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SYMPLECTIC DECOMPOSITION OF THE MASSIVE COADJOINT ORBITS OF A SEMIDIRECT PRODUCT

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Abstract. Let G be the semidirect product $V \rtimes K$ where K is a connected semisimple non-compact Lie group acting linearly on a finite-dimensional real vector space V. Let $\mathcal{O}$ be a coadjoint orbit of G whose little group K_0 is a maximal compact subgroup of K. We construct an explicit symplectomorphism between $\mathcal{O}$ and the symplectic product $\mathbb{R}^{2n} \times \mathcal{O}'$ where $\mathcal{O}'$ is a little group coadjoint orbit. We treat in details the case of the Poincaré group.

1. Introduction. Coadjoint orbits of Lie groups appear in many aeras of mathematics and physics. In particular, coadjoint orbits can be used in harmonic analysis to classify the irreducible unitary representations of Lie groups and, in physics, to describe the classical phase spaces corresponding to internal degrees of freedom of quantum particles, see [19], [16], [25], [21].

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Coadjoint orbits are basic examples of homogeneous symplectic manifolds and their geometrical structure have been intensively studied, see for instance [5], [19].

In this note, we focuse on Lie groups which are semidirect products of the form $G := V \rtimes K$ where K is a non-compact semisimple Lie group acting linearly on a finite-dimensional real vector space V. The coadjoint orbits of such groups were described by J. H. Rawnsley in [23] and the symplectic structure of these coadjoint orbits was studied by P. Baguis in [2].

Here we consider a coadjoint orbit $\mathcal{O}$ of G whose little group K_0 is a maximal compact subgroup of K. As it will be explained in Section 6, this is the direct generalization of the 'massive' coadjoint orbits of the Poincaré group, see [21], Chapter IV, Section 3 and [25], Chapter 8.

In [8], assuming that $\mathcal{O}$ is integral and then associated with a irreducible unitary representation π of G [19], [20], we introduced the Berezin-Weyl correspondence $\mathcal{W}$ from a space of functions on $\mathbb{R}^{2n} \times \mathcal{O}'$, where $\mathcal{O}'$ is a little group coadjoint orbit, onto a space of operators on the space of π. Then, by dequantizing the derived representation $d\pi$ by means of $\mathcal{W}$, that is, by computing $W^{-1}(d\pi(X))$ for each element X of the Lie algebra of G, we obtained an explicit symplectomorphism from the symplectic product $\mathbb{R}^{2n} \times \mathcal{O}'$ onto $\mathcal{O}$. Note that $\mathcal{O}$ is integral if and only if $\mathcal{O}'$ is, see [23].

The main aim of the present note is to extend this result to the case where $\mathcal{O}$ is not assumed to be integral. In other words, we show that the existence and the explicit form of the symplectomorphism $\mathbb{R}^{2n} \times \mathcal{O}' \to \mathcal{O}$ do not depend on the existence of unitary irreducible representations of G attached to $\mathcal{O}$, see Section 5. Since our strategy is to deduce the general case from the case where $\mathcal{O}$ is integral, we review some results from [8] in Section 3 and Section 4. Moreover, in Section 6, we give formulas for the symplectomorphism in the case of the (generalized) Poincaré group which is of some importance in mathematical physics, see [16], [25].

We can hope for further applications of our results, namely (1) the study of contractions of Lie group representations in the spirit of [14], [11], [10] and (2) the construction of explicit star-products on $\mathcal{O}$, see [6] and its references.

2. Generalities.

Here we use the notation of [23]. Let K be a connected, non-compact, semisimple real Lie group with finite center. Let $\mathfrak{k}$ be the Lie algebra of K. For k in K and f in the dual $\mathfrak{k}^*$ of $\mathfrak{k}$ we denote by $k \cdot f$ the coadjoint action of k on f.

We assume that K acts linearly on a finite-dimensional real vector space V and, for k in K and v in V, we denote by $k \cdot v$ the action of k on v. We also denote by $(k,p) \to k\cdot p$ the contragredient action of K on V^*. Let $(A,v) \to A\cdot v$ and $(A,p) \to A\cdot p$ the corresponding representations of $\mathfrak{k}$ on V and V^*. For each v in V and p in V^* we define $v \wedge p \in \mathfrak{k}^*$ by $(v\wedge p)(A) = p(A\cdot v) = -(A\cdot p)(v)$ for $A \in \mathfrak{k}$. Note that we have

$$k\cdot(v\wedge p) = k\cdot p \wedge k\cdot v$$

for each $k \in K$, $v \in V$ and $p \in V^*$.

We can form the semidirect product $G = V \rtimes K$. The multiplication of G is

$$(v,k)(v',k') = (v + k\cdot v', kk')$$

for each v, v' in V and k, k' in K. The Lie algebra $\mathfrak{g}$ of G is the vector space $V\times\mathfrak{k}$ equipped with the Lie bracket

$$[(a,A),\,(a',A')] = (A\cdot a' - A'\cdot a,\,[A,A'])$$

for each a, a' in V and A, A' in $\mathfrak{k}$.

We can identify $\mathfrak{g}^*$ with $V^*\times\mathfrak{k}^*$. The coadjoint action of G on $\mathfrak{g}^*$ is then given by

$$(v,k)\cdot(p,f) = (k\cdot p, k\cdot f + v\wedge k\cdot p)$$

for each $(v,k) \in G$ and $(p,f) \in \mathfrak{g}^*$. We can identify K-equivariantly $\mathfrak{k}$ to its dual $\mathfrak{k}^*$ by using the Killing form of $\mathfrak{k}$. Then $\mathfrak{g}^*$ can be identified to $V^*\times\mathfrak{k}$.

