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CONVERGENCE ANALYSIS OF SEMI-IMPLICIT EULER METHOD FOR NONLINEAR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

M. Milev, O. Farkhondeh Rouz, D. Ahmadian

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Abstract. The main purpose of this paper is to study the convergence of numerical solutions to a class of neutral stochastic delay differential equations (NSDDEs) in Itô sense. The basic idea is to reformulate the original problem eliminating the dependence on the differentiation of the solution in the past values, which leads to a stochastic differential algebraic system. It is shown that the Semi-implicit Euler (SIE) method with two parameters θ and λ is mean-square convergent with order $p = \frac{1}{2}$ for Lipschitz continuous coefficients of underlying NSDDEs. A nonlinear numerical example illustrates the theoretical results.

1. Introduction. Stochastic functional differential equations (SFDEs), as an important mathematical model, appear in science and engineering applications, especially for systems whose evolution in time is influenced by random forces as well as its history information. Both the theory and numerical methods for SFDEs have been well developed in the recent decades. If the time delay

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in SFDEs reduces to a constant, it is usually called stochastic delay differential equations (SDDEs) (see [1] and [11]). Motivated by chemical engineering systems and the theory of aeroelasticity, Kolmanovskii et al. [8] introduced a class of neutral stochastic functional differential equations (NSFDEs), which can be identified as SFDEs. For the theory of NSFDEs we refer to [7] and [12]. The scalar neutral stochastic differential equations with fixed time delay (NSDDE) has the following general form

$$\begin{cases} d[x(t) - N(x(t-\tau))] = f\big(t, x(t), x(t-\tau)\big)dt + g\big(t, x(t), x(t-\tau)\big)dW(t), \ t > 0 \\ x(t) = \psi(t) \in C([-\tau, 0]; \mathbb{R}^n), \end{cases}$$

where $\tau > 0$ is a fixed constant.

In practice, many system models are described by NSDDEs. The models involve not only time delays in the state but also has time delay included in the state derivatives (see [3] and [6]). Since most of these equations cannot be solved explicitly, numerical approximations became to be an important tool in studying stochastic systems of neutral type (see [2] and [15]).

Mean-square convergence analysis of numerical solution for system of stochastic differential equations (SDEs) is one of the key problems in stochastic analysis (see [4]). However, the study on convergence of numerical method for neutral stochastic differential systems is relatively scarce due to their technical difficulties, which is the main topic of the present paper. Also for the convergence analysis of numerical solution on SDEs, there exist mostly concerned papers. For example, Li et al. [10, 9] discussed the convergence of the numerical solutions for SDDE with jumps and SDDE with Poisson jump and Markovian switching. Zhou and Wu [17] studied the convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching. Zhang and Gan [16] considered the mean square convergence of one-step methods for NSDDEs. Also Milošević [13] studied the convergence and almost sure exponential stability of implicit numerical methods for a class of highly nonlinear NSDDEs. Tan and Wang [14] studied the convergence and stability of the split-step backward Euler method for linear stochastic delay integro-differential equations. Gan et al. [5] investigated the mean square convergence of stochastic θ-methods for nonlinear neutral stochastic differential delay equations.

Based on these papers, to the best of our knowledge, convergence analysis of semi-implicit Euler method for NSDDEs has never been considered so far. In this paper, we derive a sufficient condition of the mean-square convergence of the SIE method for NSDDEs and develop the method to two parameters θ and λ which is novel and its property is such a way can accelerate the mean-square convergence as show in the proof. Also in the numerical section we see that the parameters θ and λ can enhance the accuracy of the convergence for SIE method.

The rest of the paper is organized as follows: Section 2 begins with notations and preliminaries, then introduces the SIE method with two parameters θ and λ for NSDDEs. Section 3 proves that the SIE method is mean-square convergent to the exact solution with the strong convergence order $p = \frac{1}{2}$. Section 5 gives a nonlinear numerical example to confirm the theoretical results.

2. Preliminaries and notations. Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ denotes both the Euclidean norm in $\mathbb{R}^n$ and the trace (or Frobenius) norm in $\mathbb{R}^{n\times d}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. $a \vee b$ represents $\max\{a, b\}$ and $a \wedge b$ denotes $\min\{a, b\}$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is right continuous and satisfies that each $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets, and $W(t)$ be a d-dimensional standard Wiener process defined on this probability space. Let $N : \mathbb{R}^n \to \mathbb{R}^n$, $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$ be Borel measurable real-valued functions satisfy the global lipschitz condition.

