Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **42** (2016), 311–324

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

OVERDETERMINED STRATA FOR DEGREE 6 HYPERBOLIC POLYNOMIALS

Ezzaldine Hayssam

Communicated by V. Drensky

Abstract. A resultant-based method to calculate the overdetermined strata for degree 6 hyperbolic polynomials in one variable is revealed. This method is a new method to calculate overdetermined strata. The overdetermined strata in degree 6 have not been calculated before as the geometric method used until now can not be generalized to degree $n \geq 6$.

1. Introduction. We consider the polynomial $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n,\ x, a_i \in \mathbb{R}$. This polynomial is called (*strictly*) *hyperbolic* if all its roots are real (real and distinct). If P is (strictly) hyperbolic, then such are $P^{(1)}, \ldots, P^{(n-1)}$ as well. Examples of hyperbolic polynomials are the ones of all known orthogonal families (e.g. the Legendre, Laguerre, Hermite, Tchebychev polynomials).

2010 *Mathematics Subject Classification:* 12D10, 14P05, 26C10, 13P15, 13P10.
Key words: hyperbolic polynomials, Gegenbauer polynomials, overdetermined strata, resultants.

Some properties of hyperbolic polynomials and criteria of hyperbolicity have been studied at the beginning of the twentieth century, see [18]. The interest of hyperbolic polynomials appear in the theory of linear partial differential equations, see [17], and in the potential theory, see [1], [2], [8], [9] and [10].

If the coefficients of a polynomial depend on parameters, we say that the set of values taken by these parameters for which the polynomial is hyperbolic, is *the hyperbolicity domain* denoted by Π^*. The change $x \mapsto x - a_1/n$ reduces the study of Π^* to the case $a_1 = 0$.

Lemma 1. *In the case $a_1 = 0$ the polynomial P is hyperbolic only if $a_2 \leq 0$. If $a_1 = a_2 = 0$, then P is hyperbolic only for $a_2 = \cdots = a_n = 0$.*

Proof. All derivatives of P must be hyperbolic, in particular $P^{(n-2)}(x) = (n!/2)x^2 + (n-2)!a_2$, therefore $a_2 \leq 0$.

Let $a_1 = a_2 = 0$. As $P^{(n-3)} = (n!/6)x^3 + (n-3)!a_3$ must be hyperbolic, one has $a_3 = 0$ etc. $\square$

A second change $x = \sqrt{|a_2|}x$ can reduce the study of Π^* to the case $a_1 = 0$ and $a_2 = -1$. Denote by $\Pi = \Pi^* \cap \{a_1 = 0,\, a_2 = -1\}$. Hence, we consider from now on the family of polynomials of the form

$$P(x,a) = x^n - x^{n-2} + a_3x^{n-3} + \cdots + a_n,\ x, a_i \in \mathbb{R}$$

Notation 2. We denote by $x_1 \leq \cdots \leq x_n$ the roots of P and by $x_1^{(k)} \leq \cdots \leq x_{n-k}^{(k)}$ the ones of $P^{(k)}$. We set $x_j^{(0)} = x_j$.

Definition 3. *We call* arrangement *of the roots of $P, P', \ldots, P^{(n-1)}$ the complete system of strict inequalities and equalities that hold for these roots. We assume that the roots are arranged in a string in which any two roots occupying consecutive positions are connected with a sign $<$ or $=$. An arrangement is called* non-degenerate *if there are no equalities between any two of the roots, i.e., no equalities of the form $x_i^{(j)} = x_q^{(r)}$ for any indices $(i,j) \neq (q,r)$.*

The configurations of the roots of $P, P', \ldots, P^{(n-1)}$ are indicated on a figure by configuration vectors on which coinciding roots are put in square brackets. E.g. the configuration vector (CV) corresponding to the point $A([x_1x_2x_1^1], x_1^2, [x_2^1x_1^3], x_2^2, [x_3x_4x_3^1])$, in Fig. 1, means that $x_1 = x_2 = x_1^1 < x_1^2 < x_2^1 = x_1^3 < x_2^2 < x_3 = x_4 = x_3^1$.

Recall that, by applying the Rolle's theorem several times one gets for any $i < j < n$ the standard Rolle's restrictions

$$x_l^{(i)} \leq x_l^{(j)} \leq x_{l+j-i}^{(i)}.$$

And, from the properties of multiple roots, we have the obvious condition

$$\left(\left(x_k^{(i)} = x_k^{(i+1)}\right) \text{ or } \left(x_{k+1}^{(i)} = x_k^{(i+1)}\right)\right) \Rightarrow \left(x_k^{(i)} = x_k^{(i+1)} = x_{k+1}^{(i)}\right).$$

The absence of some of the arrangements is closely connected with the presence of overdetermined strata in any generic family of hyperbolic polynomials. See [13] and [15].