Let us consider the orbit $\mathcal{O}(\xi_0)$ of the element $\xi_0 = (p_0, f_0)$ of $\mathfrak{g}^* \simeq V^*\times\mathfrak{k}$ under the coadjoint action of G on $\mathfrak{g}^*$. Henceforth we assume that the little group $K_0 = \{k \in K \,:\, k\cdot p_0 = p_0\}$ is a maximal compact subgroup of K. Then K_0 is a connected semisimple subgroup of K [17]. Let $\mathfrak{k}_0$ be the Lie algebra of K_0. We have the Cartan decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{k}$. Then we have $\mathfrak{p} = \{v\wedge p_0 \,:\, v \in V\}$, see [8] and [23], Lemma 1. From this, we see that, without loss of generality, we can assume that $\xi_0 = (p_0, \varphi_0)$ with $\varphi_0 \in \mathfrak{k}_0$. We denote by $o(\varphi_0) \subset \mathfrak{k}_0$ the orbit of $\varphi_0 \in \mathfrak{k}_0 \simeq \mathfrak{k}_0^*$ under the (co)adjoint action of K_0.

Let n be the dimension of $\mathfrak{p}$. We know that the restriction to $\mathfrak{p}$ of the Killing form $\langle\cdot,\cdot\rangle$ of $\mathfrak{k}$ is positive definite [17]. We fix an orthonormal basis $(E_1, E_2, \ldots, E_n)$ for $\mathfrak{p}$ and we denote by $(t_1, t_2, \ldots, t_n)$ the coordinates of $T \in \mathfrak{p}$ in this basis.

Before going on, let us recall the definition of a symplectic product. Let (M_1, ω^1) and (M_2, ω^2) be two symplectic manifolds and let $p_1 : M_1\times M_2 \to M_1$ and $p_2 : M_1\times M_2 \to M_2$ be the projections. Then $p_1^*\omega^1 + p_2^*\omega^2$ is a symplectic form on $M_1\times M_2$ which is denoted by $\omega^1\otimes\omega^2$ and $(M_1\times M_2, \omega^1\otimes\omega^2)$ is called the symplectic product of (M_1, ω^1) and (M_2, ω^2), see for instance [22]. Now, consider another symplectic manifold (M_3, ω^3). If M_3 is symplectomorphic to the symplectic product $M_1 \times M_2$, we say that M_3 has symplectic decomposition $M_1 \times M_2$.

Let ω_0 and ω_1 be the Kirillov 2-forms on $\mathcal{O}(\xi_0)$ and $o(\varphi_0)$, respectively. Denote by $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_0$ the Poisson brackets associated with ω_1 and ω_0. We consider the symplectic form $\omega := \sum_{k=1}^{n} dt_k \wedge ds_k$ on $\mathfrak{p}^2$. The corresponding Poisson bracket on $C^\infty(\mathfrak{p}^2)$ is given by

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial t_k} \frac{\partial g}{\partial s_k} - \frac{\partial f}{\partial s_k} \frac{\partial g}{\partial t_k} \right).$$

We denote by $\{\cdot,\cdot\}_2$ the Poisson bracket associated with the symplectic form $\omega_2 := \omega \otimes \omega_1$ on $\mathfrak{p}^2 \times o(\varphi_0)$. Let $u, v \in C^\infty(\mathfrak{p}^2)$ and $a, b \in C^\infty(o(\varphi_0))$. Note that for $f(T,S,\varphi) = u(T,S)a(\varphi)$ and $g(T,S,\varphi) = v(T,S)b(\varphi)$ we have

$$\{f, g\}_2 = u(T,S)v(T,S)\{a, b\}_1 + a(\varphi)b(\varphi)\{u, v\}.$$

3. Representations.

The material of this section and of the next section is essentially taken from [8].

In this section (and in the next section) we assume that $o(\varphi_0)$ is associated with the unitary irreducible representation (ρ, E) of K_0 as in [28], Section 4. This correspondence goes as follows. Let T be a maximal torus of K_0 with Lie algebra $\mathfrak{t}$. We fix an ordering on the root system $\Delta(\mathfrak{g}^c, \mathfrak{t}^c)$. Now, let $\lambda \in (i\mathfrak{t})^*$ be the highest weight of (ρ, E). We then define $\varphi_0 \in \mathfrak{k}_0^*$ by $\varphi_0(X) = -i\lambda(X)$ for $X \in \mathfrak{t}$ and $\varphi_0(X) = 0$ for X in the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}_0$ with respect to the Killing form of $\mathfrak{k}_0$. The orbit of φ_0 under the (co)adjoint action of K_0 is then said to be associated with the representation (ρ, E).

Let $Z(p_0)$ be the orbit of p_0 under the action of K on V^*. By [17], Chapter VI, Theorem 1.1, we see that the map $e : T \to \exp T \cdot p_0$ is a diffeomorphism from $\mathfrak{p}$ onto $Z(p_0)$.

For $p \in Z(p_0)$ we denote by $M(p)$ the unique element of $\exp(\mathfrak{p})$ such that $M(p) \cdot p_0 = p$. Consequently, if $p = e(T)$ then $M(p) = \exp T$.

