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# EXTENDING THE APPLICABILITY OF NEWTON-SECANT METHODS FOR FUNCTIONS WITH VALUES IN A CONE

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Abstract. In this study, we consider Newton-secant method for solving the nonlinear variational inclusion problems in Banach space. Using generalized continuity conditions, we prove the convergence of the method with the following advantages: tighter error estimates on the distances involved and the information on the location of the solution is at least as precise. These advantages were obtained under the same computational cost.

**1. Introduction.** We study the variational inclusion

$$0 \in F(x) + G(x) + E(x), \tag{1.1}$$

where $X$, $Y$ are Banach space $D \subset X$ is an open set $F : D \longrightarrow Y$ is a smooth operator, $G : D \longrightarrow Y$ is continuous operator, $[\cdot, \cdot; G]$ is a divided difference of order one for $G$ [4, 19] (i.e., for $x, y \in D$ with $x \neq y$, $[x, y; G](x-y) = G(x)-G(y)$,

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if $G$ is Fréchet differentiable,then $[x,x;G]=G'(x))$ and $E:X\rightrightarrows Y$ is a set-valued operator. Many problems can be written in the form (1.1) using Mathematical Modeling [1]–[29]. The solution $x^*$ of (1.1) can rarely be found in closed form. That is why most solution methods for (1.1) are usually iterative. In particular, the inclusion

$$0\in F(x_m)+G(x_m)+(F'(x_m)+[x_{m-1},x_m;G])(x_{m+1}-x_m)+E(x_{m+1}), \tag{1.2}$$

where $x_0\in D$ is an initial point was studied in [19]. If $E=\{0\}$, the method was studied by Cătinas [4]. Special cases of the inclusion (1.2) have been studied in the notable papers by Dontchev et.al. [7, 8, 9, 10, 11]. Then, assuming $E(x)=-C$ for each $x\in X$ and $C\subseteq Y$ being a closed convex cone, Pietrus and Alexis [19] studied the algorithm

$$\begin{aligned}\text{minimize}\{\|x-x_m\|/F(x_m)+G(x_m)+(F'(x_m)\\ +[x_{m-1},x_m;G])(x-x_m)\in C\}.\end{aligned} \tag{1.3}$$

Next, we shall define the Algorithm for solving (1.1). Let $u_1,u_2\in D$. Define a set-valued operator $Q(u_1,u_2):X\longrightarrow Y$ by

$$Q(u_1,u_2)x:=(F'(u_2)+[u_1,u_2;G])x-C. \tag{1.4}$$

Then, $Q(u_1,u_2)$ is a normed convex process. The inverse, defined for each $y\in Y$ by

$$Q^{-1}(u_1,u_2)y:=\{z\in X:(F'(u_2)+[u_1,u_2;G])z\in y+C\}$$

is also a normed convex process. Recall [19], that a mapping $F$ between real linear space $X$ and $Y$ is a convex process if it satisfies

- $F(x)+F(z)\subset F(x+z),$ for all $x,z\in X$
- $F(\lambda x)=\lambda F(x),$ for all $\lambda>0$ and every $x\in X$
- $0\in F(0).$

We have the following [19].

**Algorithm.** Newton-secant-cone $(F,G,C,x_0,x_1,\varepsilon)$

1. If $Q^{-1}(x_0,x_1)[-F(x_1)-G(x_1)]=\emptyset$, stop.
2. Do while $e>\varepsilon$.

(a) Choose $x$ as a solution of the problem

$$\text{mimimize}\{\|x-x_1\|/F(x_1)+G(x_1)+(F'(x_1)+[x_0,x_1;G])(x-x_1)\in C\}.$$

(b) Compute $e=\|x-x_1\|; x_0:=x_1; x_1:=x$.