In [6] we presented a resultant-based method to calculate the overdetermined strata for degree 5 hyperbolic polynomials. This new method can be generalized to calculate the overdetermined strata in degree greater than 5. In this paper we will use it to calculate the overdetermined strata for degree 6 hyperbolic polynomials. These overdetermined strata in degree 6 were not known before, because the geometric method, used until now, can not be generalized for degree greater than 5.

The idea of our method is to use the fact that a common root between two polynomials is a root of the resultant of these polynomials. We can then transform the principal problem to a system of four-variables polynomials which we resolve using the Gröbner basis techniques.

In the next section we will give the definition and examples of overdetermined strata then we will explain the geometric method used in [5, 13, 14, 10, 11, 15, 16] to calculate them for degree 4 and 5 polynomials. In Section 3 we recall the definitions of resultants and subresultants and we recall their properties. Our main result is presented in the Section 5.

2. Geometric methods.

2.1. Definitions and properties.

Notation 4. We denote by $\mathrm{Pol}_n^{\mathbb{R}}$ the space of all monic degree n polynomials in one variable with real coefficients. Denote by $\mathcal{PP}_n^{\mathbb{R}}$ the product space $\mathrm{Pol}_n^{\mathbb{R}} \times \mathrm{Pol}_{n-1}^{\mathbb{R}} \cdots \times \mathrm{Pol}_1^{\mathbb{R}}$. A point of $\mathcal{PP}_n$ is an n-tuple of polynomials $(P_n, P_{n-1}, \cdots, P_1)$ of respective degrees.

One can decompose the space $\mathcal{PP}_n$ according to the multiplicities of the roots of the different polynomials and the presence and multiplicities of their common roots. The combinatorial objects enumerating the strata should be called coloured partitions since they are partitions of C_{n+1}^2 not necessarily distinct points on $\mathbb{R}$ divided into groups of cardinalities $n, n-1, \cdots, 1$ which we can think of as having different colours (it is easy to check that this decomposition is actually a Whitney stratification).

There is a natural embedding map $\pi : \mathrm{Pol}_n^{\mathbb{R}} \hookrightarrow \mathcal{PP}_n$ sending each monic polynomial P of degree n to $(P, P'/n, P''/n(n-1), \cdots, P^{(n-1)}/n!)$.

Let λ be a coloured partition of C^2_{n+1} coloured points, $St_\lambda \subset \mathcal{PP}_n$ be the corresponding stratum and $\pi(St\lambda) = St_\lambda \cap \pi(Pol_n^{\mathbb{R}})$ be its (probably empty) intersection with the embedded space of polynomials $\pi(\mathrm{Pol}_n^R)$. We call this intersection a stratum. Note that $\dim St_\lambda$ equals the number of parts in λ.

Definition 5. *The stratum St_λ is called* overdetermined *if the codimension of St_λ in $\mathcal{PP}_n$ is greater than the codimension of $\pi(St_\lambda)$ in $\pi(\mathrm{Pol}_n)$ (here we assume that $\pi(St_\lambda) \neq \emptyset$). We denote by ϱ the difference between these codimensions.*

Remark 1. A polynomial P such that there are $> n-2$ equalities between roots of $P, P', \cdots, P^{(n-1)}$ belongs to an overdetermined stratum. Indeed, the latter depends on $n-2$ parameters (after the normalization $a_1 = 0$, $a_2 = -1$).

Definition 6. *An overdetermined stratum is called* non-trivial *if ϱ is due not only to the presence of the multiple roots in P and in its derivatives.*

Example 7. The polynomial $(x-1)^3(x+1)^3$ has multiple roots, but it defines a non-trivial stratum because 0 is a common root of all odd-degree derivatives.

Definition 8. *An overdetermined stratum is called* old *if the embedding $\pi : \mathrm{Pol}_{n-1} \hookrightarrow \mathcal{PP}_{n-1}$ defines an overdetermined stratum in $\mathcal{PP}_{n-1}$. ("Old" is used in the sense of "previously known", i.e., known already for $n-1$). When n is odd (resp. even), an overdetermined stratum is called* odd *(resp.* even*) if it is defined by an odd (resp. even) polynomial. Such are the strata defined by Gegenbauer polynomials, see Definition* 10.

Example 9. The stratum $x^5 - x^3 + \dfrac{1}{4}x$ is old and odd. It is obtained by integrating the degree 4 and even stratum $x^4 - x^2 + \dfrac{5}{36}$ followed by a rescaling and multiplication by a non-zero constant.