Let $dT = dt_1 dt_2 \dots dt_n$ be the Lebesgue measure on $\mathfrak{p}$. Then, the K-invariant measure $d\mu$ on $Z(p_0)$ is given by $d\mu = e^*(\delta(T)dT)$ where $\delta(T) := \operatorname{Det}\left(\frac{\sinh \operatorname{ad} T}{\operatorname{ad} T}|_{\mathfrak{p}} \right)$, see [17].

Since we assume that $o(\varphi_0)$ is integral, $\mathcal{O}(\xi_0)$ is integral [23]. Then $\mathcal{O}(\xi_0)$ is associated with the unitarily induced representation $\pi = \operatorname{Ind}_{V\rtimes K_0}^{G}(e^{i\langle p_0,\cdot\rangle}\otimes\rho)$. By a result of G. Mackey, π is irreducible since ρ is [26]. The representation π is usually realized on the Hilbert space $L^2(Z(p_0),E)$ which is the completion of the space of compactly supported smooth functions $\psi : Z(p_0)\rightarrow E$ with respect to the norm defined by

$$\|\psi\|^2 = \int_{Z(p_0)} \langle \psi\,(p),\, \psi(p)\rangle_E \, d\mu\,(p)$$

as follows. For $(v,k)\in G$ the action of the operator $\pi(v,k)$ is given by

$$(\pi\,(v,k)\,\psi)\,(p) = e^{i\langle p,v\rangle}\,\rho(M(p)^{-1}\,k\,M(k^{-1}\cdot p))\,\psi\,(k^{-1}\cdot p).$$

However, having in mind to use the Weyl calculus, it is convenient to realize π on the Hilbert space $L^2(\mathfrak{p},E)$ defined as the completion of the space $C_0^\infty(\mathfrak{p},E)$ of compactly supported smooth functions $\phi : \mathfrak{p}\rightarrow E$ with respect to the norm given by

$$\|\phi\|^2 = \int_{\mathfrak{p}} \langle\phi(T),\, \phi(T)\rangle_E \, dT.$$

To this aim, we introduce the unitary operator $\phi\rightarrow\psi$ from $L^2(\mathfrak{p},E)$ to $L^2(Z(p_0),E)$ defined by $\psi(e(T)) = \delta(T)^{1/2}\phi(T)$. Let us denote by $k\cdot T$ the action of K on $\mathfrak{p}$ which corresponds to the action of K on $Z(p_0)$, that is, we have $e(k\cdot T) = k\cdot e(T)$ for $k\in K$ and $T\in\mathfrak{p}$. Then we obtain

$$(\pi\,(v,k)\,\phi)\,(T) = \left(\frac{\delta(T)}{\delta(k^{-1}\cdot T)}\right)^{1/2} e^{i\langle e(T),v\rangle}\,\rho(M(e(T))^{-1}\,k\,M(k^{-1}\,e(T)))\,\phi\,(k^{-1}\cdot T)$$

for each $(v,k)\in G$.

Now we give an explicit expression for the differential $d\pi$ of π. Let us introduce some additional notation. For $A\in\mathfrak{k}$ and $T\in\mathfrak{p}$ we define $A\cdot T := \dfrac{d}{dt}(\exp tA)\cdot T\Big|_{t=0}$. Futhermore for $p\in Z(p_0)$ and $A\in\mathfrak{k}$ we set

$$L(p,A) = \frac{d}{dt}\left(M(p)^{-1}\exp(tA)M(\exp(-tA)\cdot p)\right)\Big|_{t=0}.$$

Let $\operatorname{pr}_{\mathfrak{k}_0}$ and $\operatorname{pr}_{\mathfrak{p}}$ be the projections of $\mathfrak{k}$ onto $\mathfrak{k}_0$ and $\mathfrak{p}$ associated with the direct decomposition $\mathfrak{k} = \mathfrak{k}_0\oplus\mathfrak{p}$.

If u is an endomorphism of $\mathfrak{k}$ which leaves the space $\mathfrak{p}$ invariant, the trace and the determinant of the restriction of u to $\mathfrak{p}$ are respectively denoted by $\operatorname{Tr}_{\mathfrak{p}}u$ and $\operatorname{Det}_{\mathfrak{p}}u$.

We have the following lemma.

Lemma 3.1 ([8]). (1) *For $A \in \mathfrak{k}$ and $T \in \mathfrak{p}$, we have*

$$A \cdot T = -\operatorname{ad} T\ \mathrm{pr}_{\mathfrak{k}_0}(A) + \frac{\operatorname{ad} T}{\tanh \operatorname{ad} T}\ \mathrm{pr}_{\mathfrak{p}}\,(A).$$

(2) *For $p = e(T) \in Z(p_0)$ and $A \in \mathfrak{k}$, we have*

$$L(p, A) = \mathrm{pr}_{\mathfrak{k}_0}(A) - \tanh\Big(\frac{1}{2}\operatorname{ad}\ T\Big)\ \mathrm{pr}_{\mathfrak{p}}\,(A).$$

(3) *For $A \in \mathfrak{k}$ and $T \in \mathfrak{p}$ we have*

$$\frac{d}{dt}\delta(\exp(tA)\cdot T)\big|_{t=0} = \delta(T)\,\mathrm{Tr}_{\mathfrak{p}}\,\big(\gamma(\operatorname{ad} T)\ \operatorname{ad}\mathrm{pr}_{\mathfrak{p}}(A)\big)$$

where the function γ is defined by $\gamma(z) = \dfrac{z\cosh z - \sinh z}{z \sinh z}$ if $z \neq 0$ and by $\gamma(0) = 0$.

