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Serdica Math. J. **42** (2016), 251–260

Serdica
Mathematical Journal

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DERIVATIONS IN FINITE ENDOMORPHISM SEMIRINGS

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Communicated by V. Drensky

Abstract. The aim of this article is to construct examples of derivations in finite semirings.

1. Introduction and preliminaries. The differential algebra has been studied by many authors for the last 65 years and especially the relationships between derivations and the structure of rings. The notion of the ring with derivation is old and plays an important role in the integration of analysis, algebraic geometry and algebra. In 1950 J. Ritt [6], and in 1973 E. Kolchin [4], wrote the classical books on differential algebra.

During the last few decades there has been a great deal of works concerning derivations in rings, in Lie rings, in skew polynomial rings and other algebraic structures. About derivations in semirings it is known the definition in [2], examples and some properties in [1], examples and properties of derivations in simplicial complexes of strings, see [9], and examples and properties of derivations in triangles, see [12] and [13].

2010 *Mathematics Subject Classification:* 06A05, 12H05, 16Y60, 16W20.
Key words: Semiring, endomorphism semiring of a finite chain, differential algebra, simplicial complex.

The endomorphism semirings of a finite semilattice are well-established, see [8, 10, 11, 14]. Basic facts for semirings can be found in [2]. Concerning background of simplicial complexes and combinatorics a reader is referred to [5] and [7].

Example 3.2 in [1] shows that the map D given by $D\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, where the elements $a, b, c \in \mathbb{Z}^+ \cup \{0\}$ is a derivation. It is true only for the second order matrices. Now we present the following example.

Example. Let $UTM_n(S)$ is the semiring ot the upper triangular matrices of order n with entries from the additively idempotent semiring S. Let for $A = (a_{ij}) \in UTM_n(S)$ we define $D(A) = A \backslash \operatorname{diag}(a_{11}, \dots, a_{nn})$. For $B = (b_{ij}) \in UTM_n(S)$ the arbitrary element of $D(AB)$ is $\sum_{k=i}^{j} a_{ik}b_{kj}$, where $1 \leq i \leq k \leq j \leq n$. On the other hand the arbitrary element of the matrix $D(A)B$ is $\sum_{1<k\leq j} a_{ik}b_{kj}$. Also the arbitrary element of the matrix $AD(B)$ is $\sum_{i\leq k<j} a_{ik}b_{kj}$. Since S is an additively idempotent semiring, it follows that $\sum_{i<k<j} a_{ik}b_{kj} + \sum_{i<k<j} a_{ik}b_{kj} = \sum_{i<k<j} a_{ik}b_{kj}$. So $D(AB) = D(A)B + AD(B)$ and since D is evidently linear, it follows that D is a derivation.

This example shows the crucial role of the additively idempotent semirings for the derivations in semiring theory. In Theorem 2.2. in [3] Kim, Roush and Markowsky prove that any finite additively idempotent semiring can be represented as the endomorphism semiring of a finite chain. So, the present paper investigate the derivations in a finite endomorphism semiring.

Following [8], we fix a finite chain $\mathrm{C}_n = (\{0, 1, \dots, n-1\}, \vee)$ and denote the endomorphism semiring of this chain with $\widehat{\mathrm{E}}_{\mathrm{C}_n}$ For elements $a_0, a_1, \dots, a_{k-1} \in \mathrm{C}_n$, where $k \leq n$, $a_0 < a_1 < \dots < a_{k-1}$ we denote $A = \{a_0, a_1, \dots, a_{k-1}\}$. Now, consider endomorphisms $\alpha \in \widehat{\mathrm{E}}_{\mathrm{C}_n}$ with $Im(\alpha) \subseteq A$. The set of the all such endomorphisms α is a maximal simplex. We denote this simplex by $\sigma_k^{(n)}(A) = \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$. The endomorphisms $\alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$ such that

$$\alpha(0) = \cdots = \alpha(i_0 - 1) = a_0, \alpha(i_0) = \cdots = \alpha(i_0 + i_1 - 1) = a_1, \ \cdots$$

$$\alpha(i_0 + \cdots + i_{k-2}) = \cdots = \alpha(i_0 + \cdots + i_{k-1} - 1) = a_{k-1}$$

we denote by $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$, where $i_0 + i_1 + \dots + i_{k-1} = n$.