3. Return $x$.

**Remark 1.1.** It is worth noticing that the continuity of the linear operator $F'(x_m)$ and $G$ being closed and convex, imply that the feasible set of (1.3) is a closed convex set for all $m$. Hence, the existence of a feasible point $\bar{x}$ implies that each solution of (1.3) lies in the intersection of the feasible set of (1.3) with the closed ball of center $x_m$ and radius $\|\bar{x}-x_m\|$. Then, by [6, 7, 8, 9, 10, 20] a solution of (1.3) exists, since $X$ is reflexive and the function $\|x-x_m\|$ is weakly lower semi-continuous.

The convergence of Newton-secant method (1.3) was shown in [19] using Lipschitz continuity conditions on $F'$ and divided differences of order one and two for $G$. However, there are problems where the Lipschitz continuity of $F'$ does not hold or the divided differences of order two of $G$ do not exist (see also the numerical examples). Motivated by these constrains, we present a convergence analysis of the Newton-secant method (1.3) using generalized continuity on $F'$ and hypotheses only on the divided difference of order one for $G$. Our results are weaker even, if we specialize the conditions on $F'$ to the condition given in [19]. This way we expand the applicability of Newton-secant method (1.3).

The rest of the paper is organized as follows: Section 2 contains the convergence of Newton-secant method (1.3). Numerical examples are given in Section 3.

## 2. Convergence of the Newton-secant method.

We need an auxiliary result on majorizing sequences for Newton-secant method (1.3).

**Lemma 2.1.** *Let $a\geq 0$, $b\geq 0$ and $c>0$ be given parameters. Let also $w:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, $w_1:\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$, $w_2:\mathbb{R}_+^4\longrightarrow\mathbb{R}_+$ be continuous, nondecreasing functions. Suppose that*

$$a\leq b\leq 2a \tag{2.1}$$

*and equation*

$$(t-a)(1-q(t))+a-b=0 \tag{2.2}$$

*has zeros greater than a, where*

$$q(t)=\frac{c}{1-cp(t)}(\int_0^1 w(a\theta)d\theta+w_1(a,a,a)) \tag{2.3}$$

*and*

$$p(t)=w(t-a)+w_2(t,t-a,t,t-a). \tag{2.4}$$

*Denote by $t^*$ the smallest such zero. Then, scalar sequence $\{t_n\}$ defined by*

$$\begin{aligned} t_0 &= 0,\ t_1=a, t_2\geq b,\\ t_{n+1} &= t_n+\frac{c}{1-cp_n}\left[\int_0^1 w(\theta(t_n-t_{n-1}))d\theta\right.\\ &\left.+w_1(t_n-t_{n-1},t_n-t_{n-2},t_{n-1}-t_{n-2})\right](t_n-t_{n-1}) \end{aligned} \tag{2.5}$$

*is well defined, nondecreasing, bounded from above by $t^{**}$ given by*

$$t^{**}=\frac{t_2-t_1}{1-q}+a \tag{2.6}$$

*and converges to its unique least upper bound $t^*$ which satisfies*

$$b\leq t^*\leq t^{**}, \tag{2.7}$$

*where*

$$p_n=w(t_n-a)+w_2(t_{n-1},t_{n-1}-a,t_n,t_{n-1}-t_1),$$

$$q=\frac{c}{1-cp}$$

*and*

$$p=p(t^*).$$

*Moreover, the following estimates hold*

$$0\leq t_2-t_1\leq t_1-t_0 \tag{2.8}$$

$$0\leq t_{n+1}-t_n\leq q(t_n-t_{n-1}) \tag{2.9}$$

*and*

$$0\leq t^*-t_n\leq \frac{q^{n-1}}{1-q}(t_2-t_1)\ \ n=2,3,\ldots. \tag{2.10}$$

Proof. By (2.1), we have that $t_0 \leq t_1 \leq t_2$ and $t_2 - t_1 \leq t_1 - t_0$. It then follows by (2.5) and a simple inductive argument that

$$t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t_{k+1}$$

and

$$\begin{aligned} t_{k+1} - t_0 &= (t_{k+1} - t_k) + (t_k - t_{k-1}) + \cdots + (t_2 - t_1) + (t_1 - t_0) \\ &\leq (q^{k-1} + \cdots + 1)(t_2 - t_1) + t_1 - t_0 \\ &\leq \frac{t_2 - t_1}{1 - q} + t_1 - t_0 = t^{**}. \end{aligned}$$

