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MATRIX COEFFICIENTS OF THE IRREDUCIBLE UNITARY REPRESENTATION OF $SU(n,1)$

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ABSTRACT. This paper is devoted to presenting an explicit expression for the A-radial part of matrix coefficients of the irreducible unitary representations in terms of Gaussian hypergeometric series and some involved expressions of binomial coefficients

1. Introduction. Let G be the Lie group and (T,V) be a unitary representation of G on a complex Hilbert space V. Then, for arbitrary vectors e_n and e_m in an orthogonal basis $\{e_i\}$ of V, the following functions

$$t_{nm}: \quad G \longrightarrow \mathbb{C}$$

$$g \longmapsto \Big\langle T(g)e_n, e_m \Big\rangle$$

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are called matrix coefficients of the representation (T, V). They satisfy the Schur orthogonality relation

$$\left\langle t_{nm}, t_{kl} \right\rangle_{L^2(G)} = \begin{cases} 0, & e_n \neq e_k \\ \dfrac{1}{\dim(V)} \langle e_n, e_m \rangle \overline{\langle e_k, e_l \rangle}, & e_n = e_k. \end{cases}$$

The matrix coefficients of irreducible representations of finite groups and their profound role in harmonic analysis and physics are well known to any one who has any connection to harmonic analysis and physics. They play a prominent role in the representation theory of these groups as developed by Burnside, Frobenius, and Schur. Among the problems raised, there is the one related to finding explicit analytic formulas for them but this problem remained open except for certain cases.

Moreover, the beautiful formulas for the A-radial part of the the matrix coefficients for the higher rank Lie groups in the literature found by the specialists of representation theory are formulated unopened to avoid certain combinatorial complexity and the formulas are sometimes not effectively computable.

In this paper we want to have an explicit formula for the A-radial part of the matrix coefficients of the unitary irreducible representation $(T_h, \mathcal{H}_h)$ of $SU(n,1)$, $h \geq \dfrac{1}{2}$, where $\mathcal{H}_h$ is the weighted Bergmun space on the unit ball. We described them uniformly in terms of standard integrals with respect to the standard Cartan decomposition of the group $SU(n,1)$. More precisely, the main results of this paper can be stated as follow:

Let $\{\phi_p^h \mid p \in \mathbb{N}^n\}$ be an orthonormal basis of $\mathcal{H}_h$

$$\phi_p^h(z) = \Big[\frac{\Gamma(2h+|p|)}{p!\Gamma(2h)}\Big]^{\frac{1}{2}} z_1^{p_1} \cdots z_n^{p_n},\ p = (p_1, \ldots, p_n),\ p! = p_1! \cdots p_n! \quad \text{and } |p| = p_1 + \cdots + p_n.$$

According to the Cartan decomposition of $G = KAK$ (i.e. $g = k_1 a_t k_2$), the A-radial part of the matrix coefficient $t_{pq}^h(g) = \left\langle T_h(g)\phi_p^h, \phi_q^h \right\rangle_h$ is given by $t_{kl}^h(a_t) = \left\langle T_h(a_t)\phi_k^h, \phi_l^h \right\rangle_h$.

Theorem 1.1. *Let $h > \dfrac{1}{2}$, $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots, l_n)$. Then for $k_1 \geq l_1$, we have*

$$t^h_{kl}(a_t) = \frac{2(|l|+n)l!}{n}\frac{(-1)^{k_1-l_1}\Gamma(2h)}{\Gamma(2h+|h|+1)}\binom{k_1}{l_1}$$
$$\times \quad \cosh t^{-(2h+|k|)+k_1}\tanh t^{k_1-l_1}F(-l_1,2h+|k|,k_1-l_1+1;\tanh t^2),$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient and* $F(a,b,c;x)$ *is the classical Gauss hypergeomeric function* (*see* [2]).

For $l_1 > k_1$, we replace k_1 and l_1 by l_1 and k_1, respectively.

This paper is organized as follows:

In Section 2, we review the basis results on $SU(n,1)$. Section 3 is devoted to the proof of main result.

2. The group $SU(n,1)$ and its unitary irreducible representation. We review some basic definitions and known results of harmonic analysis on the unit ball B^n in $\mathbb{C}^n$ which will be needed in the sequel (refereing to [1] for more details on this subject). More precisely, we recall the Cartan decomposition of $SU(n,1)$. Also, we give an unitary irreductible representation of $SU(n,1)$.