Consider the n-dimensional NSDDE in Itô-sense

(1)
$$d[x(t)-N(x(t-\tau))] = f\big(t, x(t), x(t-\tau)\big)dt+g\big(t, x(t), x(t-\tau)\big)dW(t), \quad t \in [t_0, T],$$

with initial data $x(t) = \psi(t) \in C([t_0 - \tau, t_0]; \mathbb{R}^n)$, satisfying

$$\mathbb{E}\big(\sup_{t_0-\tau\leq t\leq t_0} |\psi(t)|^2\big) < +\infty,$$

where $\tau > 0$ is delay time.

Assumption 2.1 (**Contractive Mapping**). *Assume that for all $x, y \in \mathbb{R}^n$, there exists a positive constant $\kappa \in (0, 1)$ such that*

(2)
$$|N(x) - N(y)| \leq \kappa|x - y|.$$

Now we introduce the semi-implicit Euler (SIE) approximation $\{y_k\}_{k\geq 0}$ as follows:

(3)
$$y_{k+1} = y_k + N(y_{k+1-N_\tau}) - N(y_{k-N_\tau}) + \theta f(y_{k+1}, y_{k+1-N_\tau})\Delta + \lambda g(y_k, y_{k-N_\tau})\Delta W_k,$$

where stepsize $\Delta = \frac{\tau}{N_\tau}$ for a integer N_τ, $x_k = y_k = \psi(k\Delta)$ for $k = -N_\tau, -N_\tau+1, \dots, -1$, $y_0 = \psi(0)$, θ and λ are fixed parameters in interval $(0, 1]$. The Wiener increments is defined as $\Delta W_k := W((k+1)\Delta) - W(k\Delta)$, where $W(k\Delta)$ denotes the Wiener process at time $k\Delta$.

3. Mean-square convergence analysis. In this section we examine the mean-square convergence of SIE method for $\theta, \lambda \in (0,1]$. To this aim for every $0 \le t_0 \le t < T < \infty$, equation (1) has the following integral form

$$x(t) - N(x(t-\tau)) = x(t_0) - N(x(t_0-\tau)) + \int_{t_0}^{t} f(t, x(s), x(s-\tau))ds + \int_{t_0}^{t} g(t, x(s), x(s-\tau))dW(s). \tag{4}$$

Assumption 3.1. *Let f and g satisfy the following conditions: there exist positive constants $\overline{K}_1$, $\overline{K}_2$ and $\overline{K}_3$ such that, for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, and $t \in [t_0, T]$ we have*

$$|f(t, x_2, y_2) - f(t, x_1, y_1)|^2 \vee |g(t, x_2, y_2) - g(t, x_1, y_1)|^2 \le \overline{K}_1(|x_2 - x_1|^2 + |y_2 - y_1|^2), \tag{5}$$

$$|f(t, x, y)|^2 \vee |g(t, x, y)|^2 \le \overline{K}_2(1 + |x|^2 + |y|^2), \tag{6}$$

and

$$|f(t, x, y) - f(s, x, y)|^2 \vee |g(t, x, y) - g(s, x, y)|^2 \le \overline{K}_3(1 + |x|^2 + |y|^2)|t - s|, \tag{7}$$

for all $x, y \in \mathbb{R}^n$ and $t, s \in [t_0, T]$.

Proposition 3.2. *If the contractive condition* (2) *and the linear growth condition* (6) *are fulfilled, then for $p \ge 2$, we have*

$$\mathbb{E}\big(\sup_{t_0-\tau \le s \le t} |x(t)|^p\big) \le C_L\big(1 + \mathbb{E}\big(\sup_{t_0-\tau \le t \le t_0} |\psi(t)|^p\big)\big), \tag{8}$$

where C_L depends on κ, $\overline{K}_2$ and T (see [12]).

Lemma 3.3. *Let conditions* (2), (5) *and* (6) *hold. Assume that the initial function $\psi(t)$ is Hölder continuous, that is there is a positive constant L_1 such that*

$$\mathbb{E}|\psi(\bar{t}_2) - \psi(\bar{t}_1)|^2 \le L_1|\bar{t}_2 - \bar{t}_1|, \quad \text{if} \quad t_0 - \tau \le \bar{t}_1 < \bar{t}_2 \le t_0, \tag{9}$$

then there exist constant L_2 *depending on* L_1, T *and initial function* $\psi(t)$, *such that*

$$\mathbb{E}|x(t)-x(s)|^2 \leq L_2|t-s|, \qquad \textit{for all} \quad t_0 \leq s < t \leq T, \tag{10}$$

with $t - m\tau \in [-\tau, 0]$ *and* $s - m\tau \in [-\tau, 0]$, *for any positive integer* m *(see [5]).*