Definition 10. *The* Gegenbauer polynomial G_n *is defined as the unique polynomial of the kind*

$$x^n - x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_0$$

which is divisible by its second derivative. One can prove that it is strictly hyperbolic, and that it is odd or even together with n. The Gegenbauer polynomial $G_4 := x^4 - x^2 + \dfrac{5}{36}$ has by definition two roots in common with $G_4^{(2)}$ (they are equal to $\pm\dfrac{1}{\sqrt{6}}$), and $G_4^{(1)}$ has 0 as a common root with $G_4^{(3)}$. This makes three *equalities between roots from the set of* 10 *roots of* G_4, $G_4^{(1)}$, $G_4^{(2)}$ *and* $G_4^{(3)}$.

Remark 2. For all $n \geq 4$, the Gegenbauer polynomial P defines an overdetermined stratum since it is completely defined by the condition of being divisible by its second derivative, so we get the second supplementary condition that $P^{(n-1)} = n!x$ divides all its derivatives that are odd-degree polynomials. So the quantity ϱ is equal to $(n-2)/2$.

2.2. Overdetermined strata for $n = 4$.

Theorem 11. *There are no non-trivial overdetermined strata for* $n < 4$. *For* $n = 4$ *the points* A *and* B *(see Fig.* 1*) define the only non-trivial overdetermined strata, and all points (except* A*) on the boundary of the hyperbolicity domain belong to trivial overdetermined strata.*

P r o o f. For $n = 2$, there is only one arrangement with at least one equality between roots; it is $[x_1 x_1^{(1)} x_2]$ that defines a trivial overdetermined stratum.

For $n = 3$, there are 4 arrangements with at least one equality be-

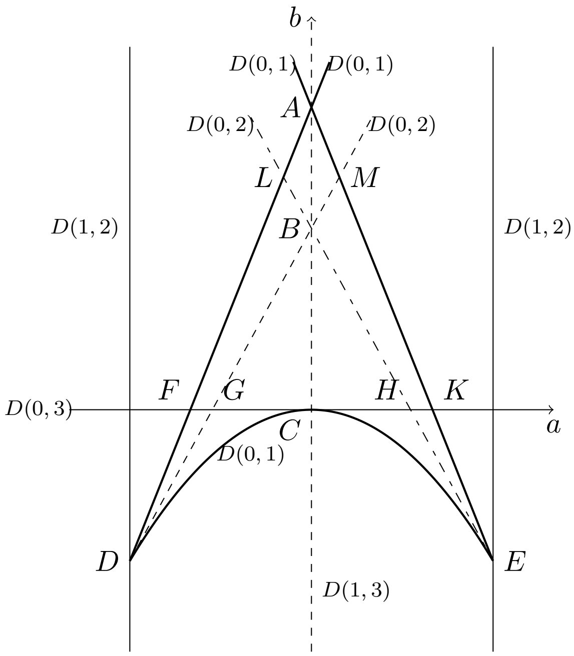

Fig. 1. The hyperbolicity domain for $n = 4$. We denote by $D(i,j) = \{(a,b) \in \Pi;\ \mathrm{Res}(P^{(i)}, P^{(j)}) = 0\}$

tween roots, they are $(x_1, x_1^{(1)}, [x_2x_1^{(2)}], x_2^{(1)}, x_3)$, $([x_1x_1^{(1)}x_2], x_1^{(2)}, x_2^{(1)}, x_3])$, $(x_1, x_1^{(1)}, x_1^{(2)}, [x_2x_2^{(1)}x_3])$ and $[x_1x_1^{(1)}x_2x_1^{(2)}x_2^{(1)}x_3]$. The first one doesn't define an overdetermined stratum since $\operatorname{codim}_{\mathcal{PP}_3} St_\lambda = \operatorname{codim}_{\pi(\mathrm{Pol}_3)} \pi(St_\lambda)$. The others define trivial overdetermined strata.

For $n = 4$, at the point B we have

$$\operatorname{codim}_{\mathcal{PP}_4} St_\lambda = 3 > \operatorname{codim}_{\pi(\mathrm{Pol}_4)} \pi(St_\lambda) = 2,$$

so there is an overdetermined stratum.

At the point A we have $\operatorname{codim}_{\mathcal{PP}_4} St_\lambda = 5 > \operatorname{codim}_{\pi(\mathrm{Pol}_4)} \pi(St_\lambda) = 2$, so there is an overdetermined stratum.

The stratum A is non-trivial because $x_2^{(1)} = x_1^{(3)}$ isn't an algebraic result of the two equalities $x_1 = x_2$ and $x_3 = x_4$, in fact, for an hyperbolic polynomial, if we have $x_1 = x_2$ and $x_3 = x_4$, then we should have also $x_2^{(1)} = x_1^{(3)}$.

