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A CHARACTERIZATION OF DUPIN HYPERSURFACES IN $\mathbb{R}^5$

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Abstract. In this paper we study Dupin hypersurfaces in $\mathbb{R}^5$ parametrized by lines of curvature, with four distinct principal curvatures. We give a local characterization of this class of hypersurfaces in terms of the principal curvatures and four vector valued functions of one variable. We prove that these vectorial functions describe plane curves or points in $\mathbb{R}^5$. We show that the Lie curvature of these Dupin hypersurfaces is constant with some conditions on the Laplace invariants and the Möbius curvature, but some Möbius curvatures are constant along certain lines of curvature. We give explicit examples of such Dupin hypersurfaces.

1. Introduction. Let M be an immersed hypersurface in Euclidean space $\mathbb{R}^n$ or the unit sphere $S^n \subset \mathbb{R}^{n+1}$. The hypersurface M is said to be *Dupin* if along each curvature surface the corresponding principal curvature is constant. The hypersurface M is called *proper Dupin* if the number g of distinct principal curvatures is constant on M.

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Dupin surfaces were first studied by Dupin in 1822 and more recently by many authors [1]–[6], [10]–[22], which studied several aspects of Dupin hypersurfaces. The class of Dupin hypersurfaces is invariant under Lie transformations [13]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations. Pinkall [14] gave a complete classification up to Lie equivalence for Dupin hypersurfaces $M^3 \subset \mathbb{R}^4$, with three distinct principal curvatures. Niebergall [11, 12], Cecil and Jensen [6] studied proper Dupin hypersurfaces with four distinct principal curvatures and constant Lie curvature. Niebergall [11, 12] proved that a connected irreducible proper Dupin hypersurface M^4 in S^5 with four principal curvatures and constant Lie curvature is Lie equivalent to an isoparametric hypersurface under an additional assumption. In [6], Cecil and Jensen proved that Niebergall's additional assumptions are unnecessary. They showed that every connected irreducible proper Dupin hypersurface M^4 in S^5 with four principal curvatures and constant Lie curvature is Lie equivalent to an isoparametric hypersurface. Later Cecil et al. [4] generalized this result to higher dimensions.

Tenenblat and Riveros [19] obtained a local characterization of the Dupin hypersurfaces in $\mathbb{R}^5$ parametrized by lines of curvature, with four distinct principal curvatures and $T_{ijkl} \neq 0$, in terms of the principal curvatures and four vector valued functions in $\mathbb{R}^5$ which are invariant by inversions and homotheties. Riveros in [17] studied Dupin hypersurfaces parametrized by lines of curvature with some conditions on the Laplace invariants.

Riveros, Rodrigues and Tenenblat [20] studied a class of proper Dupin hypersurfaces $M^n \subset$ in $\mathbb{R}^{n+1}$ parametrized by lines of curvature, with n distinct principal curvatures and constant Möbius curvature. They then showed that for $n \geq 3$ the condition of having constant Möbius curvature is equivalent to having all Laplace invariants equal to zero.

Tenenblat et al. [7], obtain a characterization for Dupin hypersurfaces in $\mathbb{R}^5$, parametrized by lines of curvature with four distinct principal curvatures and $T_{ijkl} \neq 0$, in terms of three vector values functions, this result improves the result obtained in [19].

In this paper we consider proper Dupin hypersurfaces in $\mathbb{R}^5$, parametrized by lines of curvature with four distinct principal curvatures and we ask if, it is possible to obtain a similar result to obtained in [7] with the condition $T_{ijkl} = 0$. The Theorem 3.1 gives an affirmative answer to this question, more precisely, we obtain a local characterization of a class of Dupin hypersurfaces parametrized by lines of curvature and $T_{ijkl} = 0$ in terms of the principal curvature functions and four vector valued functions of one variable. Moreover, it follows from results of

Pinkall [13] and Cecil and Jensen [4] that these hypersurfaces are either reducible or have nonconstant Lie curvature. Furthermore, this result shows that the characterization obtained in [7] is not true when $T_{ijkl} = 0$. Also, we prove that the vector valued functions in this characterization are plane curves or points in $\mathbb{R}^5$. Moreover, we prove that the Lie curvature of these hypersurfaces is constant with some assumptions on the laplace invariants and the Möbius curvature, but some Möbius curvatures are constant along certain lines of curvature. Finally, we give explicit examples of this class of Dupin hypersurfaces.

2. Properties of hypersurfaces with distinct principal curvatures. We consider Ω an open subset of $\mathbb{R}^n$ and $x = (x_1, x_2, \ldots, x_n) \in \Omega$. Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$ and $N : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ a unit normal vector field of X. Then

$$\begin{aligned} \langle X_{,i}, X_{,j} \rangle &= \delta_{ij} g_{ii},\ 1 \leq i, j \leq n, \\ N_{,i} &= -\lambda_i X_{,i}, \end{aligned} \tag{2.1}$$

where the subscript $_{,i}$ denotes the derivative with respect to x_i.

