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COPULAS ON SOBOLEV SPACES

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Abstract. Defining a copula function and investigating its properties are both non-trivial tasks, as there is no general method for constructing them. We present a method which allows us to obtain a class of copulas as a solution of a boundary value problem in an appropriate Sobolev space. Furthermore, our method allows us to reduce the otherwise complex task of checking the 2-increasing of C-volume of the copula to simple differentiation. We demonstrate the applicability of our method to a number of examples.

1. Introduction. Copulas allow explicit construction of multivariate distribution functions by one-dimensional distributions playing the role of marginals according to the Sklar theorem [17] (see also [6]), making them useful in a variety of applications. Comprehensive theory of copulas is developed in [17] as well as in [6]. In this work we will restrict ourselves to the two dimensional case for the sake of clarity and brevity. Generalisations of our constructions are readily achieved and will be presented in a future work.

Let us denote $I = [0,1]$ and $I^2 = I \times I$.

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Definition 1.1. *A two-dimensional copula (or 2-copula, or briefly, a copula) is a function $C : I^2 \to I$ with the following properties (see* [17]*):*

1. *For all u, v in I*

$$C(u,0) = 0 = C(0,v), \quad C(u,1) = u, \ C(1,v) = v; \tag{1}$$

2. *C is a 2-increasing function, i.e. for every u_1, u_2, v_1, v_2 in I such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$,*

$$V_C(B) \equiv C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geqslant 0, \tag{2}$$

where B is the rectangle $[u_1,u_2] \times [v_1,v_2]$ and the expression (2) *defines the C-volume of B.*

Let X and Y be random variables with the distribution functions $F(x)$ and $G(y)$ respectively, and a joint distribution function $H(x,y)$ (see [17, Chapter 2]), namely H is 2-increasing (i.e. $V_H(B) \geqslant 0$ for each rectangle $B \subset \mathbb{R}^2$) and

$$H(x,-\infty) = H(-\infty,y) = 0, \qquad H(\infty,\infty) = 1,$$

and H has margins F and G, i.e.

$$F(x) = H(x,\infty), \qquad G(y) = H(\infty,y).$$

Denote ranges of F and G with $\mathrm{Ran}(F)$ and $\mathrm{Ran}(G)$, respectively. Under the conditions above, the Sklar theorem [17, Theorem 2.3.3] states:

1. For given F, G and H, there exists a copula $C(u,v)$ such that for all x, y in $\overline{\mathbb{R}}$:

$$H(x,y) = C\big(F(x), G(y)\big). \tag{3}$$

   If F and G are continuous, then C is unique; otherwise C is uniquely determined on $\mathrm{Ran}(F) \times \mathrm{Ran}(G)$.

2. If C is a copula and $F(x)$ and $G(x)$ are distribution functions representing the random variables X and Y respectively, then the function $H(x,y)$, defined by (3), is a joint distribution function of X and Y with margins $F(x)$ and $G(y)$.

In fact, in Nelsen [17, Chapter 2], considerations are restricted to functions which are well defined in each point and it is proven that copulas are Lipschitz functions on their domain (see [17, Theorem 2.2.4]).

Further, any copula C is represented as a sum of an absolutely continuous component and a singular component, where the absolutely continuous component corresponds to the joint density arising from $\partial_{uv}C(u,v) := \partial^2 C(u,v)/\partial u \partial v$ (see [17, Section 2.4], see also [6, §3.2]). The partial derivatives are calculated in the sense of 'almost everywhere' which makes it difficult to check if a given function is a copula or not (we consider this problem in detail in Section 2).

In [5] authors present a new proof of the Sklar's theorem in the case when at least one of the marginals has a discrete component. The proof is based on some analytical regularization techniques (i.e., mollifiers) and on the compactness (with respect to the L_∞ norm) of the class of copulas.

The aim of this article is to construct a family of copulas C based on a given probability density $\partial_{uv}C$, using the concept of weak derivative (derivative in terms of distribution theory). Despite some restrictions imposed by Sobolev spaces, we are able to construct new copulas using this approach. The main result is to solve the boundary value problem

$$\begin{aligned}
&\partial_{uv}C(u,v) = f(u,v) \text{ in } I^2 \text{ (in weak sense);}\\
&C(u,0) = 0 = C(0,v);\\
&C(u,1) = u,\ C(1,v) = v,\ \text{for all } u,v \in I,
\end{aligned}$$

under certain conditions on f. First we consider the case when f is smooth function and later generalize it to $f \in W^{-1,p}(I^2)$, $p > 2$. This problem can be thought of as a Dirichlet problem for the wave equation. However, it is important to note here that this is an ill-posed boundary value problem for which there exist comprehensive surveys (see [7, 2, 8]). Actually, as seen below, part of the conditions over the boundary are obtained by assumptions regarding the right hand side of the equation.

Copulas are most frequently applied in the fields of finance, economics and actuarial science. For example, in [4], copulas are introduced from the viewpoint of mathematical finance applications. There copulas are used in order to describe major topics such as asset pricing, risk management and credit risk analysis. Patton [18] reviews the use of copulas in econometric modelling, Genest et al. [9] gives a nice bibliometric overview. McNeil et al. [14] contains an introduction to the realm of copulas aimed at the quantitative risk manager.

Our method is particularly suitable for problems coming from finance, economics and actuarial science that involve processing real data and will be considered in a future work.

The outline of the article is as follows. Section 2 contains the generating technique for 2-increasing functions. Section 3 provides the required knowledge on Sobolev spaces and *a priori* estimate, from which the uniqueness of solution in the smooth case follows. Section 4 discuses the smooth case and in Section 5 the general solution is considered. The applicability of our method is shown in Section 6 through an example.

A short version of this work is published in Comptes rendus de l´Académie Bulgare des Sciences [12].

2. 2-increasing functions. Examples. The notion of a “2-increasing” function is fundamental for the considered range of problems. Let $H : G \to \mathbb{R}$, be a function defined over a region $G \subset \overline{\mathbb{R}}^2 = [-\infty, +\infty] \times [-\infty, +\infty]$ with range in $\mathbb{R} = (-\infty, +\infty)$. (More specific assumptions for H will be given later.) Let $B = [x_1, x_2] \times [y_1, y_2]$ be an arbitrary rectangle all of whose vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$ are in G. Then the H-volume of B is given by

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1). \tag{4}$$

According to definition 2.12 in [17] we give the following

Definition 2.1. *A real function H is 2-increasing if $V_H(B) \geqslant 0$ for all rectangles B whose vertices lie in G.*

In the case when all the derivatives H_x, H_y, H_{xy} exist and belong to $C^0(\mathbb{R}^2)$, the following statement holds

$$V_H(B) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \frac{\partial^2 H}{\partial\xi\partial\eta}(\xi, \eta)\, d\xi\, d\eta = \frac{\partial^2 H}{\partial x\partial y}(\overline{x}, \overline{y})\, (x_2 - x_1)(y_2 - y_1), \tag{5}$$

for an appropriate point $(\overline{x}, \overline{y}) \in B$, according to the mean value theorem, as the assumptions of this latter theorem hold.

These observations bring to the following result for continuous functions:

Lemma 2.2. *Let the function $H : \mathbb{R}^2 \to \mathbb{R}$ have continuous gradient and let the derivative H_{xy} be continuous too. Then H is 2-increasing if and only if*

$$H_{xy} \geqslant 0 \ \textit{in}\ \mathbb{R}^2. \tag{6}$$

A natural generalization of the lemma above is the case when the derivative H_{xy} exists only in a weak sense or, in other words, as derivative in terms of distribution theory (see [11]). We propose the following definition.