At the point M we have a trivial overdetermined stratum since we can have $x_3 \neq x_4$ and $x_2 = x_1^{(2)}$ using the change $\mathcal{P} \longmapsto \mathcal{P} + \varepsilon(x - x_2)$, $\varepsilon \lesssim 0$. By symmetry we prove that the point L defines a trivial overdetermined stratum.

At the point K there is a trivial overdetermined stratum because we can have $x_2 = x_1^{(3)}$ and $x_3 = x_4$ using the change $\mathcal{P} \longmapsto \mathcal{P} + \varepsilon(x - x_2)$, $\varepsilon \lesssim 0$. By symmetry we prove that the point F defines a trivial overdetermined stratum.

The point C defines a trivial overdetermined stratum because we can have $x_2 \neq x_3$ and $x_2^{(1)} = x_1^{(3)}$ using the change $\mathcal{P} \longmapsto \mathcal{P} + \varepsilon$, $\varepsilon \gtrsim 0$.

Each of the open arcs AM,MK, KE, EC, CD, DF, FL, LA and the points E and D defines a trivial overdetermined stratum because the value of ϱ depends only on the presence of multiple roots of P and its derivatives. There is only one case to be treated, the case where the arrangement is of the form $([x_1x_1^{(1)}x_2x_1^{(2)}x_2^{(1)}x_3x_1^{(3)}x_2^{(2)}x_3^{(1)}x_4])$ which defines a trivial overdetermined stratum. $\square$

Corollary 12. *The overdetermined strata and their corresponding configuration vectors for $n = 4$ are*

$$x^4 - x^2 + \frac{5}{36}, \quad \textit{CV: } (x_1, x_1^{(1)}, [x_1^{(2)}x_2], [x_1^{(3)}x_2^{(1)}], [x_2^{(2)}x_3], x_3^{(1)}, x_4),$$
$$x^4 - x^2 + \frac{1}{4}, \quad \textit{CV: } ([x_1x_1^{(1)}x_2], x_1^{(2)}, [x_2^{(1)}x_1^{(3)}], x_2^{(1)}, [x_3x_3^{(1)}x_4]).$$

Remark 3. We remark that

- All of the overdetermined strata for $n = 4$ are even.

- The polynomial $x^4 - x^2 + \dfrac{5}{36}x$ is a Gegenbauer polynomial.

2.3. Overdetermined strata for $n = 5$.

Theorem 13. *The overdetermined strata and their corresponding configuration vectors for $n = 5$ are*

$$\begin{aligned}
&x^5 - x^3 + \frac{9}{100}x, && \textit{CV: } (x_1, x_1^{(1)}, x_1^{(2)}, [x_2 x_1^{(3)}], x_2^{(1)}, [x_3 x_2^{(2)} x^{(4)}], x_3^{(1)}, [x_2^{(3)} x_4], x_3^{(2)}, x_4^{(1)}, x_5),\\
&x^5 - x^3 + \frac{21}{100}x, && \textit{CV: } (x_1, x_1^{(1)}, [x_2 x_1^{(2)}], x_1^{(3)}, x_2^{(1)}, [x_3 x_2^{(2)} x^{(4)}], x_3^{(1)}, x_2^{(3)}, [x_3^{(2)} x_4], x_4^{(1)}, x_5),\\
&x^5 - x^3 + \frac{1}{4}x, && \textit{CV: } ([x_1 x_1^{(1)} x_2], x_1^{(2)}, [x_1^{(3)}, x_2^{(1)}], [x_3 x_2^{(2)} x^{(4)}], [x_3^{(1)} x_2^{(3)}], x_3^{(2)}, [x_4 x_4^{(1)} x_5]).
\end{aligned}$$

Proof. See [5] and [13] $\square$

Remark 4. We remark that

- All of the overdetermined strata for $n = 5$ are odd.
- The polynomial $x^5 - x^3 + \dfrac{21}{100}x$ is a Gegenbauer polynomial.
- The stratum $x^5 - x^3 + \dfrac{1}{4}x$ is old.

3. Resultant-based method to calculate overdetermined Strata for $n = 6$.

3.1. Resultants and subresultants. Let $P = \sum_{i=0}^{p} a_i x^i$ and $Q = \sum_{i=0}^{q} b_i x^i$ be two non-zero polynomials in one variable and of degree p and q respectively.