Any endomorphism $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$ can be represented as sum $(a_0)_{n-i_1}(a_1)_{i_1} + (a_0)_{n-i_2}(a_2)_{i_2} + \dots + (a_0)_{n-i_{k-1}}(a_{k-1})_{i_{k-1}}$. So the elements of strings $\mathrm{STR}^{(n)}\{a_0, a_m\}$, where $1 \le m \le k-1$ form an *additive base* of the simplex and these strings are called *basic strings* of the simplex.

2. Linearity of the projections. Let us consider the maps

$$\partial_m : \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\} \to \mathrm{STR}^{(n)}\{a_0, a_m\},$$

where $1 \le m \le k-1$, such that for any $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$, where $i_0 + i_1 + \dots + i_{k-1} = n$,

$$\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}.$$

These maps are called *projections* of the simplex on the basic strings.

Proposition 1. *For any $\alpha, \beta \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$ and projection*

$$\partial_m : \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\} \to \mathrm{STR}^{(n)}\{a_0, a_m\},$$

where $1 \le m \le k-1$, it follows $\partial_m(\alpha+\beta) = \partial_m(\alpha) + \partial_m(\beta)$.

Proof. We consider the endomorphisms $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$ and
$\beta = (a_0)_{j_0}(a_1)_{j_1}\dots(a_{k-1})_{j_{k-1}}$, $i_0 + i_1 + \dots + i_{k-1} = j_0 + j_1 + \dots + j_{k-1} = n$.

Case 1. Let $i_0 \le j_0$. Then $\alpha + \beta = (a_0)_{i_0}(a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, $i_0 + s_1 + \dots + s_{k-1} = n$ and $\partial_m(\alpha+\beta) = (a_0)_{i_0}(a_m)_{n-i_0} = \partial_m(\alpha)$. Since $i_0 \le j_0$ imply $\partial_m(\beta) = (a_0)_{j_0}(a_m)_{n-j_0} \le (a_0)_{i_0}(a_m)_{n-i_0} = \partial_m(\alpha)$ it follows $\partial_m(\alpha+\beta) = \partial_m(\alpha) + \partial_m(\beta)$.

Case 2. Let $j_0 \le i_0$. After the interchanging α and β it follows that $\partial_m(\alpha+\beta) = \partial_m(\alpha) + \partial_m(\beta)$. □

3. Projection on the least basic string of a simplex. Now we consider the set

$$\mathcal{D}_{\partial_1} = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_1) \le a_1\}.$$

For $\alpha, \beta \in \mathcal{D}_{\partial_1}$ we find $(\alpha+\beta)(a_1) = \alpha(a_1) + \beta(a_1) \le a_1$ and $(\alpha\beta)(a_1) = \beta(\alpha(a_1)) \le \beta(a_1) \le a_1$. So, we prove the following lemma

Lemma 1. *The set $\mathcal{D}_{\partial_1}$ is a subsemiring of the semiring*

$$\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}.$$

Lemma 2. *For any $\alpha, \beta \in \mathcal{D}_{\partial_1}$, it follows*

$$\partial_1(\alpha\beta) = \partial_1(\alpha)\beta + \alpha\partial_1(\beta). \tag{1}$$

Proof. Let $\alpha = (a_0)_{i_0}(a_1)_{i_1} \ldots (a_{k-1})_{i_{k-1}}$, $\beta = (a_0)_{j_0}(a_1)_{j_1} \ldots (a_{k-1})_{j_{k-1}}$, where $i_0 + i_1 + \cdots + i_{k-1} = j_0 + j_1 + \cdots + j_{k-1} = n$.