Definition 2.3. *A distribution $H \in \mathscr{D}'$ is weakly 2-increasing if for any test function $\varphi \geqslant 0$ in $\mathscr{D}(\mathbb{R}^2)$:*

$$(H_{xy}, \varphi) \geqslant 0. \tag{7}$$

Lemma 2.4. *Let the distribution $H \in \mathscr{D}'(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$ be weakly 2-increasing according to Definition 2.3. Then H is 2-increasing in the sense of Definition 2.1.*

P r o o f. Let $\varepsilon > 0$, $J_\varepsilon(x, y) \in C_0^\infty(\mathbb{R}^2)$ be a mollifier and $H_\varepsilon = J_\varepsilon * H$ (see [1, §2.28 and Section 3]). Then

$$(H_\varepsilon)_{xy} = (H_{xy})_\varepsilon, \quad \text{by [11, Theorem 1.6.1]},$$

and

$$V_H(B) = \lim_{\varepsilon \to 0^+} V_{H_\varepsilon}(B), \quad \text{according to [1, Theorem 2.29]}.$$

Hence, using (5) we obtain

$$V_{H_\varepsilon}(B) = \int_B (H_\varepsilon)_{xy}\, dx\, dy = \int_{\mathbb{R}^2} (H_{xy} * J_\varepsilon)\, \chi_B\, dx\, dy,$$

where χ_B is the characteristic function of B. Then by the monotone convergence theorem for Lebesgue integrals (see [1]) the last expression is equal to

$$\lim_{\rho \to 0^+} \int_{\mathbb{R}^2} (H_{xy} * J_\varepsilon)(\chi_B * J_\rho)\, dx\, dy.$$

Indeed (see [11, §1.6]) using the notation $\check{f}(x) = f(-x)$

$$\begin{aligned}
(H_{xy} * J_\varepsilon, \chi_B * J_\rho) &= \big((H_{xy} * J_\varepsilon) * \overbrace{\chi_B * J_\rho}^{\vee}\big)(0) \\
&= \big((H_{xy} * J_\varepsilon) * \check{\chi}_B * J_\rho\big)(0) \\
&= (H_{xy} * \check{\chi}_B * J_\varepsilon * J_\rho)(0) \\
&= \big(H_{xy}, \overbrace{\check{\chi}_B * J_\varepsilon * J_\rho}^{\vee}\big)
\end{aligned}$$

$$= \quad (H_{xy}, \chi_B * J_\varepsilon * J_\rho)(0) \geqslant 0,$$

where the parity of J_ε and J_ρ is taken into account, and in the last inequality we use that $\big((\chi_B)_\varepsilon\big)_\rho \in \mathscr{D}(\mathbb{R}^2)$ and is non-negative. Therefore $V_{H_\varepsilon}(B) \geqslant 0$ and hence $V_H(B)$ is non-negative. $\square$

Remark 2.5. For every distribution defined over a bounded domain there exist an extension to $\mathbb{R}^2$ which satisfies the lemma above.

Example 2.6 (Smooth copulas)**.** Applying Lemma 2.2 for the function H given by (see [17, Exercise 2.14])

$$H(x, y) = (1 + e^{-x} + e^{-y} + (1 - \theta)e^{-x-y})^{-1}, \; x, y \in \overline{\mathbb{R}},$$

it is easy to find values of the parameter θ for which H is 2-increasing. Let us note that

$$p(x, y) = H^{-1}(x, y).$$

Then

$$H_{xy}(x, y) = \frac{e^{-x-y}}{p^3(x, y)}\big[1 + \theta + (1 - \theta)(e^{-x} + e^{-y}) + (1 - \theta)^2 e^{-x-y}\big].$$

Let us consider the second factor of the above expression in the coordinate system $X = e^{-x}, Y = e^{-y}$. In quadrant I, values of θ for which the expression is non-negative, are easily obtained. Indeed, over the Y-axis, $Y = 0$ or $y = +\infty$ we need to have

$$1 + \theta + (1 + \theta)X \geqslant 0. \tag{8}$$

If we assume $\theta > 1$, then for $X > 0$ we observe

$$1 - \theta \geqslant \frac{-1 - \theta}{X},$$

and for $X \to +\infty$ we attain the contradiction $1 \geqslant \theta$. Now, if assume $\theta < -1$ then from (8) follows the contradiction $\theta \geqslant -1$ for $X \to 0^+$. Finally, for $\theta \in [-1, 1]$, we determine $H_{xy} \geqslant 0$ in $\mathbb{R}^2$.

Example 2.7 (Functions for which the weak derivative ∂_{xy} is a distribution)**.** Let us recall the rule for finding the weak derivative of a function defined and differentiable over both sides of a continuous curve S (see [19, Chapter II, §6]).

Let G and G_1 be two domains on either side of the curve S and n be the unit normal vector to S pointing towards G_1. Then by the Gauss theorem

$$\frac{\partial f}{\partial t} = \left\{\frac{\partial f}{\partial t}\right\} + \big[f\big]_S \cos(n,t)\delta_S,$$

where $t = x$ or $t = y$, $\left\{\frac{\partial f}{\partial t}\right\}$ is the usual derivative defined almost everywhere in G and G_1 (which might be well defined only on one side of S), and the distribution $\delta_S \in \mathscr{D}'(\mathbb{R}^2)$ is given by

$$(\delta_S, \varphi) = \int_S \varphi\, ds, \quad \varphi \in \mathscr{D}(\mathbb{R}^2),$$

where ds is the elementary arc length of S. Finally, the jump condition is given by

$$\big[f\big]_S(x,y) = \lim_{\substack{(x',y')\to(x,y)\\(x',y')\in G_1}} f(x',y') - \lim_{\substack{(x'',y'')\to(x,y)\\(x'',y'')\in G}} f(x'',y'').$$

a) Consider the Fréchet-Hoeffding lower bound copula (see [17]) $W(u,v) = \max(u+v-1,0)$ and let S be the curve given by

$$S = \big\{(u,v) \in I^2 \mid u+v-1=0\big\}.$$

Let n be the unit normal vector of S given by $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ at any point. Then

$$\partial_v\big(\partial_u W\big) = 0 + [1-0]\cos(n,v)\delta_S = \frac{1}{\sqrt{2}}\delta_S,$$

hence $(\partial_v\big(\partial_u W\big), \varphi) = \left(\frac{1}{\sqrt{2}}\delta_S, \varphi\right) \geqslant 0$ for any test function $\varphi \geqslant 0$ in $\mathscr{D}(\mathbb{R}^2)$. Thus W is weakly 2-increasing and by lemma 2.4 is 2-increasing.

b) Consider the function $f = \max(u,v)$. Similarly, for $S = \big\{(u,v) \in I^2 | u = v\big\}$ and the unit normal vector $n = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ to S, one has

$$\partial_v\big(\partial_u f\big) = 0 + [0-1]\cos(n,v)\delta_S = -\frac{1}{\sqrt{2}}\delta_S,$$

i.e. $(\partial_v(\partial_u f), \varphi) = \left(-\frac{1}{\sqrt{2}}\delta_S, \varphi\right) \leqslant 0$ for any test function $\varphi \geqslant 0$ in $\mathscr{D}(\mathbb{R}^2)$. Thus f is not a 2-increasing (and is not a copula).

3. Sobolev spaces. *A priori* estimate and uniqueness theorem. In this section we provide basic background on Sobolev spaces needed for our considerations. Furthermore, here we define natural extensions and restrictions of distributions from Sobolev spaces with negative exponents.