From this result, we deduce the following expression of $d\pi$.

Proposition 3.2 ([8]). *For each $(v, A) \in \mathfrak{g}$ and $\phi \in C_0(\mathfrak{p}, E)$, we have*

$$(d\pi(w, A)\phi)(T) = i\,\langle e(T), w\rangle\phi(T) + d\rho\left(\mathrm{pr}_{\mathfrak{k}_0}(A) - \tanh\Big(\frac{1}{2}\operatorname{ad}\ T\Big)\ \mathrm{pr}_{\mathfrak{p}}\,(A)\right)\phi(T)$$

$$+\, d\phi(T)\left(\operatorname{ad} T\ \mathrm{pr}_{\mathfrak{k}_0}(A) - \frac{\operatorname{ad} T}{\tanh \operatorname{ad} T}\ \mathrm{pr}_{\mathfrak{p}}\,(A)\right) + \frac{1}{2}\,\mathrm{Tr}_{\mathfrak{p}}\,\big(\gamma(T)\operatorname{ad}\mathrm{pr}_{\mathfrak{p}}(A)\big)\,\phi(T).$$

4. Dequantization.

Recall that the Berezin calculus is a one-to-one linear map which associates with each operator A on E a complex-valued function $s(A)$ on $o(\varphi_0)$ called the symbol of the operator A, see [3], [4], [12]. The Berezin calculus has various properties for which we refer the reader to [3], [12], [9], [28]. Here, we just mention the following property. Let $d\rho$ denote the derived representation of ρ.

Proposition 4.1 ([9]). *For each $X \in \mathfrak{k}_0$ and $\varphi \in o(\varphi_0)$, we have*

$$s(d\rho(X))(\varphi) = i\langle \varphi, X\rangle.$$

Now we introduce the Berezin-Weyl calculus on $\mathfrak{p}^2 \times o(\varphi_0)$ by combining the Berezin calculus with the Weyl calculus for $\mathrm{End}(E)$-valued functions.

We say that a smooth function $f : (T, S, \varphi) \to f(T, S, \varphi)$ is a symbol on $\mathfrak{p}^2 \times o(\varphi_0)$ if for each $(T, S) \in \mathfrak{p}^2$ the function $\varphi \to f(T, S, \varphi)$ is the symbol in the

Berezin calculus on $o(\varphi_0)$ of an operator $\hat{f}(T,S)$ on E. Moreover, a symbol f on $\mathfrak{p}^2 \times o(\varphi_0)$ is called a S-symbol if the function $\hat{f}$ belongs to the Schwartz space of rapidly decreasing smooth functions on $\mathfrak{p}^2$ with values in $\operatorname{End}(E)$.

Let us consider the Weyl calculus for $\operatorname{End}(E)$-valued functions, which is a slight refinement of the usual Weyl calculus for complex-valued functions [18]. For any S-symbol f on $\mathfrak{p}^2 \times o(\varphi_0)$ we define the operator $\mathcal{W}(f)$ on the Hilbert space $L^2(\mathfrak{p}, E)$ by the equation

$$(\mathcal{W}(f)\phi)(T) = (2\pi)^{-n} \int_{\mathfrak{p}^2} e^{i\langle S,Z\rangle} \hat{f}\left(T + \frac{1}{2}S, Z\right) \phi(T+S)\, dS\, dZ$$

for each $\phi \in C_0^\infty(\mathfrak{p}, E)$.

The Weyl calculus can be extended to much larger classes of symbols [18], in particular to polynomial symbols. We say that a symbol f on $\mathfrak{p}^2 \times o(\varphi_0)$ is a P-symbol if the function $\hat{f}(T,S)$ is polynomial in S. Let f be the P-symbol defined by $f(T,S,\varphi) = u(T)S^\alpha$ where $u \in C^\infty(\mathfrak{p}, E)$ and with the usual notation $S^\alpha := s^{\alpha_1} s^{\alpha_2} \dots s^{\alpha_n}$ for each multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then, by [27], we have

$$(\mathcal{W}(f)\phi)(T) = \left(i\frac{\partial}{\partial S}\right)^\alpha \left(u\left(T + \frac{1}{2}S\right) \phi(T+S)\right)\Big|_{Z=0}$$

In particular, if $f(T,S,\varphi) = u(T)$ then $(\mathcal{W}(f)\phi)(T) = u(T)\,\phi(T)$ and if $f(T,S,\varphi) = u(T)S_k$ then

$$(\mathcal{W}(f)\phi)(T) = i\left(\frac{1}{2}\partial_k u(T)\,\phi(T) + u(T)\partial_k\phi(T)\right)$$

where ∂_k denotes the partial derivative with respect to the variable t_k.

The correspondence $f \to \mathcal{W}(f)$ is called the Berezin–Weyl calculus on $\mathfrak{p}^2 \times o(\varphi_0)$. The following property of $\mathcal{W}$ can be proved by a direct computation.