By defining the function $z(t) = x(t) - N(x(t-\tau))$, equation (1) can be rewritten as follows:

$$dz(t) = \widehat{f}\big(t, z(t), x(t-\tau)\big)dt + \widehat{g}\big(t, z(t), x(t-\tau)\big)dW(t), \qquad t_0 \leq t \leq T, \tag{11}$$

with the initial condition:

$$z(t_0) = x(t_0) - N(x(t_0-\tau)), \tag{12}$$

where

$$\widehat{f}(t,z,u) = f(t, z+N(u), u), \qquad \widehat{g}(t,z,u) = g(t, z+N(u), u). \tag{13}$$

For convenience we can assume $N_\tau \geq 2$, therefore SIE method for problem (11)–(12) can be rewritten as follows:

$$\begin{cases} z_{k+1} = z_k + \theta \widehat{f}(t_{k+1}, z_{k+1}, x_{k+1-N_\tau})\Delta + \lambda \widehat{g}(t_k, z_k, x_{k-N_\tau})\Delta W_k \\ \hfill k = 0, 1, \ldots, N-1, \\ z_k = x_k - N(x_{k-N_\tau}), \quad k = 0, 1, \ldots, N, \end{cases} \tag{14}$$

where z_k is approximate solution to $z(t_k)$, and the initial values are given by $z_0 = z(t_0)$, $x_{k-N_\tau} = \psi(t_0 + t_k - \tau)$ for $k - N_\tau \leq 0$. We define the local error of SIE method (14) as follows:

$$\begin{aligned} \delta_\Delta(t_k) = z(t_{k+1}) - z(t_k) - \theta \widehat{f}\big(t_{k+1}, z(t_{k+1}), x(t_{k+1}-\tau)\big)\Delta \\ - \lambda \widehat{g}\big(t_k, z(t_k), x(t_k-\tau)\big)\Delta W_k. \end{aligned} \tag{15}$$

Theorem 3.4. *Assume that all the conditions Assumption* 3.1 *hold and by the Proposition* 3.2 *for any* $\theta, \lambda \in (0,1]$, *the algorithm defined in* (14) *is mean-square convergent with order* $p = \frac{1}{2}$, *when*

$$\max_{0 \leq k \leq N} \big(\mathbb{E}(|z(t_k) - z_k|^2)\big)^{\frac{1}{2}} \leq \overline{C}\Delta^{\frac{1}{2}} \quad \textit{as} \quad \Delta = \frac{\tau}{N_\tau} \to 0, \tag{16}$$

where the constant $\overline{C}$ *dose not depend on stepsize* Δ, *but may depend on* T *and the initial data.*

Proof. With the estimate of local error, we can make an estimate of global error

$$e(t_k) := z(t_k) - z_k.$$

Note that the global error $e(t_k)$ is $\{\mathcal{F}_{t_k}\}_{t_k\geq 0}$-measurable, since both $z(t_k)$ and z_k are $\{\mathcal{F}_{t_k}\}_{t_k\geq 0}$-measurable random variables. By using the local error (15), we have

$$e(t_{k+1}) - e(t_k) = R(t_k) + \delta_\Delta(t_k), \tag{17}$$

where

$$\begin{aligned} R(t_k) = \theta\big(&\widehat{f}(t_{k+1}, z(t_{k+1}), x(t_{k+1}-\tau)) - \widehat{f}(t_{k+1}, z_{k+1}, x_{k+1-N_\tau})\big)\Delta \\ &+ \lambda\big(\widehat{g}(t_k, z(t_k), x(t_k-\tau)) - \widehat{g}(t_k, z_k, x_{k-N_\tau})\big)\Delta W_k. \end{aligned} \tag{18}$$

By using (17) and the elementary inequality $2a^T b \leq |a|^2 + |b|^2$, $a, b \in \mathbb{R}^n$, we can obtain

$$\begin{aligned} \mathbb{E}|e(t_{k+1})|^2 \leq \mathbb{E}|e(t_k)|^2 &+ 2\mathbb{E}|R(t_k)|^2 + 2\mathbb{E}|\delta_\Delta(t_k)|^2 \\ &+ 2|\mathbb{E}\langle e(t_k), \delta_\Delta(t_k)\rangle| + 2|\mathbb{E}\langle e(t_k), R(t_k)\rangle|. \end{aligned} \tag{19}$$