Let $SU(n,1)$ be the group consisting of all matrices g in $SL(n+1,\mathbb{C})$ which leave invariant the quadratic form on $\mathbb{C}^{n+1}$

$$(z_1,\dots,z_{n+1}) \longrightarrow z_1^2+z_2^2+\cdots+z_n^2-z_{n+1}^2.$$

For any matrix g we denote by $g^* = \bar{g}^t$ its conjugate transpose. Then the group $SU(n,1)$ can be realized as

$$SU(n,1) = \Big\{g \in SL(n+1,\mathbb{C}) \mid g^*Jg = J\Big\}, \qquad J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus writing $g \in SU(n,1)$ as $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $\begin{cases} A^*A - C^*C = I_n \\ A^*B = C^*D \\ B^*B - D^*D = 1. \end{cases}$ We denote by K a maximal compact subgroup of $SU(n,1)$

$$K = \left\{ \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \middle| M \in U(n), N \in \mathbb{C} \text{ and } \det(M)\det(N){=}1 \right\} = S(U(n){\times}U(1))$$

and

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R} \right\}$$

Then the Cartan decomposition of $SU(n,1)$ is $SU(n,1) = KAK$.

Let $\mathcal{H}_h$, $h > \dfrac{1}{2}$ be the weighted Bergman space of index h

$$\mathcal{H}_h = \left\{ f : B^n \longrightarrow \mathbb{C} \quad \text{analytic} \;\middle|\; C_h \int_{B^n} (1-|z|^2)^{2h-n-1} |f(z)|^2 d\mu(z) < \infty \right\},$$

where $d\mu(z)$ being the Lebesgue measure on B^n and $C_h = \dfrac{\Gamma(2h)}{n!\Gamma(2h-n)}$.

For any multi-index $p = (p_1, \dots, p_n)$ of non negative integers, we write $|p| = p_1 + \cdots + p_n$ and $p! = p_1! \cdots p_n!$.

For any $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $z^p = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}$. The standard orthonormal basis of $\mathcal{H}_h$ is $\{\phi_p^h \mid p \in \mathbb{N}^n\}$ where

$$\phi_p^h(z) = \left[\frac{\Gamma(|p| + 2h)}{p!\Gamma(2h)} \right]^{\frac{1}{2}} z^p.$$

We denote by $\langle , \rangle_h$ the inner product of $\mathcal{H}_h$

$$\langle \phi, \psi \rangle_h = C_h \int_{B^n} (1-|z|^2)^{2h-(n+1)} \phi(z) \overline{\psi(z)} d\mu(z).$$

For any $g \in SU(n,1)$, we define the operator $T_h(g)$ on $\mathcal{H}_h$ by

$$T_h(g)F(z) = (Cz + D)^{-2h} F(g.z) = (Cz + D)^{-2h} F((Az + B)(Cz + D)^{-1}),$$

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then $(T_h, \mathcal{H}_h)$ is an irreducible unitary representation of $SU(n,1)$.
Now, we consider the coefficient matrix $t_{pq}^h = \left\langle T_h(g)\phi_p^h, \phi_q^h \right\rangle_h$ of the group $SU(n,1)$ according to the above orthonormal basis.

Since $T_h(g_1 g_2) = T_h(g_1) T_h(g_2), \quad g_1, g_2 \in SU(n,1)$, we have

$$t_{pq}^h(g_1 g_2) = \sum_{k \in \mathbb{N}^n} t_{pk}^h(g_1) t_{kq}^h(g_2).$$

According to the Cartan decomposition $SU(n,1) = KAK$ each element g in $SU(n,1)$ can be written as $g = k_1 a_t k_2$. Henceforth

$$t_{pq}^h(g) = \sum_{\substack{k \in \mathbb{N}^n \\ l \in \mathbb{N}^n}} t_{pk}^h(k_1) t_{kl}^h(a_t) t_{lq}^h(k_2).$$