Using definitions and notations in [1, Chapters 2 and 3] let us consider a mollifier ω_ε with support shifted on $(-1, -1)$ and defined for every $\varepsilon > 0$ by

$$\omega_\varepsilon(x, y) = \varepsilon^{-2} J\left(\frac{x+\varepsilon}{\varepsilon}, \frac{y+\varepsilon}{\varepsilon}\right), \quad \text{for all } x, y \text{ in } \mathbb{R}^2,$$

where J is a non-negative, real-valued function belonging to $C_0^\infty(\mathbb{R}^2)$ such that $J(x, y) = 0$ for all $\|(x, y)\| \geqslant 1$ and $\int_{\mathbb{R}^2} J(x, y)\, dx\, dy = 1$ (see [1, 2.28]).

Therefore, ω_ε is a non-negative, real-valued function belonging to $C_0^\infty(\mathbb{R}^2)$ such that $\omega_\varepsilon(x, y) = 0$ for all (x, y) in the domain $(x+\varepsilon)^2 + (y+\varepsilon)^2 \geqslant \varepsilon^2$, and $\int_{\mathbb{R}^2} \omega_\varepsilon(x, y)\, dx\, dy = 1$.

The corresponding regularization u_ε of $u \in L_1^{loc}(\mathbb{R}^2)$ is defined by

$$u_\varepsilon(x, y) = (u * \omega_\varepsilon)(x, y) = \int_{\mathbb{R}^2} u(\xi, \eta)\, \omega_\varepsilon(x - \xi, y - \eta)\, d\xi\, d\eta.$$

The adjustment above is necessary in order to ensure that regularization of a continuous function u will vanishes on the complement of the quadrant I $(x \geqslant 0, y \geqslant 0)$ and on the axes $x = 0$ and $y = 0$ together with their gradient

$$\partial_x u_\varepsilon(x, 0) = 0, \ \partial_y u_\varepsilon(0, y) = 0. \tag{9}$$

In all our considerations we assume that $p \in (2, \infty)$ (i.e. according to the embedding theorem we consider only continuous functions). The conjugate exponent q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Given a domain $\Omega \subset \mathbb{R}^2$, the space $L_p(\Omega)$, with norm $\|\cdot\|_p$, is defined as usual.

We define

$$W^{1,p} = \big(u \in L_p(\Omega) \,\big|\, \partial^\alpha u \in L_p(\Omega),\ 0 \leqslant |\alpha| \leqslant 1\big),$$

with norm

$$\|u\|_{1,p} = \Bigg(\sum_{0 \leqslant |\alpha| \leqslant 1} \big\|\partial^\alpha u\big\|_p^p \Bigg)^{\frac{1}{p}}.$$

Let $W_0^{1,p}(\Omega)$ be the complement of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$ with respect to the norm $\|\cdot\|_{1,p}$. By [1, 2.44] for each linear functional $\mathscr{L} \in \big(L_p(\Omega)\big)'$ there exists unique $v \in L_q(\Omega)$ [1, 2.43], such that

$$\mathscr{L}(u) = \mathscr{L}_v(u) \equiv \int_\Omega u(x)v(x)\,dx, \text{ for all } u \in L_p(\Omega),$$

and $\|v\|_q = \big\|\mathscr{L}; \big(L_p(\Omega)\big)'\big\|$, thus $\big(L_p(\Omega)\big)'$ is isometrically isomorphic to $L_q(\Omega)$.

By [1, 3.8],

$$\big(L_p(\Omega^{(3)})\big)' = L_q(\Omega^{(3)}), \tag{10}$$

where $\Omega^{(3)} = \Omega \times \Omega \times \Omega$. If $u = (u_0, u_1, u_2) \in L_p(\Omega^{(3)})$, then equality (10) means that

$$\mathscr{L}(u) = \sum_{|\alpha| \leqslant 1} \int\limits_\Omega u_\alpha(x)v_\alpha(x)\,dx.$$

By Hölder's inequality [1, 2.4], we obtain

$$\begin{aligned}
|\mathscr{L}(u)| &\leqslant \sum_{|\alpha| \leqslant 1} \|u_\alpha\|_{L_p(\Omega)} \, \|v_\alpha\|_{L_q(\Omega)} \\
&\leqslant \Bigg[\sum_{0 \leqslant |\alpha| \leqslant 1} \int\limits_\Omega |u_\alpha|^p \,dx \Bigg]^{\frac{1}{p}} \cdot \Bigg[\sum_{0 \leqslant |\alpha| \leqslant 1} \int\limits_\Omega |v_\alpha|^q \,dx \Bigg]^{\frac{1}{q}}.
\end{aligned}$$

In the last inequality we use an algebraic inequality (see [10, Theorem 14]).

Therefore

$$\|\mathscr{L}\| \leqslant \Bigg(\sum_{0 \leqslant |\alpha| \leqslant 1} \int\limits_\Omega |v_\alpha|^q \,dx \Bigg). \tag{11}$$

We obtain Theorem 3.9 in [1], precisely:

For every $\mathscr{L} \in \big(W^{1,p}(\Omega)\big)'$ there exist an element $v \in L_q\big(\Omega^{(3)}\big)$ such that if the restriction of v to Ω_α is v_α, then for all $u \in W^{1,p}(\Omega)$

$$\mathscr{L}(u) = \sum_{0\leqslant|\alpha|\leqslant 1} \big(\partial^\alpha u, v_\alpha\big), \tag{12}$$

moreover

$$\Big\|\mathscr{L}; \big(W^{1,p}(\Omega)\big)'\Big\| = \left(\sum_{0\leqslant|\alpha|\leqslant 1} \|v_\alpha\|^q_{L_q(\Omega)}\right)^{\frac{1}{q}}.$$

Remark 3.1. According to [1, 3.10] every functional $\mathscr{L} \in \big(W^{1,p}(\Omega)\big)'$ is an extension to $W^{1,p}(\Omega)$ of a distribution $\mathscr{T} \in \mathscr{D}'(\Omega)$, defined by

$$\mathscr{T} = v_0 - \partial_x v_1 - \partial_y v_2,$$

i.e. for every $\phi \in \mathscr{D}(\Omega) \subset W^{1,p}(\Omega)$ we have

$$\mathscr{T}(\phi) = (v_0, \phi) + (v_1, \phi_x) + (v_2, \phi_y) = \mathscr{L}(\phi).$$

Therefore $\mathscr{L}$ is an extension of $\mathscr{T}$. But $\mathscr{T}$ possesses (possibly non-unique) continuous extensions to $W^{1,p}(\Omega)$. However, $\mathscr{T}$ has a unique continuous extension to $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ (see [1, 3.11 and 3.12]).

Remark 3.2. Unlike the above remark, the one-to-one correspondence

$$\mathscr{L} \leftrightarrow (v_0, v_1, v_2), \tag{13}$$

does not have distribution on the boundary $\partial\Omega$. For that reason, it is easy to use exactly the correspondence (13), because we consider $W^{1,p}(I^2)$, and not $W_0^{1,p}(I^2)$.

Corollary 3.3. *Let $(G_0, G_1, G_2) \in L_q\big(\Omega^{(3)}\big)$ and*

$$\mathscr{L}(u) = (G_0, u) + (G_1, u_x) + (G_2, u_y),$$

be a linear continuous functional on $W^{1,p}(\Omega)$ that vanishes for all $u \in W^{1,p}(\Omega)$. Then

$$G_0 = 0,\ G_1 = 0,\ G_2 = 0 \quad \textit{almost everywhere in } \Omega.$$

The proof follows from the fact that the norm of $\mathscr{L}$ vanishes, when expressed by G_0, G_1, G_2.