Recall the important inequality for any $p \geq 1$ and $x_1, x_2, \ldots, x_n \geq 0$, and n is a positive integer, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq n^{p-1} \sum_{i=1}^{n} x_i^p. \tag{20}$$

Therefore from the relation (18), we have

$$\begin{aligned} 2\mathbb{E}|R(t_k)|^2 &\leq 4\theta^2\Delta^2\mathbb{E}|\widehat{f}(t_{k+1}, z(t_{k+1}), x(t_{k+1}-\tau)) - \widehat{f}(t_{k+1}, z_{k+1}, x_{k+1-N_\tau})|^2 \\ &\quad + 4\lambda^2\Delta\mathbb{E}|\widehat{g}(t_k, z(t_k), x(t_k-\tau)) - \widehat{g}(t_k, z_k, x_{k-N_\tau})|^2 \\ &\leq 4\theta^2\Delta^2\mathbb{E}|f(t_{k+1}, x(t_{k+1}), x(t_{k+1}-\tau)) - f(t_{k+1}, x_{k+1}, x_{k+1-N_\tau})|^2 \\ &\quad + 4\lambda^2\Delta\mathbb{E}|g(t_k, x(t_k), x(t_k-\tau)) - g(t_k, x_k, x_{k-N_\tau})|^2 \\ &\leq 4\overline{K}_1\theta^2\Delta^2\mathbb{E}\left(|x(t_{k+1}) - x_{k+1}|^2 + |x(t_{k+1}-\tau) - x_{k+1-N_\tau}|^2\right) \\ &\quad + 4\overline{K}_1\lambda^2\Delta\mathbb{E}\left(|x(t_k) - x_k|^2 + |x(t_k-\tau) - x_{k-N_\tau}|^2\right). \end{aligned} \tag{21}$$

A combination of (21) and the fact that

$$|x(t_k) - x_k|^2 = |z(t_k) + N(x(t_k-\tau)) - z_k - N(x_{k-N_\tau})|^2$$

$$\leq 2|e(t_k)|^2 + 2\kappa^2|x(t_k-\tau) - x_{k-N_\tau}|^2, \tag{22}$$

leads to the estimation

$$\begin{aligned}
2\mathbb{E}|R(t_k)|^2 &\leq 4\overline{K}_1\theta^2\Delta^2\big(2\mathbb{E}|e(t_{k+1})|^2 + (1+2\kappa^2)\mathbb{E}|x(t_{k+1}-\tau) - x_{k+1-N_\tau}|^2\big)\\
&\quad + 4\overline{K}_1\lambda^2\Delta\big(2\mathbb{E}|e(t_k)|^2 + (1+2\kappa^2)\mathbb{E}|x(t_k-\tau) - x_{k-N_\tau}|^2\big)\\
&\leq C_1\Delta\big(\mathbb{E}|e(t_{k+1})|^2 + \mathbb{E}|e(t_k)|^2 + \mathbb{E}|x(t_{k+1}-\tau) - x_{k+1-N_\tau}|^2\\
&\quad + \mathbb{E}|x(t_k-\tau) - x_{k-N_\tau}|^2\big),
\end{aligned} \tag{23}$$

where $C_1 = \max\big\{8\overline{K}_1\theta^2\nu,\ 4\overline{K}_1\theta^2\nu(1+2\kappa^2),\ 8\overline{K}_1\lambda^2,\ 4\overline{K}_1\lambda^2(1+2\kappa^2)\big\}$ with $\nu = \dfrac{\tau}{2}$.

Also by using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
2|\mathbb{E}\langle e(t_k), \delta_\Delta(t_k)\rangle| &= 2|\mathbb{E}\big(\mathbb{E}\left(\langle e(t_k), \delta_\Delta(t_k)\rangle \mid \mathcal{F}_{t_k}\right)\big)|\\
&\leq 2\mathbb{E}|\langle e(t_k), \mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})\rangle|\\
&\leq 2\big(\Delta\mathbb{E}|e(t_k)|^2\big)^{\frac{1}{2}}\big(\Delta^{-1}\mathbb{E}|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2\big)^{\frac{1}{2}}\\
&\leq \Delta\mathbb{E}|e(t_k)|^2 + \Delta^{-1}\mathbb{E}|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2.
\end{aligned} \tag{24}$$