Case 1. Let $\beta(a_1) = a_0$ and $m, 1 \leq m \leq k-1$, be the largest positive integer such that $\beta(a_m) = a_0$. Then $\alpha\beta = (a_0)_{i_0+\cdots+i_m}(a_1)_{s_1} \ldots (a_{k-1})_{s_{k-1}}$, where $i_0 + \cdots + i_m + s_1 + \cdots + s_{k-1} = n$. Now $\partial_1(\alpha\beta) = (a_0)_{i_0+\cdots+i_m}(a_1)_{n-(i_0+\cdots+i_m)}$. Clearly $\partial_1(\alpha) = (a_0)_{i_0}(a_1)_{n-i_0}$ and so $\partial_1(\alpha)\beta = \overline{a_0}$. Since $\partial_1(\beta) = (a_0)_{j_0}(a_1)_{n-j_0}$ and $a_m \leq j_0 - 1$, it follows $\alpha\partial_1(\beta) = (a_0)_{i_0+\cdots+i_m}(a_1)_{n-(i_0+\cdots+i_m)}$ and (1) holds.

Case 2. Let $\beta(a_0) = a_0$ and $\beta(a_1) = a_1$. Then
$$\alpha\beta = (a_0)_{i_0}(a_1)_{s_1} \ldots (a_{k-1})_{s_{k-1}},$$
where $i_0 + s_1 + \cdots + s_{k-1} = n$ and $\partial_1(\alpha\beta) = (a_0)_{i_0}(a_1)_{n-i_0}$. Clearly $\partial_1(\alpha) = (a_0)_{i_0}(a_1)_{n-i_0}$ and so $\partial_1(\alpha)\beta = (a_0)_{i_0}(a_1)_{n-i_0}$. Since $\partial_1(\beta) = (a_0)_{j_0}(a_1)_{n-j_0}$ and $j_0 - 1 \leq a_1$, it follows $\alpha\partial_1(\beta) = (a_0)_{i_0}(a_1)_{n-i_0}$ and (1) holds.

Case 3. Let $\beta(a_0) = \beta(a_1) = a_1$ and $m, 1 \leq m \leq k-1$ be the largest positive integer such that $\beta(a_m) = a_1$. Then $\alpha\beta = (a_1)_{i_0+\cdots+i_m}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}}$, where $i_0 + \cdots + i_m + s_2 + \cdots + s_{k-1} = n$. Now $\partial_1(\alpha\beta) = \overline{a_1}$. Clearly $\partial_1(\alpha) = (a_0)_{i_0}(a_1)_{n-i_0}$ and so $\partial_1(\alpha)\beta = \overline{a_1}$. Since $\partial_1(\beta) = (a_0)_{j_0}(a_1)_{n-j_0}$ and $a_0 > j_0 - 1$, it follows $\alpha\partial_1(\beta) = \overline{a_1}$ and (1) holds. $\square$

Theorem 1. *The map $\partial_1 : \mathcal{D}_{\partial_1} \to \mathrm{STR}^{(n)}\{a_0, a_1\}$ is a derivation. The maximal subsemiring of $\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$, closed under the derivation ∂_1 is $\mathcal{D}_{\partial_1}$.*

Proof. Using Proposition 1 and Lemmas 1 and 2 we immediately prove that $\partial_1 : \mathcal{D}_{\partial_1} \to \mathrm{STR}^{(n)}\{a_0, a_1\}$ is a derivation. To prove the second part of the theorem we consider three cases.

Case 1. Let $\beta(a_0) = a_0$ and $\beta(a_1) > a_1$. Then
$$\alpha\beta = (a_0)_{i_0}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}},$$
where $i_0 + s_2 + \cdots + s_{k-1} = n$ and $\partial_1(\alpha\beta) = (a_0)_{i_0}(a_1)_{n-i_0}$. Since $\partial_1(\alpha) = (a_0)_{i_0}(a_1)_{n-i_0}$, it follows $\partial_1(\alpha)\beta = (a_0)_{i_0}(a_2)_{r_2} \ldots (a_{k-1})_{r_{k-1}}$, where $i_0 + r_2 + \cdots + r_{k-1} = n$. Now $\partial_1(\alpha)\beta > \partial_1(\alpha\beta)$, hence (1) does not hold.

Case 2. Let $\beta(a_0) = a_1$ and $\beta(a_1) = a_p$, where $1 < p \leq k-1$. Then $\alpha\beta = (a_1)_{i_0}(a_p)_{s_p}\dots(a_{k-1})_{s_{k-1}}$, where $i_0 + s_p + \cdots + s_{k-1} = n$ and $\partial_1(\alpha\beta) = \overline{a_1}$. Clearly $\partial_1(\alpha) = (a_0)_{i_0}(a_1)_{n-i_0}$. Since $\beta(a_1) = a_p$, where $1 < p \leq k-1$, then $\partial_1(\alpha)\beta = (a_1)_{i_0}(a_p)_{n-i_0} > \overline{a_1} = \partial_1(\alpha\beta)$. So, (1) does not hold.