We can now define the restriction to $\Omega_0 \subset \Omega$ of $\mathscr{L} \in \big(W^{1,p}(\Omega)\big)'$. There exists a unique $g = (g_0, g_1, g_2) \in L_q\big(\Omega^{(3)}\big)$, such that

$$\mathscr{L}(u) = (g_0, u) + (g_1, u_x) + (g_2, u_y). \tag{14}$$

On the other hand, the restriction $\mathscr{L}|_{W_0^{1,p}(\Omega)}$ uniquely defines the distribution $\mathscr{T}$, which is represented by g and (14). The restriction

$$\widetilde{\mathscr{T}} = \mathscr{T}|_{\Omega_0},$$

is well defined and for an appropriate $\overline{g} = (\overline{g_0}, \overline{g_1}, \overline{g_2})$ in $L_q\big(\Omega_0^{(3)}\big)$ it corresponds (precisely of the extension $\widetilde{\widetilde{\mathscr{T}}}$ to an element in $\big(W_0^{1,p}(\Omega_0)\big)'$) to the representation (14):

$$\widetilde{\widetilde{\mathscr{T}}}(\phi) = (\overline{g_0}, \phi) + (\overline{g_1}, \phi_x) + (\overline{g_2}, \phi_y),$$

for every $\phi \in W^{1,p}(\Omega_0)$. But for such a ϕ the expression (14) is well defined (over Ω_0). Therefore we should determine that

$$(g_0, g_1, g_2)|_{\Omega_0} = (\overline{g_0}, \overline{g_1}, \overline{g_2}).$$

The last is true due to Corollary 3.3. This leads to the following

Definition 3.4. *Let $\Omega_0 \subset \Omega$ and $\mathscr{L} \in \big(W^{1,p}(\Omega)\big)'$ be a linear functional with unique corresponding $g \in L_q\big(\Omega^{(3)}\big)$ by (14). Then restriction of $\mathscr{L}$ to Ω_0 is the unique functional*

$$\mathscr{L}|_{\Omega_0} \in \big(W^{1,p}(\Omega_0)\big)',$$

defined by

$$g|_{\Omega_0} \equiv (g_0, g_1, g_2)|_{\Omega_0},$$

by means of

$$\mathscr{L}|_{\Omega_0}(w) = (g|_{\Omega_0}, w) = (g_0|_{\Omega_0}, w) + (g_1|_{\Omega_0}, w_x) + (g_2|_{\Omega_0}, w_y), \tag{15}$$

for every $w \in W^{1,p}(\Omega_0)$.

We use notations from [1] and [3] and we will cite in detail only facts we need for Theorem 2.

Further, in our considerations, the notion *Fourier multiplier on L_p* (see [3, Definition 6.1.1]) plays an important role as well as *the Mihlin multiplier theorem* (see [3, Theorem 6.1.6]).

Taken $\rho \in \mathscr{S}'$, ρ is called a Fourier multiplier on L_p if $(\mathscr{F}^{-1}\rho) * f \in L_p$ for all $f \in \mathscr{S}$, and if $\sup\limits_{\|f\|_{L_p}=1} \|(\mathscr{F}^{-1}\rho) * f\|_{L_p}$ is finite. The linear space of all such ρ is denoted by M_p, the norm on M_p is the above supremum.

For convenience, we list some of multipliers of M_p needed in Section 5. We define ([3], see the proof of Theorem 6.2.3.) the supporting functions:

1) $\chi \in C_0^\infty(\mathbb{R})$, which is non-negative and $\chi(x) = \begin{cases} 1, & \text{for } |x| > 2, \\ 0, & \text{for } |x| < 1. \end{cases}$

2) $\psi \in C_0^\infty(\mathbb{R}^2)$, such that $\operatorname{supp}\psi \subset \big\{(x,y) \in \mathbb{R}^2 \,\big|\, |x| \leqslant 2,\ |y| \leqslant 2\big\}$ and $\psi = 1$ on $\big\{(x,y) \in \mathbb{R}^2 \,\big|\, |x| \leqslant 1,\ |y| \leqslant 1\big\}$.

3) Functions $\chi_1(x,y)$ and $\overline{\chi}_1(x,y)$ of $C^\infty(\mathbb{R}^2)$, which are identically equal to 1 in a cone neighbourhood of rays $(0,\pm1)$ and $(\pm1,0)$, respectively, with supports slightly larger cone neighbourhoods of the corresponding rays not containing neighbourhood of $(0,0)$.

4) Finally, we put

$$\chi_2(x,y) = 1 - \chi_1(x,y) - \psi(x,y),$$
$$\overline{\chi}_2(x,y) = 1 - \overline{\chi}_1(x,y) - \psi(x,y),$$

which obviously belong to $C^\infty(\mathbb{R}^2)$.

Now we are ready to list the necessary multipliers of M_p:

$$\rho_1(x,y) = \frac{\big(1 + x^2 + y^2\big)^{\frac{1}{2}}}{1 + \chi(x)|x| + \chi(y)|y|}, \tag{16}$$

$$\rho_2(x,y) = \frac{\chi(x)|x|}{x},\ \rho_3(x,y) = \frac{\chi(y)|y|}{y}. \tag{17}$$

The function

$$\rho_4(x,y) = \chi(y)\chi_2(x,y)\frac{|y|}{x}, \tag{18}$$

with support $\operatorname{supp}\chi_2 \subset \big\{(x,y) \in \mathbb{R}^2 \,\big|\, |x| \leqslant C|y|\big\}$, where $C > 0$, is an arbitrarily large number, also is a multiplier.

The last statement is obtained by the following two steps:
Step I) The function $\chi_2(\xi,\eta)$ is a multiplier and satisfies

$$\left\|\mathscr{F}^{-1}\big(\chi_2(\xi,\eta)\cdot \widehat{f}(\xi,\eta)\big)\right\|_{L_p(\mathbb{R}^2)} \leqslant \text{const}\, \|f\|_{L_p(\mathbb{R}^2)}.$$

Step II) We define the function g by $\widehat{g} = \chi_2\widehat{f}$. As $(0,0) \notin \operatorname{supp}\widehat{g}$, the function $\dfrac{\chi(\eta)|\eta|}{\xi}$ is a multiplier on $\operatorname{supp}\widehat{g}$, i.e.

$$\left\|\mathscr{F}^{-1}\Big(\frac{\chi(\eta)|\eta|}{\xi}\cdot \widehat{g}(\xi,\eta)\Big)\right\|_{L_p(\mathbb{R}^2)} \leqslant \text{const}\, \|g\|_{L_p(\mathbb{R}^2)}.$$

Finally, we apply *Step I)*.

Analogously,

$$\rho_5(x,y) = \chi(x)\chi_2(x,y)\frac{|x|}{y},$$

is a multiplier.

We complete the section with a proof of the following *a priori estimate*:

Theorem 3.5. *Let $C \in W^{1,p}(I^2)$, $p > 2$, be a solution of the following problem*

$$\partial_{uv}C = f(u,v), \ \ (u,v) \in I^2, \tag{19}$$

where $f \in L_p(I^2)$ and the above equality attained only in a weak sense, i.e.

$$\big(C_{uv},\varphi\big) = (f,\varphi),$$

for all $\varphi \in \widetilde{W}^{1,p}(I^2) = \big\{w \in W^{1,p}(I^2)\,\big|\, w|_{u=0} = w|_{v=0} = 0\big\}$. Also, let

$$\begin{cases} C(0,v) = 0 = C(u,0) \\ C(u,1) = u, \ C(1,v) = v \end{cases}, \tag{20}$$

where $u,v \in I$.