Also,

$$X_{,ij} - \Gamma^i_{ij} X_{,i} - \Gamma^j_{ij} X_{,j} = 0,\ \ 1 \leq i \neq j \leq n, \tag{2.2}$$

$$\Gamma^i_{ij} = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i},\ \ \ 1 \leq i \neq j \leq n, \tag{2.3}$$

where Γ^k_{ij} are the Christoffel symbols. From (2.2) we have

$$\Gamma^i_{ij,k} = \Gamma^i_{ik,j},\ \ 1 \leq i \neq j \neq k \leq n. \tag{2.4}$$

Besides,

$$\Gamma^j_{jk,i} = \Gamma^i_{ik}\Gamma^j_{ji} + \Gamma^j_{jk}\Gamma^k_{ik} - \Gamma^j_{jk}\Gamma^j_{ij},\ \ 1 \leq i \neq j \neq k \leq n. \tag{2.5}$$

The Christoffel symbols in terms of the metric (2.1) are given by

$$\Gamma^k_{ij} = 0, \quad \Gamma^i_{ii} = \frac{g_{ii,i}}{2g_{ii}}, \quad \Gamma^j_{ii} = -\frac{g_{ii,j}}{2g_{jj}}, \quad \Gamma^i_{ij} = \frac{g_{ii,j}}{2g_{ii}}, \tag{2.6}$$

where i, j, k are distinct.

We now consider the higher-dimensional Laplace invariants of the system of equations (2.2) (see [8]–[9] for definitions of these invariants),

$$\begin{aligned} m_{ij} &= -\Gamma^i_{ij,i} + \Gamma^i_{ij}\Gamma^j_{ij}, \\ m_{ijk} &= \Gamma^i_{ij} - \Gamma^k_{kj},\ \ k \neq i, j,\ \ 1 \leq k \leq n. \end{aligned} \tag{2.7}$$

As a consequence of (2.3), (2.7) and the un-numbered lemma appearing in [9], we obtain for $1 \leq i, j, k, l \leq n$, i, j, k, l distinct,

$$\begin{aligned} m_{ijk} + m_{kji} &= 0, \\ m_{ijk,k} - m_{ijk}m_{jki} &= 0, \\ m_{ijk} - m_{ijl} - m_{ljk} &= 0, \\ m_{lik,j} + m_{ijl}m_{kil} + m_{ljk}m_{kij} &= 0. \end{aligned} \tag{2.8}$$

From (2.2) and (2.6), we obtain

$$X_{,ii} = \sum_j \Gamma_{ii}^j X_{,j} + g_{ii}\lambda_i N. \tag{2.9}$$

The Gauss equation for the immersion X is given by

$$\frac{\Gamma_{ij,j}^i}{g_{jj}} + \frac{\Gamma_{ij}^i}{g_{jj}}(\Gamma_{ij}^i - \Gamma_{jj}^j) + \frac{\Gamma_{ji,i}^j}{g_{ii}} + \frac{\Gamma_{ji}^j}{g_{ii}}(\Gamma_{ji}^j - \Gamma_{ii}^i) + \sum_{k \neq i \neq j} \frac{\Gamma_{ik}^i \Gamma_{jk}^j}{g_{kk}} + \lambda_i \lambda_j = 0. \tag{2.10}$$

For hypersurfaces with distinct principal curvatures, the *Möbius curvature* is defined, for distinct i, j, k, by

$$C^{ijk} = \frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_j}, \tag{2.11}$$

and the *Lie curvature* is defined, for distinct i, j, k, l by

$$\Psi = \frac{(\lambda_j - \lambda_k)}{(\lambda_j - \lambda_i)} \frac{(\lambda_l - \lambda_i)}{(\lambda_l - \lambda_k)}. \tag{2.12}$$

Since all λ_i are distinct we conclude that $C^{ijk} \neq 0$ and $C^{ijk} \neq 1$. Möbius curvatures are invariant under Möbius transformations.

The following result extends Lemma 3.3 in [7] for hypersurfaces parametrized by lines of curvature.

Lemma 2.1. *For $n \geq 3$, let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, be a hypersurface parametrized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$. Then*

$$\begin{aligned} \Gamma_{ji,ii}^j &= \Gamma_{ij}^j f_{ij} - (\Gamma_{ij}^j)^2 h_{ij} - (\Gamma_{ij}^j)^3 + 3\Gamma_{ii}^i \Gamma_{ij}^j - \lambda_{i,i}\lambda_j g_{ii} + \frac{g_{ii}}{g_{jj}}(m_{ij,j} + 2m_{ij}\Gamma_{ij}^i) \\ & \quad - \sum_{k \neq i} m_{ik} \Gamma_{jk}^j \frac{g_{ii}}{g_{kk}}, \end{aligned} \tag{2.13}$$

where

$$(2.14)\quad f_{ij} = \Gamma^i_{ii,i} - 2(\Gamma^i_{ii})^2 + 6\Gamma^i_{ii}\Gamma^j_{ij} - 3(\Gamma^j_{ij})^2 - 3\Gamma^j_{ij,i} - \lambda_i^2 g_{ii} - \sum_{k\neq i}(\Gamma^i_{ik})^2\frac{g_{ii}}{g_{kk}},$$

$$(2.15)\quad h_{ij} = 3(\Gamma^i_{ii} - \Gamma^j_{ij}).$$

Moreover, the functions f_{ij} and h_{ij} do not depend x_j.