Hence, sequence $\{t_k\}$ is nondecreasing, bounded from above by $t^{**}$ and as such it converges to $t^*$ which satisfies (2.7). Let $m \geq 1$. We can write

$$\begin{aligned} t_{n+m} - t_n &= (t_{n+m} - t_{n+m-1}) + \cdots + (t_{n+1} - t_n) \\ &\leq (q^{n+m-2} + \cdots q^{n-1})(t_2 - t_1) \\ &= \frac{1 - q^m}{1 - q} q^{n-1}(t_2 - t_1). \end{aligned}$$

By letting $m \longrightarrow \infty$ in the preceding estimate we show (2.10). □

Next, we present the semilocal convergence analysis of Newton-secant method (1.3) using the preceding notation.

**Theorem 2.2.** *Let $D, X, Y, F, G, Q$ be as defined previously. Suppose:*

*(i) There exist points $x_0, x_1 \in D$ such that $Q(x_0, x_1)$ carries $X$ onto $Y$.*

*(ii) There exist $c > 0$ and $a \geq 0$ such that*

$$\|Q^{-1}(x_0, x_1)\| \leq c$$

*and*

$$\|x_1 - x_0\| \leq a.$$

*(iii) There exist $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, w_1 : \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+, w_2 : \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ continuous, nondecreasing functions such that for each $x, y, z \in D$*

$$\|F'(x) - F'(y)\| \leq w(\|x - y\|),$$

$$\|[x, y; G] - [z, y; G]\| \leq w_1(\|x - z\|, \|x - y\|, \|z - y\|)$$

*and*

$$\|[x, y; G] - [x_0, x_1; G]\| \leq w_2(\|x - x_0\|, \|y - x_1\|, \|x - x_1\|, \|y - x_0\|).$$

*(iv)* $\|x_2 - x_1\| \leq b - a$, *where* $x_2$ *defined by the algorithm and Remark* 1.1.

*(v)* *Hypotheses of Lemma* 2.1 *hold.*

*(vi)* $\bar{U}(x_0, t^*) \subset D$.

*(vii)* $w(0) = w_1(0,0,0) = w_2(0,0,0,0) = 0$.

*Then, there exists at least a sequence* $\{x_n\}$ *generated by method* (1.3) *which is well defined in* $U(x_0, t^*)$, *remains in* $U(x_0, t^*)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to some* $x^* \in \bar{U}(x_0, t^*)$ *such that* $F(x^*) + G(x^*) \in C$. *Moreover, the following estimates hold*

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.11}$$

*where sequence* $\{t_n\}$ *is defined by* (2.5) *and* $t^* = \lim_{n \longrightarrow \infty} t_n$.

Proof. Let us define operator $A_m$ for each $m = 1, 2, \ldots$ by

$$\begin{aligned} A_m &= F(x_{m+1}) + G(x_{m+1}) - F(x^*) - G(x^*) \\ &\quad -[F(x_{m+1}) - F(x_m) - (F'(x_m) + [x_{m-1}, x_m; G])(x_{m+1} - x_m). \\ &\quad +G(x_{m+1} - G(x_m)]. \end{aligned} \tag{2.12}$$