Proposition 4.2 ([8]). *Let f and g two P-symbols on $\mathfrak{p}^2 \times o(\varphi_0)$ of the form*

$$u(T) + \langle v(T), \varphi\rangle + \sum_{k=1}^{n} w_k(T) S_k$$

where $u \in C^\infty(\mathfrak{p})$, $v \in C^\infty(\mathfrak{p}, \mathfrak{k}_0)$ and $w_k \in C^\infty(\mathfrak{p})$ for $k = 1, 2, \dots, n$. Then we have

$$[\mathcal{W}(f),\, \mathcal{W}(g)] = -i\,\mathcal{W}(\{f,\, g\}_2).$$

Also, we have the following result.

Proposition 4.3 ([8]). *For each* $X=(w,A)\in\mathfrak{g}$*, the Berezin-Weyl symbol of the operator* $-id\pi(X)$ *is the P-symbol* f_X *on* $\mathfrak{p}^2\times o(\varphi_0)$ *given by*

$$f_X(T,S,\varphi)=\langle e(T),w\rangle+\langle\varphi,L(e(T),A)\rangle+\langle A\cdot T,S\rangle.$$

Note that the map $X\to f_X(T,S,\varphi)$ is linear. Then there exists a map Ψ from $\mathfrak{p}^2\times o(\varphi_0)$ to $\mathfrak{g}^*$ such that

$$f_X(T,S,\varphi)=\langle\Psi(T,S,\varphi),\,X\rangle$$

for each $X\in\mathfrak{g}$ and each $(T,S,\varphi)\in\mathfrak{p}^2\times o(\varphi_0)$. More precisely, we have the following proposition.

Proposition 4.4 ([8]). *For* $(T,S,\varphi)\in\mathfrak{p}^2\times o(\varphi_0)$ *we have*

$$\Psi(T,S,\varphi)=\Big(e(T),\varphi+\tanh\left(\frac{1}{2}\operatorname{ad}T\right)\varphi+\left(\operatorname{ad}T+\frac{\operatorname{ad}T}{\tanh\operatorname{ad}T}\right)S\Big).$$

Moreover, Ψ *is a symplectomorphism from* $(\mathfrak{p}^2\times o(\varphi_0),\,\omega_2)$ *onto* $(\mathcal{O}(\xi_0),\,\omega_0)$.

5. Symplectic decomposition.

In this section, we retain the notation of Section 2. It is no longer assumed that $o(\varphi_0)$ (hence $\mathcal{O}(\xi_0)$) is integral. By analogy with the case when $\mathcal{O}(\xi_0)$ is integral, we introduce the map $\Psi:\mathfrak{p}^2\times o(\varphi_0)\to\mathcal{O}(\xi_0)$ defined by

$$\Psi(T,S,\varphi):=\Big(e(T),\varphi+\tanh\left(\frac{1}{2}\operatorname{ad}T\right)\varphi+\left(\operatorname{ad}T+\frac{\operatorname{ad}T}{\tanh\operatorname{ad}T}\right)S\Big).$$

Then we have the following result.

Proposition 5.1. *The map* $\Psi:\mathfrak{p}^2\times o(\varphi_0)\to\mathcal{O}(\xi_0)$ *is a bijection.*

Proof. First, we prove that Ψ takes values in $\mathcal{O}(\xi_0)$.

Let $(T,S,\varphi)\in\mathfrak{p}^2\times o(\varphi_0)$. Let $p=e(T)$. Since $\mathfrak{p}=\{v\wedge p_0\,:\,v\in V\}$ (see Section 2), there exists $v\in V$ such that

$$(M(p)^{-1}\cdot v)\wedge p_0=-\tanh\left(\frac{1}{2}\operatorname{ad}T\right)\varphi+\frac{\operatorname{ad}T}{\sinh\operatorname{ad}T}S.$$

Then we have

$$\varphi+\tanh\left(\frac{1}{2}\operatorname{ad}T\right)\varphi+\left(\operatorname{ad}T+\frac{\operatorname{ad}T}{\tanh\operatorname{ad}T}\right)S$$

$$
\begin{aligned}
&= \exp(\operatorname{ad} T)\left(\varphi - \tanh\left(\frac{1}{2}\operatorname{ad} T\right)\varphi + \frac{\operatorname{ad} T}{\sinh \operatorname{ad} T}\, S\right)\\
&= M(p)\cdot\big(\varphi + (M(p)^{-1}\cdot v)\wedge p_0\big)\\
&= M(p)\cdot\varphi + v\wedge p.
\end{aligned}
$$

hence

$$\Psi(T,S,\varphi) = (p, M(p)\cdot\varphi + v\wedge p) = (v, M(p))\cdot(p_0,\varphi)\in\mathcal{O}(\xi_0).$$

Now we prove that Ψ is a bijection from $\mathfrak{p}^2\times o(\varphi_0)$ to $\mathcal{O}(\xi_0)$. Let $\xi\in\mathcal{O}(\xi_0)$. We have to solve the equation

$$\Psi(T,S,\varphi)=\xi \tag{5.1}$$

with $(T,S,\varphi)\in\mathfrak{p}^2\times o(\varphi_0)$.