Proof. The proof is a straightforward computation, it follows from (2.3), (2.5)–(2.8) and (2.10). □

We consider $X : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$, a proper Dupin hypersurface parametrized by lines of curvature, with distinct principal curvatures $\lambda_i,\ 1 \le i \le 4$. Considering the higher-dimensional Laplace invariants satisfying (2.8), for $1 \le i \neq j \neq k \neq l \le 4$ fixed, we consider the functions T_{ijkl} and U_{ijkl} defined in [19] by

$$(2.16)\qquad T_{ijkl} = m_{jil} + \left[\log\left(\frac{m_{jik}}{m_{kil}}\right)\right]_{,i},$$

$$(2.17)\qquad U_{ijkl} = m_{kil} + \left[\log\left(\frac{m_{jik}}{m_{jil}}\right)\right]_{,i},$$

where $m_{jik} \neq 0$, $m_{jil} \neq 0$ and $m_{kil} \neq 0$.

3. Characterization of a class of Dupin hypersurfaces. In this section, the next theorems are our main results. They characterize locally a class of Dupin hypersurfaces, parametrized by lines of curvature in $\mathbb{R}^5$, with four distinct distinct principal curvatures.

The following theorem shows that when $T_{ijkl} = 0$, the theorem 4.1 obtained in [7] is not true, i.e. in this case we have four vector valued functions in $\mathbb{R}^5$ and not three as in [7].

Theorem 3.1. *Let $X : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$, be a proper Dupin hypersurface, parametrized by lines of curvature, with four distinct principal curvatures λ_r. For i, j, k, l distinct fixed indices, suppose $T_{ijkl} = 0$ and $m_{ijk} = m_{ijl} = m_{jli} = m_{kli} = 0$. Then*

$$(3.1)\qquad X = V\left[C_j - C_k - \frac{m_{jik}e^{-S}}{m_{jil}Q_k}\int G_l(x_l)dx_l\right],$$

where

$$(3.2)\quad V = \frac{e^{\int \frac{\lambda_k-\lambda_j}{\lambda_j-\lambda_i} m_{jki}dx_k}}{\lambda_j - \lambda_i},\quad C_r = \frac{1}{Q_r}\left[\int \frac{Q_r G_i(x_i)}{m_{jik}}dx_i + G_r(x_r)\right],\quad r = j, k,$$

$G_r(x_r)$, $r = i,j,k,l$ are vector valued functions of $\mathbb{R}^5$, $A_{,j} = 0$, $A = -\int m_{jki,i} dx_k$ and

$$Q_r = \begin{cases} e^{\int A dx_i} & if \;\; r = j, \\ e^{\int (A + m_{jir}) dx_i} & if \;\; r = k, \end{cases} \tag{3.3}$$

$$S = \int \left(m_{ikj} + \frac{m_{jkl} m_{jik}}{m_{jil}} \right) dx_k. \tag{3.4}$$

Moreover, considering

$$\beta^i = \left(A + \Gamma^j_{ji} \right) M + M_{,i}, \qquad \beta^s = \Gamma^i_{is} M + M_{,s}, \;\; s \neq i, \tag{3.5}$$

where $M = C_j - C_k - \dfrac{m_{jik} e^{-S}}{m_{jil} Q_k} \displaystyle\int G_l(x_l) dx_l$, the functions $G_r(x_r)$ satisfy the following properties in Ω, for $1 \leq r \neq t \leq 4$:

a) $\beta^r \neq 0$,

b) $\langle \beta^r, \beta^t \rangle = 0, \;\; r \neq t$,

c) $\lambda_r = \dfrac{\langle \beta^r_{,r}, \beta^i \times \beta^j \times \beta^k \times \beta^l \rangle}{V |\beta^r|^2 \, |\beta^i| \, |\beta^j| \, |\beta^k| \, |\beta^l|}$.

Conversely, let $\lambda_r : \Omega \subset \mathbb{R}^4 \to \mathbb{R}$, $r = 1, \ldots, 4$ be real functions, distinct at each point, such that $\lambda_{r,r} = 0$. Assume that the functions m_{rts} defined by

$$m_{rts} = \frac{\lambda_{r,t}}{\lambda_t - \lambda_r} - \frac{\lambda_{s,t}}{\lambda_t - \lambda_s}, \;\; 1 \leq r \neq t \neq s \leq 4, \tag{3.6}$$

satisfy (2.8), and for i,j,k,l distinct fixed indices, $T_{ijkl} = 0$ and $m_{ijk} = m_{ijl} = m_{jli} = m_{kli} = 0$. Then for any vector valued functions $G_r(x_r)$, satisfying properties a) b) c), where β^r is defined by (3.5), the function $X : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$ given by (3.1) describes a Dupin hypersurface, parametrized by lines of curvature, whose principal curvatures are the functions λ_r.