Then there exists a constant M, which does not depend on f, such that

$$\|C\|_{W^{1,p}(I^2)} \leqslant M\, \|f\|_{L_p(I^2)}.$$

Corollary 3.6. *The solution of the problem* (19), (20) *is unique.*

Remark 3.7. When the solution $C \in W^{1,p}(I^2)$ and $C_{uv} \in W^{-1,p}(I^2)$, the uniqueness theorem still holds, since the right side of the equation becomes $f = 0 \in L_p(I^2)$.

Proof of Theorem 3.5. Let $\widetilde{C} \in W^{1,p}(\Omega)$ be a continuous extension of C to a domain $\Omega \Supset I^2$ defined by:

$$\widetilde{C}(u,v) = \begin{cases} 0, & \text{when } u < 0 \text{ or } v < 0 \\ u, & \text{when } u \in I \text{ and } v > 1 \\ v, & \text{when } u > 1 \text{ and } v \in I \\ 1, & \text{when } u > 1 \text{ and } v > 1 \end{cases}.$$

Now, for (u,v) in a neighbourhood of I^2, define the regularization

$$z_\varepsilon(u,v) = (\widetilde{C})_\varepsilon(u,v),\ \varepsilon > 0,$$

using the modified mollifier ω_ε from the beginning of this section.

Therefore, as we are working with continuous functions

$$\begin{aligned}\int\limits_{I^2} (z_\varepsilon)_{uv} \Big\{ p(1-v).\big[(z_\varepsilon)_u\big]^{p-1} + p(1-u)\big[(z_\varepsilon)_v\big]^{p-1} \Big\}\, du\, dv \\ = \int\limits_{I^2} \Big\{ (1-v).\big(\big[(z_\varepsilon)_u\big]^p\big)_v + (1-u)\big(\big[(z_\varepsilon)_v\big]^p\big)_u \Big\}\, du\, dv,\end{aligned}$$

which by Gauss theorem is equal to

$$\begin{aligned}\int\limits_{\partial I^2} \Big\{ (1-v).\big[(z_\varepsilon)_u\big]^p.n_v + (1-u)\big[(z_\varepsilon)_v\big]^p.n_u \Big\}\, ds \\ + \int\limits_{I^2} \Big\{ \big[(z_\varepsilon)_u\big]^p + \big[(z_\varepsilon)_v\big]^p \Big\}\, du\, dv,\end{aligned}$$

here ds is the elementary arc length of ∂I^2 and $n(n_u, n_v)$ is the outward pointing unit vector normal to ∂I^2.

We immediately see that the boundary integral vanishes (for $u = 0$ and $v = 0$ use (9)). Therefore, as $\max\limits_{x \in I}(1-x) = 1$,

$$\int\limits_{I^2} \Big\{ \big[(z_\varepsilon)_u\big]^p + \big[(z_\varepsilon)_v\big]^p \Big\}\, du\, dv$$

$$\leqslant p\Big(\iint\limits_{I^2} \big|(z_\varepsilon)_{uv}\big|^p du\,dv\Big)^{\frac{1}{p}} \cdot \Big\{\iint\limits_{I^2} \big[\big|(z_\varepsilon)_u\big|^{p-1} + \big|(z_\varepsilon)_v\big|^{p-1}\big]^q du\,dv\Big\}^{\frac{1}{q}}$$

$$\leqslant 2p\Big(\iint\limits_{I^2} \big|(z_\varepsilon)_{uv}\big|^p du\,dv\Big)^{\frac{1}{p}} \cdot \Big\{\iint\limits_{I^2} \big[\big|(z_\varepsilon)_u\big|^{(p-1)q} + \big|(z_\varepsilon)_v\big|^{(p-1)q}\big] du\,dv\Big\}^{\frac{1}{q}},$$

since $\big(|A| + |B|\big)^q \leqslant 2^q\big(|A|^q + |B|^q\big)$, for any $A, B \in \mathbb{R}$. Finally:

$$\iint\limits_{I^2} \Big\{\big[(z_\varepsilon)_u\big]^p + \big[(z_\varepsilon)_v\big]^p\Big\}\, du\,dv$$

$$\leqslant 2p\,\big\|(z_\varepsilon)_{uv}\big\|_{L_p(I^2)} \cdot \Big\{\iint\limits_{I^2} \big[\big|(z_\varepsilon)_u\big|^p + \big|(z_\varepsilon)_v\big|^p\big] du\,dv\Big\}^{\frac{1}{q}},$$

as $p = q(p-1)$. Therefore

$$\iint\limits_{I^2} \Big\{\big[(z_\varepsilon)_u\big]^p + \big[(z_\varepsilon)_v\big]^p\Big\}^{\frac{1}{p}}\, du\,dv \leqslant 2p\,\big\|(z_\varepsilon)_{uv}\big\|_{L_p(I^2)},$$

and by *Fridrix inequality* (see [1, 6.30]), we obtain

$$\text{const} \cdot \|z_\varepsilon\|_{W^{1,p}(I^2)} \leqslant 2p\,\big\|(z_\varepsilon)_{uv}\big\|_{L_p(I^2)}.$$

What is left is to take the limit as $\varepsilon \to 0^+$. Let $\varepsilon_n = \dfrac{1}{2n}, n \in \mathbb{N}$, and $B_n = \Big[\dfrac{1}{n}, 1 - \dfrac{1}{n}\Big] \times \Big[\dfrac{1}{n}, 1 - \dfrac{1}{n}\Big] \subset I^2$. Certainly, when $n \to \infty$ we have

$$\big\|z_{\varepsilon_n}\big\|_{W^{1,p}(I^2)} \longrightarrow \|C\|_{W^{1,p}(I^2)},$$

by [1, Lemma 3.16]. According to the same lemma:

$$\big\|(z_\varepsilon)_{uv}\big\|_{L_p(I^2)} = \big\|\big(\widetilde{C}_{uv}\big)_{\varepsilon_n}\big\|_{L_p(I^2)}$$

$$= \lim_{n\to\infty}\Big(\iint\limits_{B_n} \big|(C_{uv})_{\varepsilon_n}\big|^p\, du\,dv\Big)^{\frac{1}{p}} = \lim_{n\to\infty}\Big(\iint\limits_{B_n} \big|f_{\varepsilon_n}\big|^p\, du\,dv\Big)^{\frac{1}{p}}$$

$$\leqslant \lim_{n\to\infty}\Big(\iint\limits_{B_n} |f|^p\, du\,dv\Big)^{\frac{1}{p}} = \|f\|_{L_p(I^2)},$$

by [1, 2.29], which concludes the proof of the theorem. □

4. Smooth case. In Vladimirov [19, §15] the Goursat problem for the hyperbolic equation $\partial_{uv}W + a\partial_u W + b\partial_u W + cW = f(u,v)$ is solved, with the initial data being prescribed differentiable functions on the characteristics $\{(u,v) \in \mathbb{R}^2 \,|\, u = 0\}$ and $\{(u,v) \in \mathbb{R}^2 \,|\, v = 0\}$ and continuous right hand side $f(u,v)$. The existence and uniqueness of the C^1-solution is proved for any rectangle with two sides being the mentioned characteristics with smooth initial data on them.

In this section we relax the smoothness conditions for f, but retain the concept of classic derivative (in the sense of almost everywhere).

We will determine the existence of a solution of the problem (19)-(20), where the function $f(u,v)$ in the right hand side of (19) satisfies the following conditions

$$f \in L_p(I^2),\ p \in (1, +\infty); \tag{21}$$

$$f(u,v) \geqslant 0,\ \text{for all } (u,v) \in I^2; \tag{22}$$

$$\int\limits_{B_{u,1}} f(\xi,\eta) d\xi\, d\eta = u,\ \text{for all } u \in [0,1],\ \text{where } B_{u,1} = [0,u] \times [0,1]; \tag{23a}$$

$$\int\limits_{B_{1,v}} f(\xi,\eta) d\xi\, d\eta = v,\ \text{for all } v \in [0,1],\ \text{where } B_{1,v} = [0,1] \times [0,v]. \tag{23b}$$

Then there exists a solution $C(u,v) \in W^{1,p}(I^2)$ of the problem (19)–(20). It is unique by Corollary 3.6.