Definition 14. *The* Sylvester matrix *of P and Q, denoted by $S_1(P, Q)$, is the matrix*

$$S_1(P,Q) = \begin{pmatrix}
a_p & \cdots & \cdots & \cdots & \cdots & a_0 & 0 & \cdots & 0\\
0 & \ddots & & & & & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & & & & & \ddots & 0\\
0 & \cdots & 0 & a_p & \cdots & \cdots & \cdots & \cdots & a_0\\
b_q & \cdots & \cdots & \cdots & b_0 & 0 & \cdots & \cdots & 0\\
0 & \ddots & & & & \ddots & \ddots & & \vdots\\
\vdots & \ddots & \ddots & & & & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & & & & \ddots & 0\\
0 & \cdots & \cdots & 0 & b_q & \cdots & \cdots & \cdots & b_0
\end{pmatrix}$$

It is a matrix of size $(p+q)\times(p+q)$*. Note that its rows are*

$$x^{q-1}P, \cdots, P, x^{p-1}Q, \cdots, Q$$

considered as vectors in the basis $(x^{p+q-1}, \cdots, 1)$*.*

The resultant of P *and* Q *is the determinant of* $S_1(P,Q)$*, it is denoted by* $Res(P,Q)$*.*

Definition 15. *For* $k = 2, \ldots, \min(p,q)$ *we define the* k*-th Sylvester matrix* $S_k(P,Q)$ *of* P *and* Q *by deleting the* $q-k+2$*-nd row, the last row and the last two columns of* $S_{k-1}(P,Q)$*. Hence* $S_k(P,Q)$ *is of size* $(p+q+2-2k)\times(p+q+2-2k)$*.*

We denote by $\Delta_k(P,Q)$ *the determinant of* $S_k(P,Q)$*.* $\Delta_k(P,Q)$ *is the* k*-th subresultant of* P *and* Q*.* $\Delta_1(P,Q) = Res(P,Q)$ *is the resultant of* P *and* Q*.*

For example if $p=4$, $q=3$ we have

$$S_1(P,Q) = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix},$$

$$S_2(P,Q) = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \\ 0 & 0 & b_3 & b_2 & b_1 \end{pmatrix}, \quad S_3(P,Q) = \begin{pmatrix} a_4 & a_3 & a_2 \\ b_3 & b_2 & b_1 \\ 0 & b_3 & b_2 \end{pmatrix}.$$

Theorem 16. *The polynomials* P *and* Q *have exactly* m *common roots, counted with their multiplicities, if and only if* $\Delta_1(P,Q) = \cdots = \Delta_m(P,Q) = 0 \neq \Delta_{m+1}(P,Q)$*.*

Proof. See Proposition 4.25 in [3]. □

3.2. Algorithm. We search a, b, c and d such that the polynomial $x^6 - x^4 + ax^3 + bx^2 + cx + d$ defines an overdetermined stratum. We will use the resultants to reduce this problem to a problem of resolution of a polynomial system of 5 polynomial equations in 4 variables. Our algorithm is summarized in the following steps:

1. There are 357 cases (see [5]), where we have at least 5 equalities between roots (and so an eventual overdetermined strata), to study . The arrangements of the Figs 2 and 3 give the example of 2 such cases.