The next result provides a geometric description of the vector valued functions $G_r(x_r)$.

Theorem 3.2. *Under the hypothesis of Theorem 3.1, the vector valued functions $Gr(x_r)$, $r = i,j,k,l$, describe plane curves or points.*

The next results provide a characterization of this class of Dupin hypersurfaces using the Lie curvature and the Möbius curvature.

Theorem 3.3. *Let X be a Dupin hypersurface as in Theorem 3.1. Then X has Lie curvature constant, if and only if, $m_{jik} + m_{kil}C^{jkl} = 0$ and*

$$\frac{m_{jki}C^{ilk}}{\lambda_i - \lambda_j} + \frac{m_{lki}C^{ikl}}{\lambda_l - \lambda_k} = 0.$$

Theorem 3.4. *Let X be a Dupin hypersurface as in Theorem 3.1. Then for any $s, r \in \{j, k, l\}$, $s \neq r$, the Möbius curvatures C^{irs} are constant along the lines of curvature corresponding to the principal curvatures λ_r or λ_s. Also, the Möbius curvature C^{kjl} is constant along the lines of curvature corresponding to the principal curvatures λ_j or λ_l.*

We will now prove two lemmas that will be used in the proof of Theorem 3.1.

Lemma 3.5. *Let X be a proper Dupin hypersurface in $\mathbb{R}^5$, parametrized by lines of curvature, with four distinct principal curvatures λ_r. For i, j, k, l distinct fixed indices, suppose $T_{ijkl} = 0$ and $m_{ijk} = m_{ijl} = m_{jli} = m_{kli} = 0$. Then*
i) The functions defined in (3.3) *and* (3.4) *satisfy*

$$Q_{j,i} = AQ_j,\ Q_{j,j} = 0,\ Q_{j,k} = m_{ikj}Q_j,\ Q_{j,l} = 0,$$

$$Q_{k,i} = (A + m_{jik})Q_k,\ \ Q_{k,j} = 0,\ Q_{k,k} = 0,\ Q_{k,l} = 0,$$

$$S_{,i} = 0,\ S_{,j} = 0,\ S_{,k} = m_{ikj} + \frac{m_{jkl}m_{jik}}{m_{jil}},\ S_{,l} = 0.$$

ii) The Dupin hypersurface is given by

$$X = \frac{V}{m_{jik}}\left[L^k - L^j\right], \tag{3.7}$$

where $L^k(x_i, x_j, x_l)$ and $L^j(x_i, x_k, x_l)$ satisfy the following systems of equations,

$$\begin{aligned} L^k_{,ij} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right) L^k_{,j} &= 0, \\ L^k_{,il} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right) L^k_{,l} &= 0, \\ L^k_{,jl} &= 0. \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
L^j_{,ik} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right) L^j_{,k} + m_{ikj}m_{jik}L^j &= 0,\\
L^j_{,il} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right) L^j_{,l} &= 0,\\
L^j_{,kl} + \frac{m_{jkl}m_{jik}}{m_{jil}} L^j_{,l} &= 0.
\end{aligned} \tag{3.9}
$$

P r o o f. $i)$ The proof it follows from (2.8), (3.3) and (3.4).

$ii)$ Observe that from (2.2), it follows that

$$X_{,sr} - \Gamma^s_{sr} X_{,s} - \Gamma^r_{sr} X_{,r} = 0, \quad 1 \leq s \neq r \leq 4. \tag{3.10}$$

For fixed distinct indices i, j, k, we consider the transformation

$$X = V\overline{X}, \tag{3.11}$$

as in Lemma 2.4 in [18], where V is given by (3.2). Then (3.10) reduces to

$$
\begin{aligned}
\overline{X}_{,ij} + A\overline{X}_{,j} &= 0,\\
\overline{X}_{,ik} + (A + m_{jik})\overline{X}_{,k} &= 0,\\
\overline{X}_{,il} + (A + m_{jil})\overline{X}_{,l} &= 0,\\
\overline{X}_{,jk} + m_{ikj}\overline{X}_{,j} &= 0,\\
\overline{X}_{,jl} &= 0,\\
\overline{X}_{,kl} + m_{ikl}\overline{X}_{,l} &= 0.
\end{aligned} \tag{3.12}
$$