By using Jensen's inequality $|\mathbb{E}(X \mid \mathcal{Y})|^2 \leq \mathbb{E}(|X|^2 \mid \mathcal{Y})$ and the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, for the estimation $|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2$, we obtain

$$\begin{aligned}
&|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2\\
&= |\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} \big(f(s,x(s),x(s-\tau)) - f(t_k,x(t_k),x(t_k-\tau))\big)ds \mid \mathcal{F}_{t_k}\Big)\\
&\quad -\Delta\mathbb{E}\big(f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_k),x(t_k-\tau)) \mid \mathcal{F}_{t_k}\big)|^2\\
&\leq 2|\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} \big(f(s,x(s),x(s-\tau)) - f(t_k,x(t_k),x(t_k-\tau))\big)ds \mid \mathcal{F}_{t_k}\Big)|^2\\
&\quad +2\Delta^2|\mathbb{E}\big(f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_k),x(t_k-\tau)) \mid \mathcal{F}_{t_k}\big)|^2\\
&\leq 2\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |f(s,x(s),x(s-\tau)) - f(t_k,x(t_k),x(t_k-\tau))|^2 ds \mid \mathcal{F}_{t_k}\Big)\\
&\quad +2\Delta^2\mathbb{E}\big(|f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_k),x(t_k-\tau))|^2 \mid \mathcal{F}_{t_k}\big).
\end{aligned} \tag{25}$$

According to conditions (5) and (7), we have

$$
\begin{aligned}
&|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2 \\
&\leq 4\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |f(s,x(s),x(s-\tau)) - f(t_k,x(s),x(s-\tau))|^2 ds \mid \mathcal{F}_{t_k}\Big) \\
&\quad +4\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |f(t_k,x(s),x(s-\tau)) - f(t_k,x(t_k),x(t_k-\tau))|^2 ds \mid \mathcal{F}_{t_k}\Big) \\
&\quad +4\Delta^2\mathbb{E}\big(|f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_{k+1}),x(t_{k+1}-\tau))|^2 \mid \mathcal{F}_{t_k}\big) \\
&\quad +4\Delta^2\mathbb{E}\big(|f(t_k,x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_k),x(t_k-\tau))|^2 \mid \mathcal{F}_{t_k}\big) \\
&\leq 4\overline{K}_3\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} (1+|x(s)|^2+|x(s-\tau)|^2)|s-t_k| ds \mid \mathcal{F}_{t_k}\Big) \\
&\quad +4\overline{K}_1\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} (|x(s)-x(t_k)|^2+|x(s-\tau)-x(t_k-\tau)|^2) ds \mid \mathcal{F}_{t_k}\Big) \\
&\quad +4\overline{K}_3\Delta^3\mathbb{E}\big(1+|x(t_{k+1})|^2+|x(t_{k+1}-\tau)|^2 \mid \mathcal{F}_{t_k}\big) \\
&\quad +4\overline{K}_1\Delta^2\mathbb{E}\big(|x(t_{k+1})-x(t_k)|^2+|x(t_{k+1}-\tau)-x(t_k-\tau)|^2 \mid \mathcal{F}_{t_k}\big).
\end{aligned} \tag{26}
$$

Subsequently, by using Proposition 3.2 and Lemma 3.3, we get the final inequality of (26)

$$
\begin{aligned}
\mathbb{E}|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2 &\leq 2\overline{K}_3\left(1+2C_L\big(1+\mathbb{E}(\sup_{t_0-\tau\leq t\leq t_0}|\psi(t)|^p)\big)\right)\Delta^3 + 4\overline{K}_1L_2\Delta^3 \\
&\quad + 4\overline{K}_3\left(1+2C_L\big(1+\mathbb{E}(\sup_{t_0-\tau\leq t\leq t_0}|\psi(t)|^p)\big)\right)\Delta^3 + 8\overline{K}_1L_2\Delta^3 \\
&= C_2\Delta^3,
\end{aligned} \tag{27}
$$

where $C_2 = 6\overline{K}_3\big(1+2C_L\big(1+\mathbb{E}(\sup_{t_0-\tau\leq t\leq t_0}|\psi(t)|^p)\big)\big) + 12\overline{K}_1L_2$, which implies

$$
\begin{aligned}
2|\mathbb{E}\langle e(t_k),\delta_\Delta(t_k)\rangle| &\leq \Delta\mathbb{E}|e(t_k)|^2 + \Delta^{-1}\mathbb{E}|\mathbb{E}(\delta_\Delta(t_k) \mid \mathcal{F}_{t_k})|^2 \\
&\leq \Delta\mathbb{E}|e(t_k)|^2 + C_2\Delta^2.
\end{aligned} \tag{28}
$$