Case 3. Let $\beta(a_0) = a_p$, where $1 < p \leq k-1$ and p is the largest positive integer with this property. Then $\alpha\beta = (a_p)_{s_p}\dots(a_{k-1})_{s_{k-1}}$, where $s_p+\cdots+s_{k-1} = n$ and in all cases $\partial_1(\alpha\beta) = \overline{a_1}$. Since $\beta(a_0) = a_p$, where $1 < p \leq k-1$, it follows $\partial_1(\alpha)\beta = (a_p)_{i_0}(a_{p+1})_{r_{p+1}}\dots(a_{k-1})_{r_{k-1}}$, where $i_0 + r_{p+1} + \cdots + r_{k-1} = n$. So, $\partial_1(\alpha)\beta > \partial_1(\alpha\beta)$.

Hence (1) does not hold again and this completes the proof. □

4. Projection on a middle basic string of a simplex.

Here, for fixed m, $1 < m < k-1$, we consider the projection

$$\partial_m : \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\} \to \mathrm{STR}^{(n)}\{a_0, a_m\}$$

such that for any $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$, where $i_0 + \cdots + i_{k-1} = n$, $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$.

Let $R_{1m} = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_m) = a_0\}$.

If $\alpha, \beta \in R_{1m}$, then $(\alpha+\beta)(a_m) = \alpha(a_m) + \beta(a_m) = a_0$ and $(\alpha\beta)(a_m) = \beta(\alpha(a_m)) = \beta(a_0) = a_0$. So, R_{1m} is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$.

Let $R_{2m} = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_1) \geq a_1, \alpha(a_m) \leq a_m\}$.

If $\alpha, \beta \in R_{2m}$, then $(\alpha+\beta)(a_1) = \alpha(a_1) + \beta(a_1) \geq a_1$, $(\alpha+\beta)(a_m) = \alpha(a_m) + \beta(a_m) \leq a_m$, $(\alpha\beta)(a_1) = \beta(\alpha(a_1)) \geq \beta(a_1) \geq a_1$ and $(\alpha\beta)(a_m) = \beta(\alpha(a_m)) \leq \beta(a_m) \leq a_m$. So, R_{2m} is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$.

Now for $\alpha \in R_{1m}$ and $\beta \in R_{2m}$ we obtain:

- $(\alpha+\beta)(a_1) = \alpha(a_1) + \beta(a_1) = a_0 + \beta(a_1) \geq a_1$, $(\alpha+\beta)(a_m) = \alpha(a_m) + \beta(a_m) = a_0 + \beta(a_m) \leq a_m$. Hence $\alpha + \beta \in R_{2m}$.
- Let $\beta(a_0) = a_0$. Then $(\alpha\beta)(a_m) = \beta(\alpha(a_m)) = \beta(a_0) = a_0$. Hence $\alpha\beta \in R_{1m}$.
- Let $\beta(a_0) \geq a_1$. Then $(\alpha\beta)(a_1) = \beta(\alpha(a_1)) = \beta(a_0) \geq a_1$ and $(\alpha\beta)(a_m) = \beta(\alpha(a_m)) = \beta(a_0) \leq \beta(a_m) \leq a_m$. Hence $\alpha\beta \in R_{2m}$.
- $(\beta\alpha)(a_m) = \alpha(\beta(a_m)) \leq \alpha(a_m) = a_0$. Hence $\beta\alpha \in R_{1m}$.

Let $\mathcal{D}_{\partial_m} = R_{1m} \cup R_{2m}$. Thus we have proved

Lemma 3. *The set $\mathcal{D}_{\partial_m}$ is a subsemiring of the semiring*

$$\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}.$$

Lemma 4. *For any $\alpha, \beta \in \mathcal{D}_{\partial_m}$, it follows*

$$\partial_m(\alpha\beta) = \partial_m(\alpha)\beta + \alpha\partial_m(\beta). \tag{2}$$

Proof. Let $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$ and $\beta = (a_0)_{j_0}(a_1)_{j_1}\dots(a_{k-1})_{j_{k-1}}$, where $i_0 + i_1 + \dots + i_{k-1} = j_0 + j_1 + \dots + j_{k-1} = n$.