$$A_{,j} = 0, \quad A_{,r} = -m_{jri,i}, \; r = k, l. \tag{3.13}$$

We observe that $m_{ijk} = m_{ijl} = m_{jli} = m_{kli} = 0$, implies

$$m_{ljk} = m_{jlk} = 0. \tag{3.14}$$

From (2.8) and (3.13), one has

$$(A + m_{jir})_{,r} = 0, \; r = k, l. \tag{3.15}$$

Also, the substitution of (3.13) and (3.15) in the first three equations of (3.12), gives

$$\overline{X}_{,i} + A\overline{X} = L^j(x_i, x_k, x_l), \tag{3.16}$$

$$\overline{X}_{,i} + (A + m_{jik})\overline{X} = L^k(x_i, x_j, x_l), \tag{3.17}$$

$$\overline{X}_{,i} + (A + m_{jil})\overline{X} = L^l(x_i, x_j, x_k), \tag{3.18}$$

where L^j, L^k and L^l are functions that do not depend on x_j, x_k and x_l, respectively. Since $m_{jik} \neq 0$, it follows from (3.16) and (3.17) that

$$\overline{X} = \frac{1}{m_{jik}}[L^k - L^j]. \tag{3.19}$$

Thus, (3.11) and (3.19) ensure that X is given by (3.7).
We will now obtain the differential equations that L^k and L^j must satisfy, by using (3.12), (3.16)-(3.18).
The substitution of $\overline{X}$ and $\overline{X}_{,i}$ into (3.16)-(3.18), gives

$$\left(A - \frac{m_{jik,i}}{m_{jik}}\right)[L^k - L^j] + [L^k - L^j]_{,i} = m_{jik}L^j, \tag{3.20}$$

$$\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)[L^k - L^j] + [L^k - L^j]_{,i} = m_{jik}L^k, \tag{3.21}$$

$$\left(A + m_{jil} - \frac{m_{jik,i}}{m_{jik}}\right)[L^k - L^j] + [L^k - L^j]_{,i} = m_{jik}L^l. \tag{3.22}$$

By a direct calculation the substitution of $\overline{X}$ and their derivatives in the system (3.12), joint with (3.14) and the condition $T_{ijkl} = 0$, we get the systems (3.8) and (3.9). $\square$

The solutions of the systems of equations (3.8) and (3.9) are given in the following lemma.

Lemma 3.6. *The solutions of the systems* (3.8) *and* (3.9) *are given by*

$$L^k = \frac{m_{jik}}{Q_j}\left[\int \frac{Q_j G_i(x_i)}{m_{jik}}dx_i + \int \widetilde{G}_l(x_l)dx_l + G_j(x_j)\right], \tag{3.23}$$

$$L^j = \frac{m_{jik}}{Q_k}\left[\int \frac{Q_k G_i(x_i)}{m_{jik}}dx_i + \int e^{-S} G_l(x_l)dx_l + G_k(x_k)\right], \tag{3.24}$$

where S is given by (3.4).

P r o o f. From equations (2.8) we can easily verify that

$$\left(A - \frac{m_{jik,i}}{m_{jik}}\right)_{,r} = 0, \quad r = j, l. \tag{3.25}$$

Substituting this expression in the first two equations of (3.8) and by integration with respect to x_i joint with (3.3) we get

$$L^k = \frac{m_{jik}}{Q_j}\left[\int \frac{Q_j G_i(x_i)}{m_{jik}}dx_i + H(x_j, x_l)\right]. \tag{3.26}$$

We will now find the expression of the function $H(x_j, x_l)$ and so L^k will be completely determined, for this, differentiating L^k, by using (2.8), (3.25) and Lemma 3.5, we have

$$L^k_{,i} = -\left(A - \frac{m_{jik,i}}{m_{jik}}\right) + G_i(x_i), \tag{3.27}$$

$$L^k_{,l} = \frac{m_{jik}}{Q_j}H_{,l}, \tag{3.28}$$

$$L^k_{,jl} = \frac{m_{jik}}{Q_j}H_{,jl}. \tag{3.29}$$

By using (3.29) in the third equation of (3.8), we obtain $H_{,jl} = 0$, whose solution is given by

$$H(x_j, x_l) = \int \widetilde{G}_l(x_l)dx_l + G_j(x_j), \tag{3.30}$$

where $G_j(x_j)$ and $\widetilde{G}_l(x_l)$ are vector valued functions in $\mathbb{R}^5$. Thus, (3.23) it follows from (3.26) and (3.30).