3. Proof of Theorem 1.1. Now we compute for

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$$

the matrix coefficient

$$\begin{aligned}
&\left\langle T_h(a_t)\phi_k^h, \phi_l^h \right\rangle_h \\
= \;& C_h \int_{B^n} (1-|z|^2)^{2h-(n+1)} (\cosh t - z_1 \sinh t)^{-2h} \phi_k^h(a_t.z)\overline{\phi_l^h(z)} d\mu(z) \\
= \;& C_h \int_{B^n} (1-|z|^2)^{2h-(n+1)} (\cosh t - z_1 \sinh t)^{-(2h+|k|)} \\
& \qquad \times (z_1 \cosh t - \sinh t)^{k_1} z_2^{k_2} \cdots z_n^{k_n} \bar{z}^l d\mu(z) \\
= \;& C_h (\cosh t)^{-(2h+|k|)+k_1} \int_{B^n} (1-|z|^2)^{2h-(n+1)} (1 - z_1 \tanh t)^{-(2h+|k|)} \\
& \qquad \times (z_1 - \tanh t)^{k_1} z_2^{k_2} \cdots z_n^{k_n} \bar{z}^l d\mu(z).
\end{aligned}$$

Since $|z_1 \tanh t| < 1$, we can use the binomial formula

$$(1-x)^{-\alpha} = \sum_{k\in\mathbb{N}} \frac{(\alpha)_k}{k!} x^k, |x| < 1$$

to rewrite the above integral as

$$\begin{aligned}
&\left\langle T_h(a_t)\phi_k^h, \phi_l^h \right\rangle_h = C_h (\cosh t)^{-(2h+|k|)+k_1} \\
\times & \sum_{p\in\mathbb{N}} \frac{(2h+|k|)_p}{p!} \int_{B^n} (1-|z|^2)^{2h-(n+1)} (-z_1 \tanh t)^p \\
& \qquad \times \sum_{q=0}^{k_1} \frac{k_1!(-\tanh t)^q}{q!(k_1-q)!} z_2^{k_2} \cdots z_n^{k_n} \bar{z}^l d\mu(z) \\
= \;& C_h (\cosh t)^{-(2h+|k|)+k_1} \sum_{p\in\mathbb{N}} \sum_{q=0}^{k_1} \frac{(-1)^{q+p} (2h+|k|)_p k_1! (\tanh t)^{p+q}}{p!q!(k_1-q)!} \\
\times & \int_{B^n} (1-|z|^2)^{2h-(n+1)} z_1^{p-q+k_1} z_2^{k_2} \cdots z_n^{k_n} \bar{z}^l d\mu(z).
\end{aligned}$$

Since

$$\int_{B^n} (1-|z|^2)^{2h-(n+1)} z_1^{p-q+k_1} z_2^{k_2} \cdots z_n^{k_n} \bar{z}^l d\mu(z)$$

$$\begin{aligned}
&= \int_0^1 (1-r^2)^{2h-(n+1)} r^{2n-1+|l|+|k|+p-q} dr \int_{\partial B^n} w_1^{p-q+k_1} w_2^{k_2} \cdots w_n^{k_n} \bar{w}^l dw \\
&= \frac{\Gamma(2h-n)\Gamma(|l|+n+1)}{\Gamma(2h+|l|+1)} \int_{\partial B^n} w_1^{p-q+k_1} w_2^{k_2} \cdots w_n^{k_n} \bar{w}^l dw,
\end{aligned}$$

the integral in the above sum equals $\frac{\Gamma(2h-n)\Gamma(|l|+n+1)}{\Gamma(2h+|l|+1)} \frac{2(n-1)!l!}{(n-1+|l|)!}$ when

$$\begin{cases} p-q+k_1 = l_1 & \\ k_j = l_j, & j = 2,\ldots,n \end{cases}$$

and vanishes otherwise. Thus

$$\begin{aligned}
&\left\langle T_h(a_t)\phi_k^h, \phi_l^h \right\rangle_h \\
&= C_h (\cosh t)^{-(2h+|k|)+k_1} \frac{2(|l|+n)\Gamma(n) l! \Gamma(2h-n)}{\Gamma(2h+|l|+1)} \\
&\quad \times \sum_{p\in\mathbb{N}} \sum_{q=0}^{k_1} \frac{(-1)^{p+k_1-l_1}(2h+|k|)_p k_1!}{p!q!(k_1-q)!} (\tanh t)^{p+q} \\
&= 2C_h (\cosh t)^{-(2h+|k|)+k_1} \frac{(-1)^{k_1-l_1}(|l|+n)\Gamma(n) l! \Gamma(2h-n)}{\Gamma(2h+|l|+1)} (\tanh t)^{k_1-l_1} \\
&\quad \times \sum_{p\in\mathbb{N}} \frac{(-1)^p (2h+|k|)_p k_1!}{p!(p+k_1-l_1)!(l_1-p)!} (\tanh t)^{2p},
\end{aligned}$$

with $k_j = l_j$, $j = 2,\ldots,n$.