Then, we get that $A_m \in C - F(x^*) - G(x^*)$ and by continuity $\lim_{n \longrightarrow \infty} A_m = 0$. It follows that $F(x^*) + G(x^*) \in C$, since $C - F(x^*) - G(x^*)$ is closed. Hence, the point $x^*$ solves (1.1). Using mathematical induction, we shall show that sequence $\{x_m\}$ is well defined and satisfies

$$\|x_{m+1} - x_m\| \leq t_{m+1} - t_m \text{ for each } m = 0, 1, 2, \ldots, \tag{2.13}$$

where sequence $\{t_m\}$ is defined by (2.5). By the second hypothesis in (ii), (iv) and (2.5), we obtain the existence of $x_2$ which solves (1.1) for $m = 1$, $\|x_1 - x_0\| \leq a = t_1 - t_0$, $\|x_2 - x_1\| \leq b - a = t_2 - t_1$, which shows (2.13) for $m = 0, 1$. Suppose that (2.13) holds for $i = 3, \ldots m$, where $x_1, x_2, \ldots x_m$ are defined by (1.3). Then, we get that $\|x_i - x_0\| \leq t_i - t_0 < t^*$. That is $x_i \in U(x_0, t^*)$. We can write for

$$B_i(x) = -(F'(x_i) - F'(x_1) + [x_{i-1}, x_i; G] - [x_0, x_1; G])x \tag{2.14}$$

that

$$Q(x_{i-1}, x_i)x = (Q(x_0, x_1) - B_i)x. \tag{2.15}$$

By (ii), (iii), (2.15), Lemma 2.1 and the induction hypotheses we obtain in turn that

$$\begin{aligned}
\|Q^{-1}(x_0,x_1)\|\|B_i\| &\leq c[w(\|x_i-x_1\|) \\
&\quad +w_2(\|x_{i-1}-x_0\|,\|x_{i-1}-x_1\|,\|x_i-x_0\|,\|x_{i-1}-x_1\|)] \\
&\leq c[w(t_i-t_1)+w_2(t_{i-1}-t_0,t_{i-1}-t_1,t_i-t_0,t_{i-1}-t_1)] \\
&\leq c[w(t^*-a)+w_2(t^*,t^*-a,t^*,t^*-a)]<1. \qquad (2.16)
\end{aligned}$$

It follows from (2.16) and the Banach perturbation Lemma [13] that

$$Q^{-1}(x_{i-1},x_i)\in L(Y,X) \qquad (2.17)$$

and

$$\begin{aligned}
\|Q^{-1}(x_{i-1},x_i)\| &\leq \frac{\|Q^{-1}(x_0,x_1)\|}{1-\|Q^{-1}(x_0,x_1)\|\|B_i\|} \\
&\leq \frac{c}{1-cp_i}. \qquad (2.18)
\end{aligned}$$

The existence of $x_i$ solving (1.3) with $i=k$ follows from the fact that $Q(x_{i-1},x_i):X\longrightarrow Y$. Next, we must solve the problem

$$\begin{aligned}
(2.19)\quad & F(x_m)+G(x_m)+[x_{m-1},x_m;G](x-x_m)\in \\
& F(x_{m-1})+(F'(x_{m-1})+[x_{m-2},x_{m-1};G](x_m-x_{m-1})+G(x_{m-1})+C.
\end{aligned}$$

The right hand side of (2.19) is contained in the cone $C$, since $x_m$ solves (ii). That is any $x$ satisfying (2.19) is feasible for (1.3). Using (2.19), we can get $x$ as the solution of

$$\begin{aligned}
(2.20)\quad x-x_m\in Q^{-1}(x_{m-1},x_m)(&-F(x_m)-G(x_m)+F(x_{m-1}) \\
&+G(x_{m-1})+(F'(x_{m-1})+[x_{m-2},x_{m-1};G](x_m-x_{m-1})).
\end{aligned}$$

The right hand side of (2.20) contains an element of least norm, so there exists $\bar{x}$ satisfying (2.18) and (2.19) such that

$$\begin{aligned}
\|\bar{x}-x_m\| &\leq \|Q^{-1}(x_{m-1},x_m)\|(\|-F(x_m)+F(x_{m-1})+F'(x_{m-1})(x_m-x_{m-1})\| \\
&\quad +\|G(x_m)-G(x_{m-1})-[x_{m-2},x_{m-1};G](x_m-x_{m-1})\|). \qquad (2.21)
\end{aligned}$$