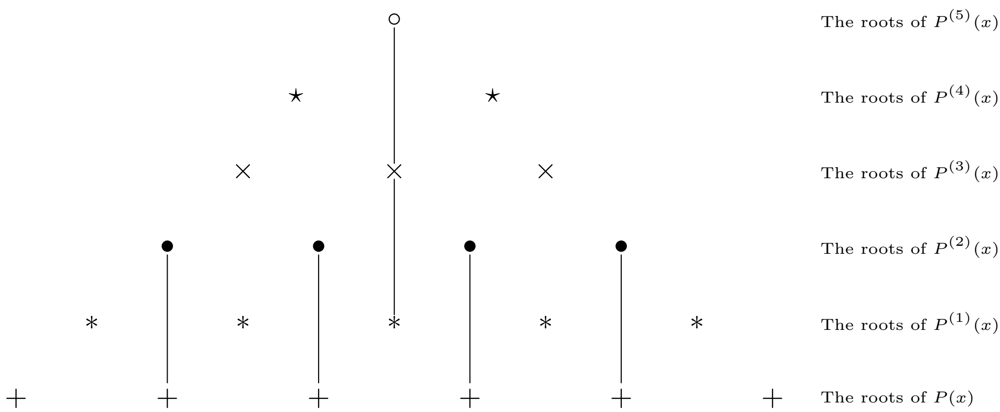

Fig. 2. Example of an eventual stratum

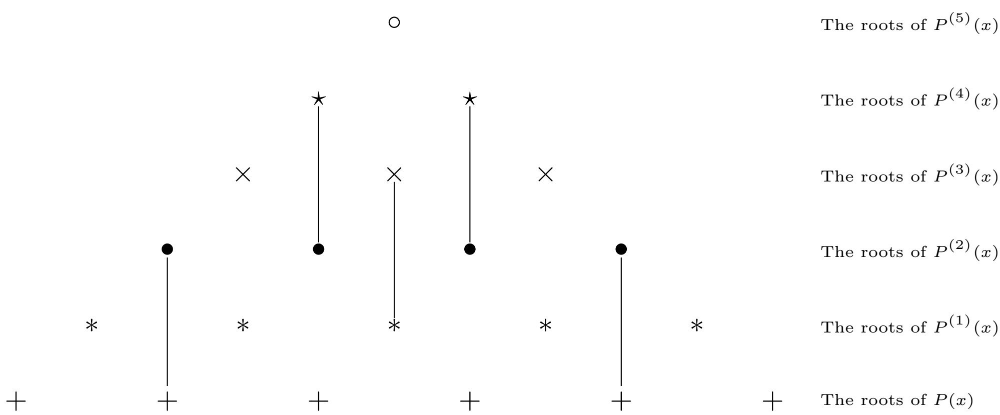

Fig. 3. Example of an eventual stratum

2. Each arrangement that can define an overdetermined stratum gives a system of 5 equations of the form $\Delta_k(P^{(i)}, P^{(j)})$ with $k \in [[1,4]]$ and $i, j \in [[0,5]]$ (for $a, b \in \mathbb{N}$ we denote by $[[a,b]]$ the set $[[a,b]] = \{n \in \mathbb{N};\ a \leq n \leq b\}$). For instance, the case of the Fig. 3 gives the following system:

$$\Delta_1(P, P^{(2)}) = \Delta_2(P, P^{(2)}) = \Delta_1(P^{(2)}, P^{(4)}) = \Delta_2(P^{(2)}, P^{(4)})$$

$$= \Delta_1(P^{(1)}, P^{(3)}) = 0.$$

In fact, there are 2 common roots between P and $P^{(2)}$, 2 common roots between $P^{(2)}$ and $P^{(4)}$, and a common root between $P^{(1)}$ and $P^{(3)}$.

3. Among the 357 final systems, we obtain some impossible systems (for example the above system given by the arrangement of the Fig. 3 is an impossible system), this means that there are no hyperbolic polynomials which realize this arrangement, and we obtain some systems that are repeated. See [6] for examples of such systems.

   Finally, it still, among these 357 cases only 19 present overdetermined strata.

Theorem 17. *There are exactly* 19 *non-trivial overdetermined strata for degree* 6 *hyperbolic polynomials. All of these strata are even. The polynomials that realize them are*

$$x^6 - x^4 + \frac{1}{4}x^2 \tag{1}$$

$$x^6 - x^4 + \frac{1}{3}x^2 - \frac{1}{27} \tag{2}$$

$$x^6 - x^4 + \frac{3}{25}x^2 \tag{3}$$

$$x^6 - x^4 + \frac{3}{25}x^2 - \frac{13}{3375} \tag{4}$$

$$x^6 - x^4 + \frac{3}{25}x^2 + \frac{13}{625} - \frac{32}{3125}\sqrt{5} \tag{5}$$

$$x^6 - x^4 + \frac{4}{25}x^2 \tag{6}$$

$$x^6 - x^4 + \frac{9}{35}x^2 - \frac{9}{875} \tag{7}$$

$$x^6 - x^4 + \frac{7}{25}x^2 - \frac{29}{1875} - \frac{16}{28125}\sqrt{30} \tag{8}$$

$$x^6 - x^4 + \frac{7}{25}x^2 - \frac{3}{125} \tag{9}$$

$$x^6 - x^4 + \frac{7}{25}x^2 - \frac{49}{3375} \tag{10}$$

$$x^6 - x^4 + \frac{8}{25}x^2 - \frac{4}{125} \tag{11}$$

$$x^6 - x^4 + \frac{45}{196}x^2 \tag{12}$$

$$x^6 - x^4 + \frac{47}{225}x^2 - \frac{11}{1125} \tag{13}$$

$$x^6 - x^4 + \frac{14}{225}x^2 \tag{14}$$

$$x^6 - x^4 + \frac{99}{361}x^2 - \frac{81}{6859} \tag{15}$$

$$x^6 - x^4 + \frac{123 - 3\sqrt{113}}{490}x^2 - \frac{223}{12250} + \frac{3\sqrt{113}}{2450} \tag{16}$$

$$x^6 - x^4 + \frac{123 + 3\sqrt{113}}{490}x^2 - \frac{223}{12250} - \frac{3\sqrt{113}}{2450} \tag{17}$$

$$x^6 - x^4 + \left(\frac{25}{147} - \frac{8\sqrt{78}}{735}\right)x^2 - \frac{1189}{165375} + \frac{8\sqrt{78}}{11025} \tag{18}$$

$$x^6 - x^4 + \left(\frac{25}{147} + \frac{8\sqrt{78}}{735}\right)x^2 - \frac{1189}{165375} - \frac{8\sqrt{78}}{11025} \tag{19}$$

Remarks.