Henceforce, by using the following equality

$$\frac{(-1)^p (k_1-l_1)! l_1!}{(l_1-p)!(k_1+p-l_1)!} = \frac{(-l_1)_p}{(k_1-l_1+1)_p}$$

we have

$$\begin{aligned}
t_{kl}^h(a_t) &= \left\langle T_h(a_t)\phi_k^h, \phi_l^h \right\rangle_h = \frac{2(|l|+n)l!}{n} \frac{(-1)^{k_1-l_1}\Gamma(2h)}{\Gamma(2h+|h|+1)} \binom{k_1}{l_1} \\
&\quad \times \cosh t^{-(2h+|k|)+k_1} (\tanh t)^{k_1-l_1} \sum_{p\in\mathbb{N}} \frac{(2h+|k|)_p (-l_1)_p}{p!(k_1-l_1+1)_p} (\tanh t)^{2p} \\
&= \frac{2(|l|+n)l!}{n} \frac{(-1)^{k_1-l_1}\Gamma(2h)}{\Gamma(2h+|h|+1)} \binom{k_1}{l_1} \\
&\quad \times \cosh t^{-(2h+|k|)+k_1} (\tanh t)^{k_1-l_1} F(-l_1, 2h+|k|, k_1-l_1+1; (\tanh t)^2).
\end{aligned}$$

Remark 3.1. For $k \in K = S(U(n) \times U(1))$ we are not yet able to give a simple explicit expression for the matrix coefficient $t^h_{pq}(k) = \left\langle T_h(k)\phi^h_p, \phi^h_q \right\rangle_h$. But we can rewrite it as a series. Indeed:

Let $k = \begin{pmatrix} A & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in K$, $A = (a_{ij})_{i,j}$ and $z = (z_1, \dots, z_n) = r(w_1, \dots, w_n) \in B^n$. Then,

$$Az = r \begin{pmatrix} \sum_{j=1}^{n} a_{1j} w_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} w_j \end{pmatrix}.$$

Thus

$$U^h_{pq}(k) = e^{i\theta(2h+|p|)} \int_0^1 (1-r^2)^{2h-(n+1)} r^{2n-1+|p|+|k|} dr \int_{\partial B^n} \prod_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} w_j)^{p_i} \bar{w}^q dw.$$

Making use of the identity

$$(z_1 + \cdots + z_n)^m = \sum_{k_1+\cdots+k_n=m} \frac{m!}{k_1! \cdots k_n!} z_1^{k_1} \cdots z_n^{k_n}, \quad m \in \mathbb{N},$$

we obtain for $p = (p_1, \dots, p_n)$, and $q = (q_1, \dots, q_n)$ that

$$\begin{aligned} t^h_{pq}(k) &= e^{i\theta(2h+|p|)} \frac{\Gamma(2h-n)\Gamma(n+|p|+1)}{\Gamma(2h+|p|+1)} \\ &\quad \times \sum_{\substack{\sum_{j=1}^n p_i^j = p_i, \\ \sum_{j=1}^n q_j^i = q_i \\ 1 \le i,j \le n}} \frac{|p_1|! \cdots |p_n|! \prod_{1 \le i,j \le n} a_{ij}^{p_i^j}}{p!} \\ &\quad \times \int_{\partial B^n} w_1^{p_1^1 + p_2^1 + \cdots + p_n^1} w_2^{p_1^2 + p_2^2 + \cdots + p_n^2} \cdots w_n^{p_1^n + p_2^n + \cdots + p_n^n} \bar{w}^q dw \\ &= \frac{e^{i\theta(2h+|p|)}\Gamma(2h)\Gamma(n+|p|+1)}{n(n-1+|p|)!\Gamma(2h+|p|+1)} \sum_{\substack{\sum_{j=1}^n p_i^j = p_i, \\ \sum_{j=1}^n q_j^i = q_i \\ 1 \le i,j \le n}} \frac{q!p! \prod_{1 \le i,j \le n} a_{ij}^{p_i^j}}{\prod_{1 \le i,j \le n} p_i^j!}. \end{aligned}$$

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