In view of (2.5), (iii), (2.18), (2.21) and the induction hypotheses we get in turn that

$$\|\bar{x}-x_m\| \leq \frac{c}{1-cp_m}\left[\int_0^1 w(\theta\|x_m-x_{m-1}\|)d\theta\|x_m-x_{m-1}\|\right.$$

$$
\begin{aligned}
& +\|[x_m, x_{m-1}; G] - [x_{m-2}, x_{m-1}; G]\| \|x_m - x_{m-1}\|\Big] \\
\leq \; & \frac{c}{1 - cp_m}\left[\int_0^1 w(\theta\|x_m - x_{m-1}\|)d\theta\right. \\
& \left. + w_1(\|x_m - x_{m-1}\|, \|x_m - x_{m-2}\|, \|x_{m-1} - x_{m-2}\|)\right] \|x_m - x_{m-1}\| \\
\leq \; & \frac{c}{1 - cp_m}\left[\int_0^1 w(\theta(t_m - t_{m-1}))d\theta\right. \\
& \left. + w_1(t_m - t_{m-1}, t_m - t_{m-2}, t_{m-1} - t_{m-2})\right] |t_m - t_{m-1}| \\
= \; & t_{m+1} - t_m. \qquad (2.22)
\end{aligned}
$$

By Lemma 2.1, sequence $\{t_m\}$ is complete. In view of (2.22), sequence $\{x_m\}$ is complete in a Banach space $X$ and

$$\|x_{m+1} - x_m\| \leq \|\bar{x} - x_m\| \leq t_{m+1} - t_m.$$ □

**Remark 2.3.**

(a) Our results can specialize to the corresponding ones in [19]. Choose $w(t) = Lt$, $w_1(t) = L_1$ and $w_2(t, t, t, t) = L_2$. But even in this case, our results are weaker, since we do not use the hypothesis on the divided difference of order two

$$\|[x, y, z; G]\| \leq K$$

but use instead the second hypothesis in (iii) which involves only the divided difference of order one. Moreover, the results in [19] cannot be used to solve the numerical examples at the end of the paper, since $F'$ is not Lipschitz continuous. However, our results can apply to solve inclusion problems. Hence, we have expanded the applicability of Newton-secant method (1.3).

(b) Our results can be improved even further as follows: Let $r_0$ be defined as the smallest positive zero of equation

$$w_2(t, t, t, t) = 1.$$

Define $D_0 = D \cap U(x_0, r_0)$. Suppose that the first and second hypotheses in (iii) are replaced by

$$\|F'(x) - F'(y)\| \leq \bar{w}(\|x - y\|)$$

and

$$\|[x,y;G]-[z,y;G]\| \le \bar{w}_1(\|x-z\|,\|x-y\|,\|z-y\|)$$

for each $x,y,z \in D_0$. Denote the new conditions by (iii)'. Then, clearly condition (iii)' can replace (iii) in Theorem 2.2, since the iterates $\{x_m\}$ lie in $D_0$ which is a more accurate location than $D$. Moreover, since $D_0 \subseteq D$, we have that

$$\bar{w}(t) \le w(t)$$

and

$$\bar{w}_1(t,t,t) \le w_1(t,t,t)$$

for each $t \in [0,r_0)$ hold. Notice that the definition of functions $\bar{w}$ and $\bar{w}_1$ depends on $w$ and $r_0$. Another way of extending our results is to consider instead of $D_0$ the set $D_1 = D \cap U(x_1, r_0 - \|x_1 - x_0\|)$. Then, corresponding functions $\bar{\bar{w}}$ and $\bar{\bar{w}}_1$ will be at least as small as $\bar{w}, \bar{w}_1$, respectively, since $D_1 \subseteq D_0$. Notice that the construction of the $\|w\|$ functions is based on the initial data $F, G, C, x_0, x_1$.