Similarly, from equations (2.8) we get

$$\left(A + m_{jir} - \frac{m_{jir,i}}{m_{jir}}\right)_{,r} = m_{jir}m_{irj}, \quad r = k, l. \tag{3.31}$$

By using this expression in the first two equations of (3.9) and by integration with respect to x_i joint with (3.3) we get

$$L^j = \frac{m_{jik}}{Q_k}\left[\int \frac{Q_k \widetilde{G_i}(x_i)}{m_{jik}}dx_i + \widetilde{H}(x_k, x_l)\right]. \tag{3.32}$$

Now to get the function $\widetilde{H}(x_k, x_l)$ we derive L^j, using (2.8), (3.3), (3.31) joint with Lemma 3.5 to obtain

$$L^j_{,i} = -\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)L^j + \widetilde{G_i}(x_i), \tag{3.33}$$

$$
\begin{aligned}
(3.34)\qquad L^j_{,l} &= \frac{m_{jik}}{Q_k}\widetilde{H}_{,l},\\
(3.35)\qquad L^j_{,kl} &= -m_{jki}L^j_{,l} + \frac{m_{jik}}{Q_k}\widetilde{H}_{,kl}.
\end{aligned}
$$

On the other hand, by using (3.34), (3.35) in the third equation of (3.9) yield

$$\widetilde{H}_{,kl} + \left(m_{ikj} + \frac{m_{jkl}m_{jik}}{m_{jil}}\right)\widetilde{H}_{,l} = 0,$$

whose solution is given by

$$\widetilde{H}(x_k, x_l) = \int e^{-S} G_l(x_l)dx_l + G_k(x_k), \tag{3.36}$$

where $G_l(x_l)$ and $G_k(x_k)$ are vector valued functions in $\mathbb{R}^5$ and S is defined in (3.4). Thus, from (3.32) and (3.36), we conclude that the solution of (3.9) is given by (3.24). Finally, we observe that the expressions (3.20), (3.27) and (3.33) ensure that $G_i(x_i) = \widetilde{G_i}(x_i)$. Thus, the proof of Lemma is complete. □

3.1. Proof of the main results.

Proof of Theorem 3.1 From Lemmas 3.5 and 3.6 we obtain that the Dupin hypersurface X is given by (3.7), where L^k and L^j are given for (3.23) and (3.24) respectively.

The next step is to show that there exist a relation between the vector valued functions $\widetilde{G}_l$ and G_l given in (3.23) and (3.24) respectively, for this, the expressions (3.16)–(3.18) ensure that

$$L^k - L^j = \frac{m_{jik}}{m_{kil}}[L^l - L^k]. \tag{3.37}$$

Now, differentiating (3.37) with respect to x_l, and by using (2.8), (3.14), (3.28) and (3.34) we obtain

$$\frac{1}{Q_j}H_{,l} = \frac{m_{kil}}{m_{jil}Q_k}\widetilde{H}_{,l}. \tag{3.38}$$

By using (3.3), (3.30), (3.36) in (3.38) we obtain the following relation between $\widetilde{G}_l$ and G_l

$$\widetilde{G}_l(x_l) = \frac{m_{kil}}{m_{jil}} e^{-\int m_{jik}dx_i} e^{-S} G_l(x_l). \tag{3.39}$$

On the other hand, it follows from (2.8), (3.14) and Lemma 3.5, that $\left(\frac{m_{kil}}{m_{jil}} e^{-\int m_{jik} dx_i} e^{-S}\right)_{,l} = 0$, consequently, using this fact in (3.23) we get

(3.40)
$$L^k = \frac{m_{jik}}{Q_j}\left[\int \frac{Q_j G_i(x_i)}{m_{jik}} dx_i + \frac{m_{kil}}{m_{jil}} e^{-\int m_{jik} dx_i} e^{-S} \int G_l(x_l) dx_l + G_j(x_j)\right].$$

Thus, substituting (3.24) and (3.40) in (3.7) we obtain (3.1).

Now, considering β^i and β^t, $t = j, k, l$ defined by (3.5), it follows from (2.7) and Lemmas 2.3, 2.4 in [18]

$$X_{,r} = V\beta^r \ , \ r = i, j, k, l, \tag{3.41}$$

and as a consequence of this expression we obtain

$$X_{,rr} = V_{,r}\beta^r + V\beta^r_{,r} \ , \ r = i, j, k, l. \tag{3.42}$$

Observe that from (3.41) it follows that $\beta^r \neq 0$, which proves item a).
Also, it follows from (3.41) that the metric of X is given by

$$g_{rr} = (V)^2|\beta^r|^2 \ , \ g_{rt} = 0 \ , \ r \neq t, \tag{3.43}$$

(which proves item b)) and

$$N = \frac{\beta^i \times \beta^j \times \beta^k \times \beta^l}{|\beta^i|\,|\beta^j|\,|\beta^k|\,|\beta^l|}. \tag{3.44}$$

is a unit vector field normal to X.

Since X is a Dupin hypersurface parametrized by orthogonal curvature lines, with λ_t, as principal curvature we get, for $1 \leq r \neq t \leq 4$

$$\langle N, X_{,rt}\rangle = 0, \qquad \lambda_t = \frac{\langle X_{,rr}, N\rangle}{g_{rr}} \ .$$

Hence, from the above expressions joint with (3.42) and (3.44) we obtain the item c).

The converse is a straightforward calculation. Therefore, the proof of the Theorem 3.1 is complete. □

Remark 3.7. It is easy to show that the vector valued functions $G_r(x_r)$ in the Theorem 3.1 are invariant under inversions and homotheties of $\mathbb{R}^5$.