Case 1. Let $\beta \in R_{1m}$. Then $\beta(a_m) = a_0$. Let p, $m \leq p \leq k-1$, be the largest positive integer such that $\beta(a_p) = a_0$. Since $a_p \leq j_0 - 1 < a_{p+1}$, it follows $\alpha\beta = (a_0)_{i_0+\dots+i_p}(a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, where $i_0 + \dots + i_p + s_1 + \dots + s_{k-1} = n$. Hence $\partial_m(\alpha\beta) = (a_0)_{i_0+\dots+i_p}(a_m)_{n-(i_0+\dots+i_p)}$. Clearly $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$ and then $\partial_m(\alpha)\beta = \overline{a_0}$. On the other hand $\partial_m(\beta) = (a_0)_{j_0}(a_m)_{n-j_0}$. Since $a_p \leq j_0 - 1 < a_{p+1}$, it follows $\alpha\partial_m(\beta) = (a_0)_{i_0+\dots+i_p}(a_m)_{n-(i_0+\dots+i_p)}$ and (2) holds.

Case 2. Let $\beta \in R_{2m}$. Then $\beta(a_1) \geq a_1$ and $\beta(a_m) \leq a_m$.

Case 2-1. Let $\beta(a_0) = a_0, \beta(a_1) \geq a_1, \beta(a_m) = a_p \leq a_m$. Then $\alpha\beta = (a_0)_{i_0}(a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, $i_0 + s_1 + \dots + s_{k-1} = n$ and $\partial_m(\alpha\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$. Clearly $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$ and so $\partial_m(\alpha)\beta = (a_0)_{i_0}(a_p)_{n-i_0}$. Since $\partial_m(\beta) = (a_0)_{j_0}(a_m)_{n-j_0}$ and $a_0 \leq j_0 - 1 < a_1$, it follows $\alpha\partial_m(\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$ and (2) holds.

Case 2-2. Let $\beta(a_0) \geq a_1, \beta(a_m) = a_q \leq a_m$. Then $\alpha\beta = (a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, where $s_1 + \dots + s_{k-1} = n$, and $\partial_m(\alpha\beta) = \overline{a_m}$. Clearly $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$ and so $\partial_m(\alpha)\beta = (a_p)_{i_0}(a_q)_{n-i_0}$, where $\beta(a_0) = a_p$ and $p \leq q$. Since $\partial_m(\beta) = \overline{a_m}$, it follows $\alpha\partial_m(\beta) = \overline{a_m}$ and (2) holds. $\square$

Theorem 2. *The map $\partial_m : \mathcal{D}_{\partial_m} \to \mathrm{STR}^{(n)}\{a_0, a_1\}$, where $1 < m < k-1$, is a derivation. The semiring $\mathcal{D}_{\partial_m}$ is the maximal subsemiring of the simplex $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$, closed under the derivation ∂_m.*

Proof. From Proposition 1 and Lemmas 3 and 4 we prove that $\partial_m : \mathcal{D}_{\partial_m} \to \mathrm{STR}^{(n)}\{a_0, a_m\}$ is a derivation. In the second part we consider four cases.

Case 1. Let $p, 1 \leq p < m$ be the largest positive integer such that $\beta(a_p) = a_0$ and $q, p < q \leq m$ be the least positive integer such that $\beta(a_m) = a_q$. Then $\alpha\beta = (a_0)_{i_0+\dots+i_p}(a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, where $i_0+\dots+i_p+s_1+\dots+s_{k-1} = n$ and $\partial_m(\alpha\beta) = (a_0)_{i_0+\dots+i_p}(a_m)_{n-(i_0+\dots+i_p)}$. Since $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$ then $\partial_m(\alpha)\beta = (a_0)_{i_0}(a_q)_{n-i_0}$. On the other hand $\partial_m(\beta) = (a_0)_{j_0}(a_m)_{n-j_0}$. Since $a_p \leq j_0 - 1 < a_{p+1}$, it follows $\alpha\partial_m(\beta) = (a_0)_{i_0+\dots+i_p}(a_m)_{n-(i_0+\dots+i_p)}$.