To obtain the above assertion, let us first consider the case when $f \in C^0(I^2)$. By substituting

$$C(u,v) = \int\limits_{B_{u,v}} f(\xi,\eta) d\xi\, d\eta,$$

where $B_{u,v} = [0,u] \times [0,v]$, $(u,v) \in I^2$, immediately (19) is satisfied together with the first row of equalities in (20). The remaining two ones follow by (23a) (23b).

In general, when $f \in L_p(I^2)$, the condition (19) is satisfied almost everywhere. Indeed, according to the Embedding Theorem (see [1, Theorem 2.14]) for L_p spaces, the function f belongs to $L_1(I^2)$. Taking into account condition (22) and Fubini's theorem (see [1, Theorem 1.54]) the functions $f(\cdot,v)$ of u and $f(u,\cdot)$ of v belong to $L_1[0,1]$ for almost all $v \in [0,1]$ and $u \in [0,1]$,respectively. Now, according to [16, Chapter IX, §4, Theorem 2] the derivative of

$$\int_0^u f(\xi, v)\, d\xi,$$

is equal to $f(u,v)$ for almost all $u \in I$.

Example 4.1. The copulas with quadratic sections in, say u, have the representation $C(u,v) = uv + \psi(v)u(1-u)$, where ψ is a function such that C is 2-increasing and $\psi(0) = \psi(1) = 0$ (see [17]).

If we suppose that C is symmetric, then $C(u,v) = uv + \theta u(1-u)v(1-v)$, where θ is parameter. Now, $C_{uv} = 1 + \theta(1-2u)(1-2v)$ and it is easy to check that $C_{uv} \geqslant 0$ in I^2 (and hence C is a copula) if and only if $\theta \in [-1,1]$. This family is known as the Farlie-Gumbel-Morgenstern family.

If C is non-symmetric a number of cases there exists. Let us suppose that ψ is a continuous and partially differentiable function. Then, using our approach, it is easy to verify the results from [17, Example 3.25]. For example, the function $\psi(v) = \dfrac{\theta}{k\pi}\sin(k\pi v)$ gives copula if and only if $\theta \in [-1,1]$.

5. Generalized solution. In this section we extend the notion of solution of the problem (19)–(20) by using the concept of a weak derivative (i.e. derivative in the sense of distribution theory). We prove the existence of $W^{1,p}(I^2)$-solution of this problem in the case when f belongs to $W^{-1,p}(I^2)$ and satisfies (respectively modified) conditions (22), (23a), (23b) together with additional condition for its Fourier transform $\widehat{f}$. The solution is unique due to the uniqueness Theorem 3.5.

As we assumed $p > 2$, the solution is continuous according to the embedding theorem (see [1, Theorem (2.14)]) and conditions (20) make sense. The condition (22) now is in the form

$$(f, \varphi) \geqslant 0, \ \text{for all } \varphi \in \mathscr{D}(I^2), \varphi \geqslant 0,$$

and guarantees that the solution C will be 2-increasing function. Furthermore, as per the boundary conditions (20) it is a copula.

In order to specify the corresponding conditions of (23a), (23b) let us extend the function f to a function $\widetilde{f} \in \big(W^{1,p}(\mathbb{R}^2)\big)'$. For this purpose, let us extend $f_0, f_1, f_2 \in L_q(I^2)$, which are the functions corresponding to f by the formulae (13), by setting them to 0 outside I^2.

The regularization defined through

$$\big(\widetilde{f}\,\big)_\varepsilon(u,v) = \big(\widetilde{f} * J_\varepsilon\big)(u,v),$$

belongs to $C_0^\infty(\mathbb{R}^2)$. Then the integral of (23a) is approximated by

$$\lim_{\varepsilon\to 0^+}\int\limits_{B_{u,1}} \big(\widetilde{f}\,\big)_\varepsilon(u,v)\,du\,dv,$$

or by

$$\lim_{\varepsilon\to 0^+}\int\limits_{\mathbb{R}^2} \big(\widetilde{f}\,\big)_\epsilon(u,v)\;\cdot\chi_{B_{u,1}}\,du\,dv,$$

where $\chi_{B_{u,1}}$ is the characteristic function of the rectangle $B_{u,1}$. The last integral has the representation

$$\begin{aligned}\Big(\chi_{B_{u,1}},\big(\widetilde{f}\,\big)_\varepsilon\Big) &= \Big(\chi_{B_{u,1}} * \big(\check{\widetilde{f}}\,\big)_\varepsilon\Big)(0,0) = \Big(\chi_{B_{u,1}} * \big(\check{\widetilde{f}} * J_\varepsilon\big)\Big)(0,0)\\ &= \Big(\check{\widetilde{f}} * \big(\chi_{B_{u,1}} * J_\varepsilon\big)\Big)(0,0) = \Big(\check{\widetilde{f}},\big(\check{\chi}_{B_{u,1}}\big)_\varepsilon\Big)(0,0)\\ &= \Big(\widetilde{f},\big(\chi_{B_{u,1}}\big)_\varepsilon\Big),\end{aligned}$$

by $\check{\widetilde{f}}(\varphi) = \widetilde{f}(\check{\varphi})$ and $\check{\check{\varphi}} = \varphi$ for continuous functions φ.

Thus the required conditions instead of (23a) and (23b) read

$$\lim_{\varepsilon\to 0^+}\Big(\widetilde{f},\chi_{B_{u,1}} * J_\varepsilon\Big) = u,\ \text{for all } u\in I, \tag{24a}$$

$$\lim_{\varepsilon\to 0^+}\Big(\widetilde{f},\chi_{B_{1,v}} * J_\varepsilon\Big) = v,\ \text{for all } v\in I. \tag{24b}$$

We will prove the following theorem:

Theorem 5.1. *Let $f\in W^{-1,p}(I^2)$, $p>2$ and $f\geqslant 0$ in a weak sense. Suppose the conditions* (24a) *and* (24b) *are satisfied. Then there exists a unique solution $C\in W^{1,p}(I^2)$ of the problem:*

$$C_{uv}(u,v) = f(u,v) \text{ in } I^2 \textit{ (in a weak sense)};$$

$$C(u,0) = 0 = C(0,v);$$

$$C(u,1) = u,\ C(1,v) = v,\ \textit{for all } u,v\in I,$$

under the conditions

$$\left\| \mathscr{F}^{-1}\left\{ \frac{\chi_1(\xi,\eta)|\eta|}{\xi} \cdot \frac{\widehat{f}(\xi,\eta)}{(1+\xi^2+\eta^2)^{\frac{1}{2}}} \right\} \right\|_{L_p} < +\infty, \tag{25a}$$

$$\left\| \mathscr{F}^{-1}\left\{ \frac{\overline{\chi}_1(\xi,\eta)|\xi|}{\eta} \cdot \frac{\widehat{f}(\xi,\eta)}{(1+\xi^2+\eta^2)^{\frac{1}{2}}} \right\} \right\|_{L_p} < +\infty, \tag{25b}$$

where χ_1 and $\overline{\chi}_1$ are smooth regularization functions with the following properties:

a) $\operatorname{supp} \chi_1 \subset \{$*cone neighbourhood of* $(0, \pm 1)\} \backslash \{$*neighbourhood of* $(0,0)\}$*;*

b) $\operatorname{supp} \overline{\chi}_1 \subset \{$*cone neighbourhood of* $(\pm 1, 0)\} \backslash \{$*neighbourhood of* $(0,0)\}$.