For the estimation $\mathbb{E}|\delta_\Delta(t_k)|^2$, we can easily derive that

$$
\begin{aligned}
2\mathbb{E}|\delta_\Delta(t_k)|^2 &\leq 12\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |f(s,x(s),x(s-\tau)) - f(t_k,x(s),x(s-\tau))|^2 ds\Big) \\
&\quad + 12\Delta\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |f(t_k,x(s),x(s-\tau)) - f(t_k,x(t_k),x(t_k-\tau))|^2 ds\Big)
\end{aligned}
$$

$$+ 12\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |g(s,x(s),x(s-\tau)) - g(t_k,x(s),x(s-\tau))|^2 ds \Big)$$
$$+ 12\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} |g(t_k,x(s),x(s-\tau)) - g(t_k,x(t_k),x(t_k-\tau))|^2 ds \Big)$$
$$+ 12\Delta^2\mathbb{E}\big(|f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_{k+1}),x(t_{k+1}-\tau))|^2\big)$$
$$+ 12\Delta^2\mathbb{E}\big(|f(t_k,x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_k,x(t_k),x(t_k-\tau))|^2\big). \tag{29}$$

Again, by using Proposition 3.2 and Lemma 3.3, we get from (29) the following inequality:

$$2\mathbb{E}|\delta_\Delta(t_k)|^2 \le 12\overline{K}_3(1+\Delta)\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} (1+|x(s)|^2+|x(s-\tau)|^2)|s-t_k| ds \Big)$$
$$+ 12\overline{K}_1(1+\Delta)\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} (|x(s)-x(t_k)|^2+|x(s-\tau)-x(t_k-\tau)|^2) ds \Big)$$
$$+ 12\overline{K}_3\Delta^3\big(1+\mathbb{E}|x(t_{k+1})|^2+\mathbb{E}|x(t_{k+1}-\tau)|^2\big)$$
$$+ 12\overline{K}_1\Delta^2\big(\mathbb{E}|x(t_{k+1})-x(t_k)|^2+\mathbb{E}|x(t_{k+1}-\tau)-x(t_k-\tau)|^2\big), \tag{30}$$

which implies

$$2\mathbb{E}|\delta_\Delta(t_k)|^2 \le C_3\Delta^2, \tag{31}$$

with $C_3 = 24\overline{K}_3\big(1+2C_L\big(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\psi(t)|^p)\big)\big)+24\overline{K}_1L_2$.
It follows from (5), (18) and (22) that

$$|\mathbb{E}(R(t_k) \mid \mathcal{F}_{t_k})|^2$$
$$= |\theta\Delta\mathbb{E}\big(\widehat{f}(t_{k+1},z(t_{k+1}),x(t_{k+1}-\tau)) - \widehat{f}(t_{k+1},z_{k+1},x_{k+1-N_\tau}) \mid \mathcal{F}_{t_k}\big)|^2$$
$$\le \theta^2\Delta^2\mathbb{E}\big(|f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau)) - f(t_{k+1},x_{k+1},x_{k+1-N_\tau})|^2 \mid \mathcal{F}_{t_k}\big)$$
$$\le \overline{K}_1\theta^2\Delta^2\mathbb{E}\big(|x(t_{k+1})-x_{k+1}|^2+|x(t_{k+1}-\tau)-x_{k+1-N_\tau}|^2 \mid \mathcal{F}_{t_k}\big)$$
$$\le \overline{K}_1\theta^2\Delta^2\mathbb{E}\big(2|e(t_{k+1})|^2+(1+2\kappa^2)|x(t_{k+1}-\tau)-x_{k+1-N_\tau}|^2 \mid \mathcal{F}_{t_k}\big)$$
$$\le C_4\Delta^2\big(\mathbb{E}(|e(t_{k+1})|^2 \mid \mathcal{F}_{t_k})+\mathbb{E}|e(t_k)|^2+\mathbb{E}|x(t_{k+1}-\tau)-x_{k+1-N_\tau}|^2$$
$$+ \mathbb{E}|x(t_k-\tau)-x_{k-N_\tau}|^2\big), \tag{32}$$

where $C_4 = \max\big\{2\overline{K}_1\theta^2, \overline{K}_1\theta^2(1+2\kappa^2)\big\}$.