We will now prove Theorem 3.2.

Proof of Theorem 3.2. We will prove that $G_r(x_r)$ describes a plane curve by showing that the vector valued functions G_r', G_r'' and G_r''' are linearly dependent. Differentiating (3.1) with respect x_j and using Lemma 3.5 we get

$$G_j' = \frac{Q_j}{V}\left(X_{,j} - \Gamma_{ij}^i X\right). \tag{3.45}$$

Diferentiating (3.45) with respect x_j and using (2.9) we get

$$G_j'' = \frac{Q_j}{V}\left([\Gamma_{jj}^j - 2\Gamma_{ij}^i]X_{,j} - \sum_{k\neq j}\Gamma_{jk}^j \frac{g_{jj}}{g_{kk}}X_{,k} + [(\Gamma_{ij}^i)^2 - \Gamma_{ij,j}^i]X + \lambda_j g_{jj}N\right).$$

Differentiating once again and using Lemma 2.1 we obtain

$$G_j''' = h_{ji}G_j'' + f_{ji}G_j'.$$

Thus, G_l''', G_l'' and G_l' are linearly dependent. Similarly, differentiating (3.1) with respect x_l and using Lemma 3.5 we get

$$G_l = -\frac{m_{jil}e^S Q_k}{m_{jik}V}[X_{,l} - \Gamma_{il}^i X]$$

Differentiating with respect x_l and using (2.9) and the fact that $\left(\frac{m_{jil}e^S Q_k}{m_{jik}}\right)_{,l} = 0$, we obtain

$$G_l' = -\frac{m_{jil}e^S Q_k}{m_{jik}V}\left([\Gamma_{ll}^l - 2\Gamma_{il}^i]X_{,l} - \sum_{j\neq l}\Gamma_{lj}^l \frac{g_{ll}}{g_{jj}}X_{,j} + [(\Gamma_{il}^i)^2 - \Gamma_{il,l}^i]X + \lambda_l g_{ll}N\right).$$

Differentiating once again and using Lemma 2.1 we obtain

$$G_l'' = h_{li}G_l' + f_{li}G_l. \tag{3.46}$$

From (3.46) we obtain

$$G_l''' = G_l''\left(h_{li} + \frac{f_{li,l}}{f_{li}}\right) + G_l'\left(f_{li} + h_{li,l} + \frac{h_{li}f_{li,l}}{f_{li}}\right).$$

Therefore, G_l''', G_l'' and G_l' are linearly dependent. Similar arguments prove that G_i and G_k describes a plane curve. Since X is a parametrized hypersurface in $\mathbb{R}^5$, at least two of the vector valued functions $G_r(x_r)$ of Theorem 3.1 are nonzero on open sets, therefore, some vector valued functions can be degenerated in a point (see examples 4.1 and 4.2). □

Now, we will prove Theorem 3.3.

Proof of Theorem 3.3. From (2.11) and (2.12) it follows that the Lie curvature of X is given by

$$\Psi = C^{kji}C^{ilk}, i, j, k, l \ \text{ distinct.} \tag{3.47}$$

Differentiating (3.47) we get

$$\begin{aligned} \Psi_{,i} &= -C^{kij}C^{ilk}\left(m_{jik} + m_{kil}C^{jkl}\right), \\ \Psi_{,j} &= 0, \\ \Psi_{,k} &= (\lambda_k - \lambda_i)C^{kji}\left(\frac{m_{jki}C^{ilk}}{\lambda_i - \lambda_j} + \frac{m_{lki}C^{ikl}}{\lambda_l - \lambda_k}\right), \\ \Psi_{,l} &= 0. \end{aligned} \tag{3.48}$$

The result it follows from (3.48). □

Finally, we will prove Theorem 3.4.

Proof of Theorem 3.4. Since

$$m_{jki}m_{kji} = 0, \ m_{lji}m_{jli} = 0, \ m_{kli}m_{lki} = 0,$$

i.e.

$$m_{sri}m_{rsi} = 0, \ \forall \, r, s \in \{j, k, l\}, \ r \neq s. \tag{3.49}$$

From (2.7) and (3.49) we obtain

$$\left(\log\left(\frac{\lambda_i - \lambda_r}{\lambda_s - \lambda_r}\right)\right)_{,r}\left(\log\left(\frac{\lambda_i - \lambda_s}{\lambda_r - \lambda_s}\right)\right)_{,s} = 0.$$

Hence,

$$\left(\frac{\lambda_i - \lambda_r}{\lambda_s - \lambda_r}\right)_{,r}\left(\frac{\lambda_i - \lambda_r}{\lambda_s - \lambda_r}\right)_{,s} = 0,$$

i.e. the Möbius curvatures C^{irs} are constant along the lines of curvature corresponding to the principal curvatures λ_r or λ_s.