1. We implemented this algorithm in Maple to make exact calculation.
2. The polynomial ($p(x) = x^6 - x^4 + \frac{9}{35}x^2 - \frac{9}{875}$) that realizes the arrangement in Fig. 1 is a Gegenbauer polynomial. Since the Gegenbauer polynomial is unique, so we can remove 15 systems among the 357 systems. For example, we cannot find any hyperbolic polynomial that realizes the arrangement of Fig. 4.
3. Each ideal $\mathcal{I} \neq 0$ has (for a given total order of monomials) a unique reduced Gröbner basis. (See [4]).
4. Let (S) be the following polynomial equations system:

$$S : \begin{cases} p_1(x_1, \dots, x_n) = 0 \\ \vdots \\ p_q(x_1, \dots, x_n) = 0 \end{cases}$$

and let $\mathcal{I}$ be the ideal generated by $p_1, \dots, p_q$, and G be a Gröbner basis of $\mathcal{I}$.

The system (S) has a solution if and only if $1 \notin G$. In this case, to solve (S) we solve the system given by the Gröbner basis of $p_1, \dots, p_q$. (See [4]).

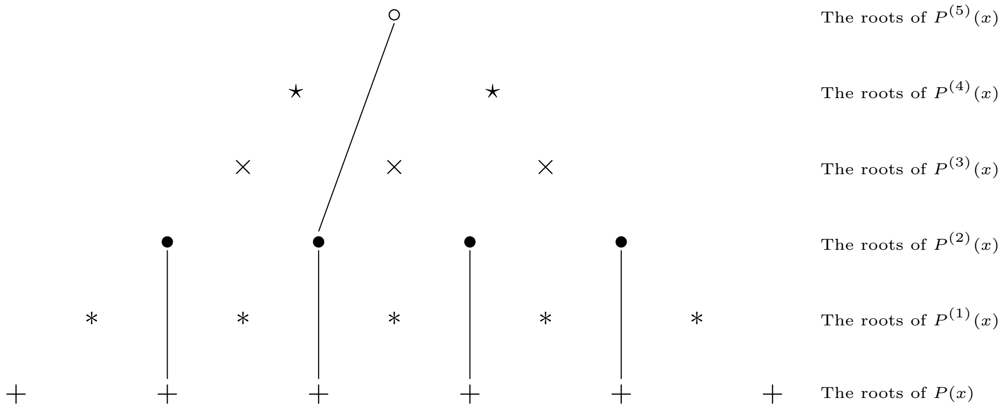

Fig. 4. An example of impossible arrangement due to the uniqueness of Gegenbauer polynomial

5. There are 7 old strata from the strata of degree 6:
   - the stratum (2) is obtained by integrating the stratum $x^5 - x^3 + \frac{1}{4}x$ (which is himself old, it is obtained by integrating the stratum $x^4 - x^2 + \frac{5}{36}$),
   - the strata (3), (4) and (6) are obtained by integrating the stratum $x^5 - x^3 + \frac{9}{100}x$,
   - the strata (8), (9) and (10) are obtained by integrating the stratum $x^5 - x^3 + \frac{21}{100}x$.
6. We can use our algorithm to obtain the configuration vectors of the overdetermined strata. For example the configuration vectors of the vectors (16), (17) and (18), (19) are respectively

$$(x_1, x_1^{(1)}, x_1^{(2)}, x_2, x_1^{(3)}, x_2^{(1)}, x_1^{(4)}, [x_3 x_2^{(2)}], [x_3^{(1)} x_2^{(3)} x^{(5)}], [x_3^{(2)} x_4], x_2^{(4)}, x_4^{(1)}, x_3^{(3)}, x_5, x_4^{(2)}, x_5^{(1)}, x_6)$$

$$(x_1, x_1^{(1)}, [x_1^{(2)} x_2], x_2^{(1)}, [x_3 x_1^{(3)}], x_1^{(4)}, x_2^{(2)}, [x_3^{(1)} x_2^{(3)} x^{(5)}], x_3^{(2)}, x_2^{(4)}, [x_3^{(3)} x_4], x_4^{(1)}, [x_5 x_4^{(2)}], x_5^{(1)}, x_6)$$

and

$$(x_1, x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, [x_1^{(4)} x_2], x_2^{(1)}, [x_3 x_2^{(2)}], [x_3^{(1)} x_2^{(3)} x^{(5)}], [x_3^{(2)} x_4], x_4^{(1)}, [x_5 x_2^{(4)}], x_3^{(3)}, x_4^{(2)}, x_5^{(1)}, x_6)$$

$$(x_1, x_1^{(1)}, [x_1^{(2)} x_2], x_1^{(3)}, x_2^{(1)}, x_1^{(4)}, x_3, x_2^{(2)}, [x_3^{(1)} x_2^{(3)} x^{(5)}], x_3^{(2)}, x_4, x_2^{(4)}, x_4^{(1)} x_3^{(3)}, [x_5 x_4^{(2)}], x_5^{(1)}, x_6)$$

7. We can remark that the overdetermined strata are all even (the overdetermined strata in the case $n = 5$ are all odd and in the case $n = 4$ are all even).