Therefore by using the Cauchy-Schwarz inequality for the last term of relation (19), we have

$$\begin{aligned}
2|\mathbb{E}\langle e(t_k), R(t_k)\rangle| &= 2|\mathbb{E}\left(\mathbb{E}\big(\langle e(t_k), R(t_k)\rangle \mid \mathcal{F}_{t_k}\big)\right)| \\
&\le 2\mathbb{E}|\langle e(t_k), \mathbb{E}(R(t_k) \mid \mathcal{F}_{t_k})\rangle| \\
&\le 2\big(\Delta\mathbb{E}|e(t_k)|^2\big)^{\frac{1}{2}}\big(\Delta^{-1}\mathbb{E}|\mathbb{E}(R(t_k) \mid \mathcal{F}_{t_k})|^2\big)^{\frac{1}{2}} \\
&\le \Delta\mathbb{E}|e(t_k)|^2 + C_4\Delta\big(\mathbb{E}|e(t_{k+1})|^2 + \mathbb{E}|e(t_k)|^2 + \mathbb{E}|x(t_{k+1}-\tau) - x_{k+1-N_\tau}|^2 \\
&\qquad + \mathbb{E}|x(t_k-\tau) - x_{k-N_\tau}|^2\big) \\
&\le C_5\Delta\big(\mathbb{E}|e(t_{k+1})|^2 + \mathbb{E}|e(t_k)|^2 + \mathbb{E}|x(t_{k+1}-\tau) \\
&\qquad - x_{k+1-N_\tau}|^2 + \mathbb{E}|x(t_k-\tau) - x_{k-N_\tau}|^2\big),
\end{aligned} \tag{33}$$

where $C_5 = C_4 + 1$. Inserting relations (23), (28), (31) and (33) into (19), it yields that

$$\begin{aligned}
\mathbb{E}|e(t_{k+1})|^2 &\le (C_1 + C_5)\Delta\mathbb{E}|e(t_{k+1})|^2 + \big(1 + (1 + C_1 + C_5)\Delta\big)\mathbb{E}|e(t_k)|^2 \\
&\quad + (C_1 + C_5)\Delta\big(\mathbb{E}|x(t_{k+1}-\tau) - x_{k+1-N_\tau}|^2 + \mathbb{E}|x(t_k-\tau) - x_{k-N_\tau}|^2\big) \\
&\quad + (C_2 + C_3)\Delta^2.
\end{aligned} \tag{34}$$

Let $m_T = \left[\dfrac{T - t_0}{\tau}\right] + 1$, and repeat of recursive relation of (22) we have

$$(1 - C_6\Delta)\mathbb{E}|e(t_{k+1})|^2 \le (1 + C_7\Delta)\mathbb{E}|e(t_k)|^2 + C_8\Delta \max_{0\le j\le k} \mathbb{E}|e(t_j)|^2 + C_9\Delta^2, \tag{35}$$

where

$$C_6 = C_1 + C_5, C_7 = 1 + C_1 + C_5, C_8 = 4\big(1 + 2\kappa^2 + \cdots + (2\kappa^2)^{m_T - 1}\big)(C_1 + C_5)$$

and $C_9 = C_2 + C_3$.

Let $\beta_k := \max\limits_{0\le j\le k} \mathbb{E}|e(t_j)|^2$ with $\beta_0 := 0$, and from (35) we get

$$\beta_{k+1} \le \frac{1 + (C_7 + C_8)\Delta}{1 - C_6\Delta}\beta_k + 2C_9\Delta^2 \le (1 + C_{10}\Delta)\beta_k + 2C_9\Delta^2, \tag{36}$$

where $C_{10} = 2(C_6 + C_7 + C_8)$. By induction, it can be achieved that

$$\beta_{k+1} \le (1 + C_{10}\Delta)^{k+1}\beta_0 + 2C_9\Delta^2 \sum_{j=0}^{k}(1 + C_{10}\Delta)^j \le \frac{2C_9}{C_{10}}\big(e^{C_{10}(T-t_0)} - 1\big)\Delta, \tag{37}$$

and by taking the square root, inequality (37) implies that

$$\max_{0\leq k\leq N}\big(\mathbb{E}(|z(t_k)-z_k|^2)\big)^{\frac{1}{2}}\leq \overline{C}\Delta^{\frac{1}{2}},$$

where $\overline{C}=\sqrt{2C_9(e^{C_{10}(T-t_0)}-1)/C_{10}}$. This completes the proof of Theorem 3.4. □