Case 1-1. Let $q \leq m$. Then

$$\partial_m(\alpha)\beta + \alpha\partial_m(\beta) = (a_0)_{i_0}(a_q)_{n-i_0} + (a_0)_{i_0+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} =$$

$$(a_0)_{i_0}(a_q)_{i_1+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} > (a_0)_{i_0+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} = \partial_m(\alpha\beta).$$

Case 1-2. Let $q > m$. Then

$$\partial_m(\alpha)\beta + \alpha\partial_m(\beta) = (a_0)_{i_0}(a_q)_{n-i_0} + (a_0)_{i_0+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} =$$

$$(a_0)_{i_0}(a_q)_{n-i_0} > (a_0)_{i_0+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} = \partial_m(\alpha\beta).$$

Hence, (2) does not hold.

Case 2. Let $\beta(a_0) = a_0$, $\beta(a_1) \geq a_1$ and $\beta(a_m) = a_p$, where $m < p \leq k-1$. Then $\alpha\beta = (a_0)_{i_0}(a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, $i_0 + s_1 + \cdots + s_{k-1} = n$ and $\partial_m(\alpha\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$. Since $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$, it follows $\partial_m(\alpha)\beta = (a_0)_{i_0}(a_p)_{n-i_0} > (a_0)_{j_0}(a_m)_{n-j_0} = \partial_m(\alpha\beta)$ and (2) does not hold.

Case 3. Let $\beta(a_0) = a_p$, where $1 \leq p \leq m$ and $\beta(a_m) = a_q$, where $m < q \leq k-1$. Then $\alpha\beta = (a_p)_{s_p}\dots(a_{k-1})_{s_{k-1}}$, where $s_p + \cdots + s_{k-1} = n$, and $\partial_m(\alpha\beta) = \overline{a_m}$. Clearly $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$ and so $\partial_m(\alpha)\beta = (a_p)_{i_0}(a_q)_{n-i_0}$. Since $\partial_m(\beta) = \overline{a_m}$, it follows $\alpha\partial_m(\beta) = \overline{a_m}$. Thus we have

$$\partial_m(\alpha)\beta + \alpha\partial_m(\beta) = (a_p)_{i_0}(a_q)_{n-i_0} + \overline{a_m} = (a_m)_{i_0}(a_q)_{n-i_0} > \overline{a_m} = \partial_m(\alpha\beta).$$

Hence, (2) does not hold.

Case 4. Let $\beta(a_0) = a_p$, where $m < p \leq k-1$ and $\beta(a_m) = a_q$, where $p \leq q \leq k-1$. Then $\alpha\beta = (a_p)_{s_p}\dots(a_{k-1})_{s_{k-1}}$, where $s_p + \cdots + s_{k-1} = n$, and $\partial_m(\alpha\beta) = \overline{a_m}$. Since $\partial_m(\alpha) = (a_0)_{i_0}(a_m)_{n-i_0}$, it follows $\partial_m(\alpha)\beta = (a_p)_{i_0}(a_q)_{n-i_0} > \overline{a_m} = \partial_m(\alpha\beta)$. Hence (2) does not hold again and this completes the proof of the theorem. □

5. Projection on the biggest basic string of a simplex.

Let

$$S_1 = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_{k-1}) = a_0\}.$$

For $\alpha, \beta \in S_1$ we have $(\alpha + \beta)(a_{k-1}) = \alpha(a_{k-1}) + \beta(a_{k-1}) = a_0$ and $(\alpha\beta)(a_{k-1}) = \beta(\alpha(a_{k-1})) = \beta(a_0) = a_0$. So, S_1 is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$.

Let us consider the set $S_2 = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_1) \geq a_1\}$.

If $\alpha, \beta \in S_2$, it follows $(\alpha+\beta)(a_1) = \alpha(a_1) + \beta(a_1) \geq a_1$ and $(\alpha\beta)(a_1) = \beta(\alpha(a_1)) \geq \beta(a_1) \geq a_1$. So, S_2 is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$.