As two of the boundary conditions are obtained from the right hand side of the equation we will focus on the following Goursat problem in a weak sense:

$$\begin{cases} \text{Find a solution } h \in W_0^{1,p}(K_1), \text{such that } h_{uv} = f \in W^{-1,p}(\mathbb{R}^2), \\ \text{in a weak sense, i.e. } (h_{uv}, \varphi) = (f, \varphi), \text{ for all } \varphi \in \mathscr{D}(K_1), \\ \text{where } \operatorname{supp} f \text{ is bounded and } K_1 = \big\{(u,v) \in \mathbb{R}^2 \mid u > 0,\ v > 0\big\}. \end{cases} \tag{26}$$

Let us recall, that the requirements for f assume the existence of the functions $f_0, f_1, f_2 \in L_p(K_1)$, such that

$$(f, \varphi) = \int\limits_{\mathbb{R}^2} \big(f_0 \varphi + f_1 \varphi_u + f_2 \varphi_v\big) du\, dv,$$

for all $\varphi \in W^{1,q}(\mathbb{R}^2)$.

In fact, as we are interested in the case when $f \in W^{-1,p}(I^2)$, we assume that $\operatorname{supp} f$ (i.e. $\operatorname{supp} f_i$, $i = 0,1,2$) is contained within the square

$$Q_M = \big\{(u,v) \in K_1 \,\big|\, 0 \leqslant u \leqslant M,\ 0 \leqslant v \leqslant M\big\},$$

where $M > 0$ is a constant. When f is defined only on $I^2 = Q_1$, we assume that f is extended as 0 on K_1 (or on $\mathbb{R}^2$), i.e. f_i are extended as 0 outside I^2. Certainly

$$\big|(f, \varphi)\big| \leqslant \text{const}\, \|\varphi\|_{W^{1,q}(K_1)}, \tag{27}$$

where the above constant depends on the L_p-norm of f_i, $i = 0,1,2$.

Let us fix $\mu \in C_0^\infty(\mathbb{R}^2)$, such as $\mu \equiv 1$ on the square

$$\Pi_M = \big\{(u,v) \in \mathbb{R}^2 \,\big|\, |u| \leqslant M,\ |v| \leqslant M\big\},$$

and $\operatorname{supp}\mu \subset \Pi_{M+1},\ 0 \leqslant \mu \leqslant 1$.

Let $\widetilde{h}$ be the zero extension of $h \in W_0^{1,p}(K_1)$ on $\mathbb{R}^2 \backslash K_1$, i.e.

$$\widetilde{h} = \begin{cases} h, & (u,v) \in K_1, \\ 0, & (u,v) \in \mathbb{R}^2 \backslash K_1. \end{cases}$$

Remark 5.2. Next step in our considerations is to give analogous formulation of the problem (26) on $\mathbb{R}^2$, not only on K_1. Thus Fourier transforms and corresponding Sobolev spaces are applicable.

In this remark we observe several facts which motivate us to give the next definition.

By [1, Lemma 3.27], if $h \in W_0^{1,p}(K_1)$, then $\widetilde{h} \in W_0^{1,p}(\mathbb{R}^2) = W^{1,p}(\mathbb{R}^2)$.

Let us observe the special case when $h \in C^1(\overline{K_1})$ and in addition

$$\text{either } h_u\big|_{\Gamma_2} = 0, \ \text{ or } h_v\big|_{\Gamma_1} = 0,$$

where with $\Gamma_i,\ i = 1, 2$, we denote the sides of K_1, i.e.

$$\Gamma_1 \subset \big\{(u,v) \in \mathbb{R}^2 \,\big|\, u = 0\big\}, \ \ \Gamma_2 \subset \big\{(u,v) \in \mathbb{R}^2 \,\big|\, v = 0\big\}.$$

Then the outward pointing unit normal vectors to Γ_1 and Γ_2 are $n = (-1, 0)$ and $\widehat{n} = (0, -1)$, respectively. We find that the derivative h_{uv} (in a weak sense) coincides with the derivative $(\widetilde{h})_{uv}$. In other words, h_{uv} is defined on restrictions of $C_0^\infty(\mathbb{R}^2)$ on K_1.

Indeed, let us take $\psi \in C_0^\infty(\mathbb{R}^2)$. Then

$$\int\limits_{K_1} h_{uv}\, \psi \, du\, dv = \big((\widetilde{h})_{uv}, \psi\big).$$

Having in mind Remark 5.2, we replace the formulation (26) by a stronger one on $\mathbb{R}^2$. To emphasize that this is not the equivalent formulation but from it follows the previous one (26). It is obtained by substituting $\widetilde{h}$ with H. Thus we have to deal with the following problem:

$$\text{(28)} \qquad \begin{cases} \text{Find a solution } H \in W^{1,p}(\mathbb{R}^2),\ p > 2,\ \operatorname{supp} H \subset \overline{K_1}, \\ \text{such that } (H, \varphi_{uv}) = (f, \varphi), \ \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2). \end{cases}$$

Indeed, in (28) we can take any $\varphi \in W_0^{1,q}(\mathbb{R}^2) = W^{1,q}(\mathbb{R}^2)$. The advantage of the new formulation is obvious: if we find such a H, then his traces vanish on ∂K_1, ant hence this will be the wanted solution h.

Definition 5.3. *The weak solution of problem* (28) *is a function* $H \in L_p(\mathbb{R}^2)$, *with* $\operatorname{supp} H \subset \overline{K_1}$, *such that*

$$(H, \varphi_{uv}) = (f, \varphi), \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2).$$

To show the existence of a weak solution H of (28) we follow the procedure based on Hahn-Banach theorem (see [15, §4.2]). Note that such a solution is not unique.

Let T denote the image of $C_0^\infty(\mathbb{R}^2)$ under the operator $\Phi = \partial_{uv}$, i.e.

$$\Phi\big(C_0^\infty(\mathbb{R}^2)\big) = T \subset C_0^\infty(\mathbb{R}^2).$$

Therefore, for all $\psi \in T$ there exists a unique function $\varphi = \Phi^{-1}(\psi) \in C_0^\infty(\mathbb{R}^2)$. Let $S \subset L_q(K_1)$ denote the set of all restrictions of elements of T on $\overline{K_1}$. Let us define a continuous, linear functional over S (which elements we denote again by ψ) given by

$$\Psi_f(\psi) = \Big(f, \mu(u,v) \int_u^{M+1}\int_v^{M+1} \psi(\xi,\eta)\, d\xi\, d\eta\Big).$$

Recall that $f = f\big|_{K_1}$ as $\operatorname{supp} f \subset \overline{K_1}$. By (27),

$$\big|\Psi_f(\psi)\big| \leqslant \operatorname{const}(f)\, \|\rho\|_{W^{1,q}(K_1)}, \tag{29}$$

where $\rho = \mu(u,v) \int_u^{M+1}\int_v^{M+1} \psi(\xi,\eta)\, d\xi\, d\eta$.

