$\mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_1}, V(\mathrm{ad}a_0)_{-\lambda_1}]\} \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\} \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{-\lambda_1}\}$$

is a graded ideal of L.

Therefore, $L_0 = \mathrm{span}_F\{V(\mathrm{ad}a_0)_0\} = \mathrm{span}_F\{[V(\mathrm{ad}a_0)_{\lambda_1}, V(\mathrm{ad}a_0)_{-\lambda_1}]\}$ and the subspace L_1 is equal to $\mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\} \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{-\lambda_1}\}$.

Notice that $\mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$ is an irreducible L_0-module, because L is G-simple. Moreover, it is not difficult to see that $L_0 \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$ is a monolithic metabelian Lie algebra with monolith $\mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$ when viewed as ordinary Lie algebra. Notice that

$$[L_0 \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}, L_0 \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}] = \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$$

cannot be represented by the sum of two ideals strictly contained within it. By Theorem 3.4, $L_0 \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$ is critical when viewed as an ordinary Lie algebra. Thus, it is critical when viewed as a graded algebra as well.

Following the arguments of Lemma 4.10, we can prove that

$$\mathrm{Id}_G(L_0 \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}) = \langle [y_1,y_2], [z_1,y_1] - [z_1, y_1^q], [z_1,z_2] \rangle_T.$$

Consequently, it follows from Proposition 3.6 that $\mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\}$ is a one-dimensional vector space. Analogously, we have $\dim(\mathrm{span}_F\{V(\mathrm{ad}a_0)_{-\lambda_1}\}) = 1$. Therefore, $L_0 \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{\lambda_1}\} \oplus \mathrm{span}_F\{V(\mathrm{ad}a_0)_{-\lambda_1}\}$ is a three-dimensional G-simple Lie algebra. So, L is simple and isomorphic to $sl_2(F)$ (as ordinary Lie algebras). Hence, by Proposition 4.6, $L \cong sl_2(F)$ (as graded Lie algebras), where $sl_2(F)$ is naturally graded by $\mathbb{Z}_2$. The proof is complete. $\square$

5. Main theorem.

We now prove the main theorem of this paper.

Theorem 5.1. *Let F be a field of* $\mathrm{char}(F) > 3$ *and size $|F| = q$. The $\mathbb{Z}_2$-graded identities of $sl_2(F)$ follow from*

$$[y_1, y_2], \mathrm{Sem}_1(y_1 + z_1, y_2 + z_2), \mathrm{Sem}_2(y_1 + z_1, y_2 + z_2), \text{ and } [z_1, y_1] - [z_1, y_1^q].$$

Proof. It is clear that $\mathrm{var}_G(sl_2(F)) \subset \mathcal{V}(S)$. To prove that the reverse inclusion holds, it is sufficient to prove that all critical algebras of $\mathcal{V}(S)$ are also critical algebras of $\mathrm{var}_G(sl_2(F))$. According to Corollary 4.12, $\mathcal{A}^2 \cap \mathcal{V}(S) = \mathcal{A}^2 \cap \mathrm{var}_G(sl_2(F))$. By Lemma 4.13, any critical solvable Lie algebra of $\mathcal{V}(S)$ is metabelian. By Lemma 4.19, any critical nonsolvable Lie algebra of $\mathcal{V}(S)$ is isomorphic to $sl_2(F)$. Therefore, $\mathcal{V}(S) \subset \mathrm{var}_G(sl_2(F))$, and the theorem is proved. $\square$

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Departamento de Matemática
Universidade Federal de Viçosa – Campus Florestal
Rodovia LMG 818, km 06, Florestal, MG, Brazil
e-mail: `luisfelipe@ufv.br`

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