**3. Numerical examples.** In this section we suggest examples where the operator $F'$ is not Lipschitz. Consequently the earlier results cannot apply to solve (1.1), where as ours can apply.

**Example 3.1.** Let $X = Y = C[0,1]$ and consider the nonlinear integral equation of the mixed Hammerstein-type defined by

$$x(s) = \int_0^1 Q(s,t)\left(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\right)dt, \tag{3.1}$$

where the kernel $Q$ is the Green's function defined on the interval $[0,1]\times[0,1]$ by

$$Q(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases} \tag{3.2}$$

Define $F, G : C[0,1] \longrightarrow C[0,1]$ by

$$F(x)(s) := x(s) - \int_0^1 G(s,t)\left(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\right)dt, \tag{3.3}$$

and

$$Q(x)(s) = 0.$$

Notice that $x^*(s) = 0$ is one of the solutions of (1.1). Using (3.2), we obtain

$$\| \int_0^1 Q(s,t)dt\| \leq \frac{1}{8}. \tag{3.4}$$

Then, by (3.2)–(3.4), we have that

$$\|F'(x) - F'(y)\| \leq \frac{1}{8}(\frac{3}{2}\|x-y\|^{\frac{1}{2}} + \|x-y\|). \tag{3.5}$$

In view of (3.5), the earlier results requiring $F'$ to be Lipschitz (such as [19, 20]) cannot apply. However, our results can apply, if we choose $w(t) = \frac{1}{8}\left(\frac{3}{2}t^{\frac{1}{2}} + t\right)$, $w_1 = 0$ and $w_2 = 0$.

**Application 3.2.** *Let*

$$A(y_n) = F'(y_n) + [y_{n-1}, y_n; G], \quad (9n \geq 0)$$

*and consider Newton-like method in the form*

$$y_{n+1} = y_n - (F'(y_n) + [y_{n-1}, y_n; G])^{-1}(F(y_n) + G(y_n)) \ \ (n \geq 0). \tag{3.6}$$

*This method has order* $\frac{1+\sqrt{5}}{2}$ *(see* [4]*) (same as the method of Chord), but higher than the order of*

$$z_{n+1} = z_n - F'(z_n)^{-1}(F(z_n) + G(z_n)) \ \ (n \geq 0) \tag{3.7}$$

*considered in* [5] *and the method of Chord*

$$w_{n+1} = w_n - [w_{n-1}, w_n; G]^{-1}(F(w_n) + G(w_n)) \ \ (n \geq 0), \tag{3.8}$$

*where* $[x, y; G]$ *denotes the divided difference of* $G$ *at the points* $x$ *and* $y$ *considered in* [13].

*Now, we shall provide an example for this case.*

**Example 3.3.** Let $X = Y = (\mathbb{R}^2, \|\cdot\|_\infty)$. Consider the system

$$\begin{aligned} 3x^2y + y^2 - 1 + |x-1| &\leq 0 \\ x^4 + xy^3 - 1 + |y| &\leq 0. \end{aligned}$$

Set $\|x\|_\infty = \|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}, F = (F_1, F_2), G = (G_1, G_2)$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we choose $F_1(x_1, x_2) = 3x_1^2x_2 + x_2^2 - 1, F_2(x_1, x_2) = x_1^4 + x_1x_2^3 - 1$, $G_1(x_1, x_2) = |x_1 - 1|, G_2(x_1, x_2) = |x_2|$. We shall take $[x, y; G] \in M_{2\times 2}(\mathbb{R})$ as

$$[x, y; G]_{i,1} = \frac{G_1(y_1, y_2) - G_i(x_1, y_2)}{y_1 - x_1},$$

$$[x, y; G]_{i,2} = \frac{G_1(x_1, y_2) - G_i(x_1, x_2)}{y_2 - x_2}, \; i = 1, 2,$$

provided that $y_1 \neq x_1$ and $y_2 \neq x_2$. Otherwise define $[x, y; G]$ to be the zero matrix in $M_{2\times 2}(\mathbb{R})$. Moreover, using method (3.7) with $z_0 = (1, 0)$ we obtain Comparison Table 1. Furthermore, using the method of Chord (3.8) with $w_{-1} = (1, 0)$ and $w_0 = (5, 5)$, we obtain Comparison Table 2. Finally, using our method (3.6) with