Now, for $\alpha \in S_1$ and $\beta \in S_2$ it follows

• $(\alpha+\beta)(a_1) = \alpha(a_1) + \beta(a_1) = a_0 + \beta(a_1) \geq a_1$, hence $\alpha + \beta \in S_2$.

• Let $\beta(a_0) = a_0$. Now, $(\alpha\beta)(a_{k-1}) = \beta(\alpha(a_{k-1})) = \beta(a_0) = a_0$, so $\alpha\beta \in S_1$.

• Let $\beta(a_0) \geq a_1$. Now, $(\alpha\beta)(a_1) = \beta(\alpha(a_1)) = \beta(a_0) \geq a_1$, so $\alpha\beta \in S_2$.

• Let $\alpha(\beta(a_1)) = \alpha(a_1)$. Now, $(\beta\alpha)(a_1) = \alpha(\beta(a_1)) = \alpha(a_1) = a_0$, so, $\alpha\beta \in S_1$.

• Let $\alpha(\beta(a_1)) > \alpha(a_1)$. Now, $(\beta\alpha)(a_1) = \alpha(\beta(a_1)) > \alpha(a_1) = a_0$, so, $(\beta\alpha)(a_1) \geq a_1$ and $\alpha\beta \in S_2$.

Let $\mathcal{D}_{\partial_{k-1}} = S_1 \cup S_2$. Thus we have proved

Lemma 5. *The set $\mathcal{D}_{\partial_{k-1}}$ is a subsemiring of the semiring*

$$\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}.$$

Lemma 6. *For any $\alpha, \beta \in \mathcal{D}_{\partial_{k-1}}$, it follows*

$$\partial_{k-1}(\alpha\beta) = \partial_{k-1}(\alpha)\beta + \alpha\partial_{k-1}(\beta). \tag{3}$$

P r o o f. Let $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$ and $\beta = (a_0)_{j_0}(a_1)_{j_1}\dots(a_{k-1})_{j_{k-1}}$, where $i_0 + i_1 + \dots + i_{k-1} = j_0 + j_1 + \dots + j_{k-1} = n$.

Case 1. Let $\beta \in S_1$. Then $\beta(a_0) = \dots = \beta(a_{k-1}) = a_0$. Then $\alpha\beta = \overline{a_0}$ and $\partial_{k-1}(\alpha\beta) = \overline{a_0}$. Clearly $\partial_{k-1}(\alpha) = (a_0)_{i_0}(a_{k-1})_{n-i_0}$ and so $\partial_{k-1}(\alpha)\beta = \overline{a_0}$. Since $\partial_{k-1}(\beta) = \overline{a_0}$, it follows $\alpha\partial_{k-1}(\beta) = \overline{a_0}$ and (3) holds.

Case 2. Let $\beta \in S_2$. Then $\beta(a_1) \geq a_1$.

Case 2-1. Let $\beta(a_0) = a_0$, $\beta(a_1) \geq a_1$ and $\beta(a_{k-1}) = a_m$, where $1 \leq m \leq k-1$. Then $\alpha\beta = (a_0)_{i_0}(a_1)_{s_1}\dots(a_{k-1})_{s_{k-1}}$, where $i_0 + s_1 + \dots + s_{k-1} = n$ and $\partial_{k-1}(\alpha\beta) = (a_0)_{i_0}(a_{k-1})_{n-i_0}$. Clearly $\partial_{k-1}(\alpha) = (a_0)_{i_0}(a_{k-1})_{n-i_0}$ and then $\partial_{k-1}(\alpha)\beta = (a_0)_{i_0}(a_m)_{n-i_0}$. On the other hand $\partial_{k-1}(\beta) = (a_0)_{j_0}(a_{k-1})_{n-j_0}$ and since $j_0 - 1 < a_1$, it follows $\alpha\partial_{k-1}(\beta) = (a_0)_{i_0}(a_{k-1})_{n-i_0}$ and (3) holds.