We can write $\xi = (v,k)\cdot(p_0,\varphi_0) = (k\cdot p_0, k\cdot\varphi_0 + v\wedge k\cdot p_0)$ for some $(v,k)\in G$. Let $p = k\cdot p_0$. Then we can decompose k as $k = M(p)u$ with $u\in K_0$. Thus Equation 5.1 implies that $p = e(T)$ hence T is uniquely determined. Moreover, Equation 5.1 also gives

$$\varphi - \tanh\left(\frac{1}{2}\operatorname{ad} T\right)\varphi + \frac{\operatorname{ad} T}{\sinh\operatorname{ad} T}\, S = u\cdot\varphi_0 + (M(p)^{-1}\cdot v)\wedge p_0.$$

Taking projections on $\mathfrak{k}_0$ and $\mathfrak{p}$, we get $\varphi = u\cdot\varphi_0$ and

$$-\tanh\left(\frac{1}{2}\operatorname{ad} T\right)\varphi + \frac{\operatorname{ad} T}{\sinh\operatorname{ad} T}\, S = (M(p)^{-1}\cdot v)\wedge p_0.$$

Hence φ and S are uniquely determined. This ends the proof. □

From classical representation theory of compact Lie groups, we deduce the following lemma.

Lemma 5.2. *Let $\mathcal{U}\subset\mathfrak{k}_0^*$ be the union of all integral coadjoint orbits of K_0. Then the linear span of $\mathcal{U}$ is $\mathfrak{k}_0^*$.*

Proof. As at the beginning of Section 3, let T be a maximal torus of $\mathfrak{k}_0$ with Lie algebra $\mathfrak{t}$. For each $\lambda\in(i\mathfrak{t})^*$, let $\varphi_\lambda\in\mathfrak{k}_0^*$ defined $\varphi_\lambda(X) = -i\lambda(X)$ for $X\in\mathfrak{t}$ and $\varphi_\lambda(X)=0$ for X in the orthogonal complement $\mathfrak{t}^\perp$ of $\mathfrak{t}$ (with respect to the Killing form of $\mathfrak{k}_0$). Then the map $\lambda\to\varphi_\lambda$ is a linear isomorphism from $(i\mathfrak{t})^*$ onto $\{\varphi\in\mathfrak{k}_0^* : \varphi|_{\mathfrak{t}^\perp}\equiv 0\}$.

Clearly, $\mathcal{U}$ contains φ_λ for each $\lambda \in (i\mathfrak{t})^*$ which is analytically integral and dominant. Then, taking into account the action of the Weyl group, we see that $\mathcal{U}$ also contains φ_λ for each $\lambda \in (i\mathfrak{t})^*$ which is analytically integral (and not necessarily dominant). In particular, for each λ in the root lattice, we have $\varphi_\lambda \in \mathcal{U}$, see [24], p. 130. This implies that the linear span of $\mathcal{U}$, says $\mathcal{V}$, contains $\sum_\alpha \mathbb{R}\varphi_\alpha$, where the sum is over all roots α. Consequently, we have $\{\varphi_\lambda : \lambda \in (i\mathfrak{t})^*\} \subset \mathcal{V}$. Finally, since $\mathcal{U}$ -hence $\mathcal{V}$- is stable under the coadjoint action of K_0, we get $\mathcal{V} = \mathfrak{k}_0^*$. $\square$

For each $X \in \mathfrak{g}$, let f_X be the function on $\mathfrak{p}^2 \times o(\varphi_0)$ defined by

$$f_X(T, S, \varphi) = \langle \Psi(T, S, \varphi),\, X\rangle.$$

Then we can easily verify that we have

$$f_X(T, S, \varphi) = \langle e(T), w\rangle + \langle \varphi, L(e(T), A)\rangle + \langle A \cdot T, S\rangle.$$

Proposition 5.3. *For each $X, Y \in \mathfrak{g}$, we have $\{f_X, f_Y\}_2 = f_{[X,Y]}$.*

Proof. For $X, Y \in \mathfrak{g}$, we consider the fonction $h := \{f_X, f_Y\}_2 - f_{[X,Y]}$ on $\mathfrak{p}^2 \times o(\varphi_0)$. Then by an easy computation, we see that h is of the form

$$h(T, S, \varphi) = \langle a(T, S), \varphi\rangle + b(T, S)$$

where $a \in C^\infty(\mathfrak{p}^2, \mathfrak{k})$ and $b \in C^\infty(\mathfrak{p}^2)$.

Now, an immediate consequence of Proposition 4.4 is that $h(T, S, \varphi) = 0$ for each $(T, S) \in \mathfrak{p}^2$ and each $\varphi \in \mathcal{U}$. In particular, one has

$$\langle a(T, S), \varphi_\lambda\rangle + b(T, S) = 0$$

for each $(T, S) \in \mathfrak{p}^2$ and each λ in the root lattice, then we get $b = 0$ and, consequently, we have $\langle a(T, S), \varphi\rangle = 0$ for each $(T, S) \in \mathfrak{p}^2$ and $\varphi \in \mathcal{U}$. By Lemma 5.2, this is also true for each $\varphi \in \mathfrak{k}_0^*$, hence we get $h = 0$. $\square$

Finally, from Proposition 5.1 and Proposition 5.3, we can deduce the following result.

Proposition 5.4. *The map $\Psi : \mathfrak{p}^2 \times o(\varphi_0) \to \mathcal{O}(\xi_0)$ is a symplectomorphism. Consequently, $\mathcal{O}(\xi_0)$ has symplectic decomposition $\mathfrak{p}^2 \times o(\varphi_0)$.*

6. The Poincaré group.

Here, let $V = \mathbb{R}^{n+1}$ and let $K = SO_0(n, 1)$ be the identity component of $SO(n, 1)$. Then G is the (generalized) Poincaré group. The usual Poincaré group corresponds to the case $n = 3$.