4. Conclusion. We calculated in this paper the overdetermined strata for degree 6 by transforming the geometric problem to an algebraic one. This method can be used for any degree n.

Will try to see, in future work, if the following property remains true: the overdetermined strata are odd when n is odd and they are even when n is even. By proving this conjecture, we reduce the number of parameters from $n - 2$ to $\frac{n-2}{2}$ if n is even and $\frac{n-3}{2}$ if n is odd.

REFERENCES

[1] V. I. Arnold. Hyperbolic polynomials and Vandermonde mappings. *Funct. Anal. Appl.* **20**, *2* (1986), 52–53.

[2] V. I. Arnold. On the Newton potential of hyperbolic layers. *Tr. Tbilis. Univ.* **232/233**, Ser. Mat. Mekh. Astron. *13/14* (1982), 23–29 (in Russian); English translation in: *Sel. Math. Sov.* **4** (1985), 103–106.

[3] S. Basu, R. Pollack, M.-F. Roy. Algorithms in Real Algebraic Geometry, 2nd ed. Algorithms and Computation in Mathematics, vol. **10**, Berlin, Springer, 2006.

[4] D. Cox, J. Little, D. OShea. Ideals, varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra, 2nd ed. Undergraduate Texts in Mathematics, New York, Springer, 1996.

[5] H. Ezzaldine. Strates surdéterminées dans les familles de polynômes à une variable de degrés 5 et 6. Ph.D thesis, Laboratoire J. A. Dieudonné, Université de Nice Sophia-Antipolis, 2009.

[6] H. Ezzaldine, H. Khalil, M. Sarrage, M. Hossein. Overdetermined strata for degree 5 hyperbolic polynomials. *Vietnam J. Math.* **43**, *1* (2015), 139–150.

[7] H. Ezzaldine, V. P. Kostov. Even and old overdetermined strata for degree 6 hyperbolic polynomials. *Serdica Math. J.* **34**, *4* (2008), 743–770.

[8] A. B. Givental'. Moments of random variables and the equivariant Morse lemma. *Uspekhi Mat. Nauk* **42**, *2(254)* (1987), 221–222 (in Russian); English translation in *Russian Math. Surveys* **42**, *2* (1987), 275–276.

[9] V. P. Kostov. On the geometric properties of Vandermonde's mapping and on the problem of moments. *Proc. R. Soc. Edinb., Sect. A* **112**, *3–4* (1989), 203–211.

[10] V. P. Kostov. On the hyperbolicity domain of the polynomial $x^n+a_1x^{n-1}+\ldots+a^n$. *Serdica Math. J.* **25**, *1* (1999), 47–70.

[11] V. P. Kostov. On arrangements of the roots of a hyperbolic polynomial and of one of its derivatives. *Séminaires et Congrès* **10** (2005), 139–153, Singularités Franco-Japonaises (Eds J.-P. Brasselet et al.) Paris, Société Mathématique de France.

[12] V. P. Kostov, B. Z. Shapiro. On arrangements of roots for a real hyperbolic polynomial and its derivatives. *Bull. Sci. Math.* **126**, *1* (2002), 45–60.

[13] V. P.Kostov. Discriminant sets of families of hyperbolic polynomials of degree 4 and 5. *Serdica Math. J.* **28** *2* (2002), 117–152.

[14] V. P. Kostov. Root configurations of hyperbolic polynomials of degree 3, 4 and 5. *Funktsional Anal. i Prilozhen.* **36**, *4* (2002), 71–74 (in Russian); English translation in *Funct. Anal. Appl.* **36**, *4* (2002), 311–314.

[15] V. P. Kostov. Overdetermined strata in general families of polynomials. *Serdica Math. J.* **29**, *4* (2003), 377–386.

[16] V. P. Kostov. On root arrangements of polynomial-like functions and their derivatives. *Serdica Math. J.* **31**, *3* (2005), 201–216.

[17] W. Nuij. A note on hyperbolic polynomials. *Math. Scand.* **23** (1968), 69–72.

[18] G. Pólya, G. Szegő. Problems and theorems in analysis, vol. 2. New York-Heidelberg, Springer-verlag, 1976.

Laboratory of Mathematics and its Applications
LaMA–Lebanon, Lebanese University
e-mail: haysam82@hotmail.com

Received June 22, 2016