Since we have,

$$\|\rho\|^q_{W^{1,q}(K_1)} = \|\rho\|^q_{L_q(K_1)} + \|\partial_u\rho\|^q_{L_q(K_1)} + \|\partial_v\rho\|^q_{L_q(K_1)},$$

we estimate consecutively each term of the above equality. Therefore, by (29),

$$\big|\Psi_f(\psi)\big| \leqslant \operatorname{const}(f)\, \|\psi\|_{L_q(K_1)}.$$

Extend Ψ_f on $L_q(K_1)$ without increasing its norm. By [1, Theorem. 2.44], there exists $H \in L_p(K_1)$, such that

$$\Psi_f(\psi) = \int_{K_1} H(u,v)\cdot\psi(u,v)\, du\, dv, \text{ for all } \psi \in L_q(K_1). \tag{30}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be an arbitrary function and let $\psi = \Phi(\varphi) = \varphi_{uv}$. Then (30) has the representation

$$\left(f, \mu(u,v) \int_u^{M+1} \int_v^{M+1} \varphi_{uv}(\xi,\eta)\, d\xi\, d\eta\right) = \int_{K_1} H(u,v) \cdot \varphi_{uv}(u,v)\, du\, dv.$$

The proof for the existence of a weak solution is completed since the left hand side is equal to (f,φ).

Let

$$\widetilde{H}(u,v) = \begin{cases} H(u,v), & \text{for } (u,v) \in K_1, \\ 0, & \text{for } (u,v) \in \mathbb{R}^2 \backslash K_1. \end{cases}$$

This function belongs to $L_p(\mathbb{R}^2)$ and

$$(31) \qquad \big(\widetilde{H}, \varphi_{uv}\big) = (f,\varphi), \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2).$$

To complete the proof of Theorem 5.1 we need to determine that $\widetilde{H} \in W^{1,p}(\mathbb{R}^2)$.

Remark 5.4. If $\widetilde{H} \in W^{1,p}(\mathbb{R}^2)$, the trace of $\widetilde{H}$ on $\{u = 0\} \cup \{v = 0\}$ vanishes and

$$\big(H, \varphi_{uv}\big) = \big(\widetilde{H}, \varphi_{uv}\big) = (f,\varphi), \text{ for all } \varphi \in C_0^\infty(K_1),$$

therefore this is the wanted generalized solution of the Goursat problem.

By (31) and [11, Theorem 1.7.5] it follows that

$$(32) \qquad \xi\eta\widehat{\widetilde{H}}(\xi,\eta) = \widehat{f}(\xi,\eta),$$

for $\varphi(u,v) = e^{-iu\xi - iv\eta}$. Thus we have

$$\begin{aligned} \|\widetilde{H}_v\|_{L_p} &= \big\|\mathscr{F}^{-1}(\eta\widehat{\widetilde{H}})\big\|_{L_p} \\ &= \left\|\mathscr{F}^{-1}\left[\frac{(1+\xi^2+\eta^2)^{\frac{1}{2}}}{1+\chi(\xi)|\xi|+\chi(\eta)|\eta|} \cdot \big(1+\chi(\xi)|\xi|+\chi(\eta)|\eta|\big)\frac{\eta\widehat{\widetilde{H}}}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\right]\right\|_{L_p} \\ &\leqslant \left\|\mathscr{F}^{-1}\left(\frac{\eta}{(1+\xi^2+\eta^2)^{\frac{1}{2}}} \cdot \widehat{\widetilde{H}}\right)\right\|_{L_p} + \left\|\mathscr{F}^{-1}\left(\frac{\chi(\xi)|\xi|}{\xi} \cdot \frac{\xi\eta \cdot \widehat{\widetilde{H}}}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\right)\right\|_{L_p} \end{aligned}$$

$$+\, C\left\| \mathscr{F}^{-1}\left(\chi(\eta)[\psi(\xi,\eta)+\chi_1(\xi,\eta)+\chi_2(\xi,\eta)]\cdot \frac{|\eta|\eta\cdot \widehat{\widetilde{H}}}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\right)\right\|_{L_p},$$

where we used that (16) and (17) are multipliers.

Triangle inequality shows that

$$\begin{aligned}
&C\left\| \mathscr{F}^{-1}\left(\chi(\eta)[\psi(\xi,\eta)+\chi_1(\xi,\eta)+\chi_2(\xi,\eta)]\cdot \frac{|\eta|\eta\cdot \widehat{\widetilde{H}}}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\right)\right\|_{L_p}\\
&\leqslant \left\| \mathscr{F}^{-1}\left(\psi(\xi,\eta)\frac{\chi(\eta)|\eta|\eta}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\widehat{\widetilde{H}}\right)\right\|_{L_p}\\
&+\left\| \mathscr{F}^{-1}\left(\chi(\eta)\chi_1(\xi,\eta)\frac{|\eta|}{\xi}\frac{\xi\eta}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\widehat{\widetilde{H}}\right)\right\|_{L_p}\\
&+\left\| \mathscr{F}^{-1}\left(\chi(\eta)\chi_2(\xi,\eta)\frac{|\eta|}{\xi}\frac{\xi\eta}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\widehat{\widetilde{H}}\right)\right\|_{L_p}.
\end{aligned}$$

In the first norm the factor of $\widehat{\widetilde{H}}$ is a multiplier as $\operatorname{supp}\psi$ is bounded and $\chi(\eta)|\eta|$ is a smooth function. For the second norm we apply (25a) as $\chi(\eta)\chi_1(\xi,\eta)=\chi_1(\xi,\eta)$. The last norm is estimated by $\|f\|_{W^{-1,p}(K_1)}$ by (18).

Thus Theorem 5.1 is proved.

6. Example: Sklar theorem. We will use a simple example to show how our approach works.

Let $H(x,y)$ be a joint distribution function of two random variables X and Y, with margins F and G, respectively, given by

$$F(x)=H(x,\infty),\ G(y)=H(\infty,y).$$

We assume that H is 2-increasing in the extended plane $[-\infty,\infty]\times[-\infty,\infty]$ and such that

$$H(x,-\infty)=H(-\infty,y)=0,\ H(+\infty,+\infty)=1.$$

Let the corresponding probability density functions, given by

$$h(x,y)=\frac{\partial^2 H}{\partial x\partial y},\ f(x)=\frac{d}{dx}F(x),\ g(y)=\frac{d}{dy}G(y),$$

be continuous. Moreover, if $f(x)$ and $g(y)$ do not vanish, then the functions $u = F(x)$ and $v = G(y)$ are invertible and their inverse functions are once differentiable. Let us denote by $x = F^{-1}(u)$ and $y = G^{-1}(v)$, $u, v \in I$, the corresponding inverses. Then the continuous function defined on I^2 and given by

$$p(u,v) = h\big(F^{-1}(u), G^{-1}(v)\big)\,\frac{1}{f\big(F^{-1}(u)\big)}\frac{1}{g\big(G^{-1}(v)\big)},$$

allows us to determine a copula $C(u,v)$ as a solution in I^2 of the problem

$$\left|\begin{array}{l} C_{uv}\,(u,v) = p(u,v) \\ C(0,v) = C(u,0) = 0 \\ C(1,v) = v,\ C(u,1) = u. \end{array}\right.$$

Since we have

$$C(u,v) = \int_0^u \int_0^v p(\xi,\eta)\, d\xi\, d\eta,$$

by applying the substitutions to the above integral

$$\begin{aligned} \xi &= u = F(x), \\ \eta &= v = G(y), \end{aligned}$$

we obtain

$$\begin{aligned} C\big(F(x), G(y)\big) &= \int_{-\infty}^{x} \int_{-\infty}^{y} h(x,y)\,\frac{1}{f(x)}\,\frac{1}{g(y)}\,\left|\,\begin{vmatrix} f(x) & 0 \\ 0 & g(y) \end{vmatrix}\,\right| dx\, dy = \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} h(x,y)\, dx\, dy = H(x,y). \end{aligned}$$

The above expression defines a copula as in the Sklar theorem. Choosing different functions h allows us to generate a variety of copulas not observed in [17].

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