Recall that $SO(n,1)$ is the group of all real $(n+1)\times(n+1)$ matrices of determinant 1 leaving invariant the bilinear form on V defined by

$$\langle p,p'\rangle = -\left(\sum_{k=1}^{n} p_i\,p'_i\right) + p_{n+1}\,p'_{n+1}.$$

We can identify V^* to V by using this bilinear form.

Let $(e_1,e_2,\ldots,e_{n+1})$ be the standard basis of $\mathbb{R}^{n+1}$. We take $p_0 = me_{n+1}$ where $m>0$. Then K_0 is the subgroup of K consisting of all matrices of the form $\begin{pmatrix} k_0 & 0\\ 0 & 1\end{pmatrix}$ for $k_0\in SO(n,\mathbb{R})$ and the orbit $Z(p_0)$ is then the sheet of the hyperboloid $\langle p,p\rangle = m^2$ defined by $p_{n+1}>0$.

For each $1\le i,\, j\le n+1$, let E_{ij} be the matrix whose ij-th entry is 1 and all of the other entries are 0. The matrices $A_{ij} = E_{ji}-E_{ij}\quad(1\le i<j\le n)$ form a basis for $\mathfrak{k}_0$ and the matrices $E_k = E_{kn+1}+E_{n+1k}\quad(1\le k\le n)$ a basis for $\mathfrak{p}$.

We can identify $\mathfrak{k}^*$ with $\mathfrak{k}$ by using the form defined on $\mathfrak{k}$ by $\langle X,Y\rangle = \dfrac{1}{2}\operatorname{Tr}(XY)$ which is a multiple of the Killing form. Note that the basis $(E_k)_{1\le k\le n}$ of $\mathfrak{p}$ is orthonormal with respect to $\langle\cdot,\cdot\rangle$. Moreover, in the identification $\mathfrak{k}^*\simeq\mathfrak{k}$, the matrix A_{ij} $(1\le i<j\le n)$ corresponds to the element $e_i\wedge e_j$ of $\mathfrak{k}^*$ and the matrix E_k $(1\le k\le n)$ to $e_k\wedge e_{n+1}$.

Let j be the isomorphism from $\mathbb{R}^n$ onto $\mathfrak{p}$ defined by $j(t)=\displaystyle\sum_{k=1}^{n} t_kE_k$. For $T=j(t)\in\mathfrak{p}$, we denote $|T| := \langle T,T\rangle^{1/2} = |t|$. Then, since for each $T=j(t)\in\mathfrak{p}$ we have

$$\exp T = I_n + \frac{\sinh|T|}{|T|}T + \frac{\cosh|T|}{|T|^2}T^2,$$

we get

$$e(T) = m\left(\frac{\sinh|T|}{|T|}t_1,\ldots,\frac{\sinh|T|}{|T|}t_n,\cosh|T|\right).$$

On the other hand, from the equality

$$(\operatorname{ad}T)^{2n}S = |T|^{2n-2}(\operatorname{ad}T)^2S = |T|^{2n-2}(|T|^2S-\langle T,S\rangle T)$$

for $T,S\in\mathfrak{p}$ and $n\ge1$, we easily deduce the following formula for Ψ

$$\Psi(T,S,\varphi)=\left(e(T),\varphi+[T,S]+\frac{\tanh\frac{1}{2}|T|}{|T|}[T,\varphi]+\frac{|T|}{\tanh|T|}S-\frac{|T|-\tanh|T|}{|T|^2\tanh|T|}\langle T,S\rangle T\right).$$

However, this expression of Ψ is rather complicated. So, we aim to find a more simple symplectomorphism. This can be done by proceeding as follows. First, we solve the equation

$$x \wedge p = \left(\operatorname{ad} T + \frac{\operatorname{ad} T}{\tanh \operatorname{ad} T} \right) S, \qquad x \in \mathbb{R}^n \times (0)$$

for $T = j(t), S = j(s) \in \mathfrak{p}$, $p = e(T)$ and we easily find the solution

$$x = x(t,s) := \frac{1}{m} \left(\frac{|t|}{\sinh |t|} s + \frac{|t| \cosh |t| - \sinh |t|}{|t|^2 \sinh |t| \cosh |t|} \langle t, s \rangle t \right).$$

Now, a tedious but easy computation shows that the map $\sigma : (t,s) \to (\tilde{p}, \tilde{q})$ defined by the equations $\tilde{q} = x(t,s)$ and $e(j(t)) = (\tilde{p}, p_{n+1})$ is a symplectomorphism of $\mathbb{R}^{2n}$. Then the map Ψ' defined by $\Psi'(\tilde{p}, \tilde{q}, \varphi) := \Psi(\sigma^{-1}(\tilde{p}, \tilde{q}), \varphi)$ is a symplectomorphism from $\mathbb{R}^{2n} \times o(\varphi_0)$ onto $\mathcal{O}(\xi_0)$. Moreover, it is clear that

$$\Psi'(\tilde{p}, \tilde{q}, \varphi) = \left(p, \varphi + \frac{\tanh \frac{1}{2} |T(p)|}{|T(p)|} [T(p), \varphi] + \tilde{q} \wedge p \right)$$

where $p = (\tilde{p}, p_{n+1}) \in Z(p_0)$ and $T(p)$ is the unique element T of $\mathfrak{p}$ such that $e(T) = p$.