4. Numerical experiments. This section is devoted to present our theoretical estimates obtained by a numerical experiment. We consider the following nonlinear NSDDE:

$$d\left[x(t)-\frac{1}{4}\sin(x(t-1))\right] = \big(-6x(t)+x(t-1)\big)dt+x(t)\cos(x(t-1))dW(t),\ t\in[t_0,T], \tag{38}$$

with the initial data $x(t)=\psi(t)=1$ for $t\in[-1,0]$. We consider the mean-square convergence of equation (38). When $t\in[0,2]$, it is easy to see that the drift and diffusion coefficients satisfy the conditions (5) and (6). To illustrate the convergence of the SIE method, 10^3 sample trajectories are simulated. A set of 10 blocks each containing 100 outcomes $(i,j: 1\leq i\leq 10, 1\leq j\leq 100)$ are applied and for each block the estimator is defined as

$$e_i=\frac{1}{100}\sum_{j=1}^{100}|\overline{x}_{T,j,i}-x_{T,j,i}|^2, \tag{39}$$

where $\overline{x}_{T,j,i}$ denotes the approximate solution obtained using the SIE method for stepsizes $\Delta=2^{-9},2^{-8},2^{-7},2^{-6},2^{-5}$ and 2^{-4}. Then, the mean of estimator (39), which is itself estimated in the usual way

$$\text{Error}=\frac{1}{10}\sum_{i=1}^{10}e_i. \tag{40}$$

Since the exact solution $x_{T,j,i}$ needed in (39) is not available, a very accurate estimation of it is obtained as the reference solution by using a very large number of time steps or equivalently the stepsize so small ($\Delta=2^{-14}$). In order to evaluate the time convergence rate, the number of time steps employed N is progressively doubled. We show the Error results in Table 1 and Figure 1 for different cases of θ and λ as well different values of stepsize Δ.

Table 1. Numerical results for SIE method with $T = 2$.

Error	$\Delta = 2^{-9}$	$\Delta = 2^{-8}$	$\Delta = 2^{-7}$	$\Delta = 2^{-6}$	$\Delta = 2^{-5}$	$\Delta = 2^{-4}$
$\theta, \lambda = 0.1$	$5.72e^{-6}$	$1.68e^{-5}$	$2.30e^{-5}$	$5.15e^{-5}$	$1.05e^{-4}$	$2.60e^{-4}$
$\theta, \lambda = 0.5$	$4.64e^{-6}$	$6.71e^{-6}$	$7.50e^{-6}$	$1.18e^{-5}$	$2.34e^{-5}$	$3.37e^{-5}$
$\theta, \lambda = 1$	$3.55e^{-6}$	$3.96e^{-6}$	$5.31e^{-6}$	$1.07e^{-5}$	$1.86e^{-5}$	$2.71e^{-5}$

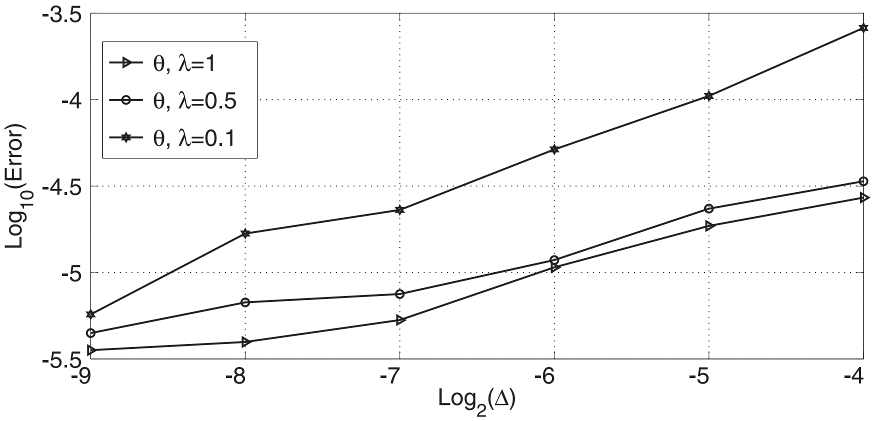

Fig. 1. The convergence rate of the SIE method

Conclusion. In this paper, we have investigated SIE method for nonlinear NSDDEs. In this regard we examined the mean-square convergence for these kind of equations. The parameters θ and λ can enhance the accuracy of the convergence for SIE method. We obtained the strong convergence of order $p = \frac{1}{2}$ and we show in Table 1 that get till 5 digit accuracy.

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M. Milev
University of Food Technology – Plovdiv
Head of Mathematics and Physics
4000 Plovdiv, Bulgaria
e-mail: marianmilev2002@gmail.com

Faculty of Mathematical Sciences
University of Tabriz
Tabriz, Iran
e-mail: omid_farkhonde_7088@yahoo.com *(O. Farkhondeh Rouz)*
e-mail: d.ahmadian@tabrizu.ac.ir *(D. Ahmadian)*

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