Comparison Table 1

| $n$ | $z_n^{(1)}$ | $z_n^{(2)}$ | $\|z_n - z_{n-1}\|$ |
|---|---|---|---|
| 0 | 1 | 0 | |
| 1 | 1 | 0.333333333333333 | 3.333E-1 |
| 2 | 0.906550218340611 | 0.354002911208151 | 9.344E-2 |
| 3 | 0.885328400663412 | 0.338027276361322 | 2.122E-2 |
| 4 | 0.891329556832800 | 0.326613976593566 | 1.141E-2 |
| 5 | 0.895238815463844 | 0.326406852843625 | 3.909E-3 |
| 6 | 0.895154671372635 | 0.327730334045043 | 1.323E-3 |
| 7 | 0.894673743471137 | 0.327979154372032 | 4.809E-4 |
| 8 | 0.894598908977448 | 0.327865059348755 | 1.140E-4 |
| 9 | 0.894643228355865 | 0.327815039208286 | 5.002E-5 |
| 10 | 0.894659993615645 | 0.327819889264891 | 1.676E-5 |
| 11 | 0.894657640195329 | 0.327826728208560 | 6.838E-6 |
| 12 | 0.894655219565091 | 0.327827351826856 | 2.420E-6 |
| 13 | 0.894655074977661 | 0.327826643198819 | 7.086E-7 |
| ... | | | |
| 39 | 0.894655373334687 | 0.327826521746298 | 5.149E-19 |

Comparison Table 2

| $n$ | $w_n^{(1)}$ | $w_n^{(2)}$ | $\|w_n - w_{n-1}\|$ |
|---|---|---|---|
| -1 | 5 | 5 | |
| 0 | 1 | 0 | 5.000E+00 |
| 1 | 0.989800874210782 | 0.012627489072365 | 1.262E-02 |
| 2 | 0.921814765493287 | 0.307939916152262 | 2.953E-01 |
| 3 | 0.900073765669214 | 0.325927010697792 | 2.174E-02 |
| 4 | 0.894939851625105 | 0.327725437396226 | 5.133E-03 |
| 5 | 0.894658420586013 | 0.327825363500783 | 2.814E-04 |
| 6 | 0.894655375077418 | 0.327826521051833 | 3.045E-04 |
| 7 | 0.894655373334698 | 0.327826521746293 | 1.742E-09 |
| 8 | 0.894655373334687 | 0.327826521746298 | 1.076E-14 |
| 9 | 0.894655373334687 | 0.327826521746298 | 5.421E-20 |

$y_{-1} = (1, 0), y_0 = (5, 5)$, we obtain Comparison Table 3.

Comparison Table 3

| $n$ | $y_n^{(1)}$ | $y_n^{(2)}$ | $\|y_n - y_{n-1}\|$ |
|---|---|---|---|
| -1 | 5 | 5 | |
| 0 | 1 | 0 | 5 |
| 1 | 0.909090909090909 | 0.363636363636364 | 3.636E-01 |
| 2 | 0.894886945874111 | 0.329098638203090 | 3.453E-02 |
| 3 | 0.894655531991499 | 0.327827544745569 | 1.271E-03 |
| 4 | 0.894655373334793 | 0.327826521746906 | 1.022E-06 |
| 5 | 0.894655373334687 | 0.327826521746298 | 6.089E-13 |
| 6 | 0.894655373334687 | 0.327826521746298 | 2.710E-20 |

The solution is

$$x^* = (0.894655373334687, 0.327826521746298)$$

chosen from the lists of the tables displayed above. Hence, method (3.6) converges faster than (3.7) and (3.8).

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