In particular, we recover the symplectomorphism introduced in [7] which is well known for $n = 3$, see for instance [13].

REFERENCES

[1] S. T. Ali, M. Englis. Quantization methods: a guide for physicists and analysts. *Rev. Math. Phys.* **17**, *4* (2005), 391–490.

[2] P. Baguis. Semidirect products and the Pukansky condition. *J. Geom. Phys.* **25**, *3–4* (1998), 245–270.

[3] F. A. Berezin. Quantization. *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 1116–1175 (in Russian); English translation in *Math. USSR Izv.* **8**, *5* (1974), 1109–1165.

[4] F. A. Berezin. Quantization in complex symmetric domains. *Izv. Akad. Nauk SSSR Ser. Mat.* **39**, *2* (1975), 363–402, 472 (in Russian); English translation in *Math. USSR Izv.* **9**, *2* (1975), 341–379.

[5] J. Bernatska, P. Holod. Geometry and topology of coadjoint orbits of semisimple Lie groups. Ninth International Conference on Geometry, Integrability and Quantization, June 8–13, 2007, Varna, Bulgaria, (Ed. I. M. Mladenov), SOFEX, Sofia 2008, pp 146–166.

[6] B. Cahen. Deformation Program for Principal Series Representations. *Lett. Math. Phys.* **36**, *1* (1996), 65–75.

[7] B. Cahen. Quantification d'une orbite massive d'un groupe de Poincaré généralisé. *C. R. Acad. Sci. Paris Sér I Math.* **325**, *7* (1997), 803–806.

[8] B. Cahen. Weyl quantization for semidirect products. *Differential Geom. Appl.* **25**, *2* (2007), 177–190.

[9] B. Cahen. Berezin quantization on generalized flag manifolds. *Math. Scand.* **105**, *1* (2009), 66–84.

[10] B. Cahen. Contraction of compact semisimple Lie groups via Berezin quantization. *Illinois J. Math.* **53**, *1* (2009), 265–288.

[11] B. Cahen. A contraction of the principal series by Berezin-Weyl quantization. *Rocky Mountain J. Math.* **43**, *2* (2013), 417–441.

[12] J. Rawnsley, M. Cahen, S. Gutt. Quantization on Kähler manifolds. I. Geometric interpretation of Berezin's quantization. *J. Geom. Phys.* **7**, *1* (1990), 45–62.

[13] J. F. Cariñena, J. M. Gracia-Bondìa, J. C. Vàrilly. Relativistic quantum kinematics in the Moyal representation. *J. Phys. A* **23**, *6* (1990), 901–933.

[14] P. Cotton, A. H. Dooley. Contraction of an adapted functional calculus. *J. Lie Theory* **7**, *2* (1997), 147–164.

[15] G. B. Folland. Harmonic Analysis in Phase Space. Annals of Mathematics Studies, vol. **122**. Princeton, NJ, Princeton University Press, 1989.

[16] V. Guillemin, S. Sternberg. Symplectic Techniques in Physics. Cambrige, Cambrige University Press, 1984.

[17] S. Helgason. Differential Geometry, Lie Groups and Symmetric Spaces. Graduate Studies in Mathematics, vol. **34**, Providence, RI, American Mathematical Society, 2001.

[18] L. Hörmander. The analysis of linear partial differential operators. III. Pseudodifferential operators, Section 18.5. Fundamental Principles of Mathematical Sciences, vol. **274**. Berlin, Heidelberg, New-York, Springer-Verlag, 1985.

[19] A. A. Kirillov. Lectures on the orbit method. Graduate Studies in Mathematics, vol. **64**. Providence, RI, American Mathematical Society, 2004.

[20] B. Kostant. Quantization and unitary representations. I. Prequantization. Lectures in modern analysis and applications, III, pp. 87–208. Lecture Notes in Math., vol. **170**. Berlin, Springer, 1970.

[21] N. P. Landsman. Mathematical Topics Between Classical and Quantum Mechanics. Springer Monographs in Mathematics. Berlin, Heidelberg, New-York, Springer-Verlag, 1998.

[22] P. Libermann, C.-M. Marle. Symplectic Geometry and Analytical Mechanics. Mathematics and its Applications, vol. **35**. Dordrecht, D. Reidel Publishing Co., 1987.

[23] J. H. Rawnsley. Representations of a semi direct product by quantization. *Math. Proc. Camb. Phil. Soc.* **78**, *2* (1975), 345–350.

[24] M. R. Sepanski. Compact Lie groups. Graduate Texts in Mathematics, vol. **235**. New York, Springer, 2007.

[25] D. J. Simms. Lie Groups and Quantum Mechanics. Lecture Notes in Mathematics, vol. **52**, Berlin, Heidelberg, New-York, Springer-Verlag, 1968.

[26] M. E. Taylor. Noncommutative Harmonis Analysis. Mathematical Surveys and Monographs, vol. **22**. Providence, RI, American Mathematical Society, 1986.

[27] A. Voros. An algebra of pseudo differential operators and the asymptotics of quantum mechanics. *J. Funct. Anal.* **29**, *1* (1978), 104–132.

[28] N. J. Wildberger. On the Fourier transform of a compact semisimple Lie group. *J. Austral. Math. Soc. A* **56**, *1* (1994), 64–116.

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