Case 2-2. Let $\beta(a_0) \geq a_1$. Then $\alpha\beta \geq \overline{a_1}$ and $\partial_{k-1}(\alpha\beta) = \overline{a_{k-1}}$. Since $\partial_{k-1}(\beta) = \overline{a_{k-1}}$, it follows $\alpha\partial_{k-1}(\beta) = \overline{a_{k-1}}$. Hence

$$\partial_{k-1}(\alpha)\beta + \alpha\partial_{k-1}(\beta) = \partial_{k-1}(\alpha)\beta + \overline{a_{k-1}} = \overline{a_{k-1}} = \partial_3(\alpha\beta)$$

and (3) holds. □

Theorem 3. *The map* $\partial_{k-1} : \mathcal{D}_{\partial_{k-1}} \to \mathrm{STR}^{(n)}\{a_0, a_{k-1}\}$ *is a derivation. The maximal subsemiring of* $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$*, closed under the derivation* ∂_{k-1} *is* $\mathcal{D}_{\partial_{k-1}}$.

Proof. Using Proposition 1 and Lemmas 5 and 6 we prove that $\partial_{k-1} : \mathcal{D}_{\partial_{k-1}} \to \mathrm{STR}^{(n)}\{a_0, a_{k-1}\}$ is a derivation. In the second part we consider two cases.

Case 1. Let $\beta(a_{k-1}) = a_{k-1}$ and $m, 1 \leq m \leq k-1$, be the largest positive integer such that $\beta(a_m) = a_0$. Then $\alpha\beta = (a_0)_{i_0+\dots+i_m}(a_1)_{s_1} \dots (a_{k-1})_{s_{k-1}}$, where $i_0+\dots+i_m+s_1+\dots+s_{k-1} = n$. Now $\partial_{k-1}(\alpha\beta) = (a_0)_{i_0+\dots+i_m}(a_{k-1})_{n-(i_0+\dots+i_m)}$. Since $\partial_{k-1}(\alpha) = (a_0)_{i_0}(a_{k-1})_{n-i_0}$, it follows $\partial_{k-1}(\alpha)\beta = (a_0)_{i_0}(a_{k-1})_{n-i_0} > \partial_{k-1}(\alpha\beta)$, hence, (3) does not hold.

Case 2. Let $\beta(a_p) = a_0$ and $\beta(a_{k-1}) < a_q$, where $1 \leq p < k-1$ and $1 \leq q < k-1$. Then $\alpha\beta = (a_0)_{i_0+\dots+i_p}(a_1)_{s_1} \dots (a_q)_{s_q}$, where $i_0 + \dots + i_p + s_1 + \dots + s_q = n$ and $\partial_{k-1}(\alpha)\beta = (a_0)_{i_0+\dots+i_p}(a_{k-1})_{n-(i_0+\dots+i_p)}$. Clearly $\partial_{k-1}(\alpha) = (a_0)_{i_0}(a_{k-1})_{n-i_0}$. Then $\partial_{k-1}(\alpha)\beta = (a_0)_{i_0}(a_q)_{n-i_0}$. Since $\partial_{k-1}(\beta) = (a_0)_{j_0}(a_{k-1})_{n-j_0}$ and $a_p \leq j_0 - 1 < a_{p+1}$, then

$$\alpha\partial_{k-1}(\beta) = (a_0)_{i_0+\dots+i_p}(a_{k-1})_{n-(i_0+\dots+i_p)}.$$

Thus we have

$$\partial_{k-1}(\alpha)\beta + \alpha\partial_{k-1}(\beta) = (a_0)_{i_0}(a_q)_{n-i_0} + (a_0)_{i_0+\dots+i_p}(a_{k-1})_{n-(i_0+\dots+i_p)} =$$

$$(a_0)_{i_0}(a_q)_{i_1+\dots+i_p}(a_{k-1})_{n-(i_0+\dots+i_p)} > (a_0)_{i_0+\dots+i_p}(a_{k-1})_{n-(i_0+\dots+i_p)} = \partial_{k-1}(\alpha\beta),$$

hence (3) does not hold and this completes the proof. □

Remark. It is possible to extend Section 2 for $1 \leq m \leq k-1$. Then $R_{11} \cup R_{21} = \mathcal{D}_{\partial_1}$ and also $R_{1\,k-1} = S_1$, $R_{2\,k-1} = S_2$. But, now the new Lemma 4 and new Theorem 2 will be consisting much more cases and we "can't see the wood for the trees".

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Received October 7, 2015
Revised December 8, 2016