Similarly, from $m_{ljk}m_{jlk} = 0$ we have

$$\left(\frac{\lambda_k - \lambda_j}{\lambda_l - \lambda_j}\right)_{,j}\left(\frac{\lambda_k - \lambda_j}{\lambda_l - \lambda_j}\right)_{,l} = 0,$$

therefore, the Möbius curvature C^{kjl} is constant along the lines of curvature corresponding to the principal curvatures λ_j or λ_l. $\square$

4. Examples. In this section using the Theorem 3.1, for fixed indices $i = 1, j = 2, k = 3$ and $l = 4$, we give examples of Dupin hypersurfaces in $\mathbb{R}^5$, parametrized by lines of curvature with four distinct principal curvatures and nonconstant Lie curvature.

Example 4.1. We consider the Dupin hypersurface parametrized by lines of curvature given by

$$\begin{aligned} X(x_1, x_2, x_3, x_4) &= \frac{(1+x_4)}{R}\left(2a\cos x_2, 2a\sin x_2, 2x_1, 2\left(a^2+x_1^2\right)x_3,\right. \\ &\qquad \left. -1+\left(a^2+x_1^2\right)\left(1-x_3^2\right)\right), \end{aligned}$$

defined in $\Omega = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4/\ x_1 > 0,\ 0 < x_2 < 2\pi,\ x_3 \in \mathbb{R},\ x_4 > -1\}$, $R = 1 + (a^2 + x_1^2)(1 + x_3^2),\ a > 0$.
The principal curvatures of X are given by

$$\lambda_1 = \frac{a(1+x_3^2)}{1+x_4}, \lambda_2 = \frac{-1+(1+x_3^2)(a^2-x_1^2)}{2a(1+x_4)}, \lambda_3 = -\frac{a}{(1+x_4)(a^2+x_1^2)}, \lambda_4 = 0.$$

From (2.3) and (2.7) we get

$$m_{213} = -\frac{2x_1}{a^2+x_1^2} \neq 0,\ m_{214} = -\frac{2x_1(1+x_3^2)}{R} \neq 0,\ m_{314} = \frac{2x_1}{(a^2+x_1^2)R} \neq 0. \tag{4.1}$$

Moreover, $m_{123} = m_{124} = m_{241} = m_{341} = 0$ and using (4.1) we obtain $T_{1234} = 0$. Thus, from Theorem 3.1, the Dupin hypersurface X is given by (3.1), where

$$V = -\frac{2a(1+x_4)}{R},\ Q_2 = 1,\ Q_3 = \frac{1}{a^2+x_1^2},\ s^3 = -\ln(1+x_3^2).$$

The vector valued functions are given by

$$\begin{aligned} G_1(x_1) &= \left(0, 0, \frac{1}{ax_1}, 0, 0\right), \\ G_2(x_2) &= (-\cos x_2, -\sin x_2, 0, 0, 0), \\ G_3(x_3) &= \left(0, 0, 0, \frac{x_3}{a}, \frac{1-x_3^2}{2a}\right), \\ G_4(x_4) &= (0, 0, 0, 0, 0). \end{aligned}$$

Observe that $m_{213} + m_{314}C^{234} = -\dfrac{x_1}{a^2} \neq 0$, hence, from Theorem 3.3, the Lie curvature of X is nonconstant.

Example 4.2. We consider the Dupin hypersurface parametrized by lines of curvature

$$X(x_1,x_2,x_3,x_4)=\left(\frac{(a+r\cos x_1)\cos x_2}{B},\frac{(a+r\cos x_1)\sin x_2}{B},\frac{r\sin x_1}{B},\frac{x_3}{B},x_4\right)$$

defined in $\Omega=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4/0<x_1<\pi,\ 0<x_2<2\pi,\ x_3>0,\ x_4\in\mathbb{R}\}$, $B=a^2+r^2+x_3^2+2ar\cos x_1,\quad a>r>0$.
The principal curvatures of X are given by

$$\begin{aligned}\lambda_1&=\frac{a^2-r^2+x_3^2}{r}, &\lambda_2&=-\frac{2ar+(a^2+r^2-x_3^2)\cos x_1}{a+r\cos x_1},\\ \lambda_3&=-2(r+a\cos x_1), &\lambda_4&=0.\end{aligned}$$

From (2.3) and (2.7) we get

$$\begin{aligned}m_{213}&=-\frac{r\sin x_1}{a+r\cos x_1}\neq 0,\quad m_{214}=-\frac{r(a^2-r^2-x_3^2)\sin x_1}{(a+r\cos x_1)B}\neq 0,\\ m_{314}&=\frac{2ar\sin x_1}{B}\neq 0.\end{aligned}\tag{4.2}$$

Moreover, $m_{123}=m_{124}=m_{241}=m_{341}=0$ and using (4.2) we obtain $T_{1234}=0$. Thus, from Theorem 3.1, the Dupin hypersurface X is given by (3.1), where

$$V=-\frac{r(a+r\cos x_1)}{aB},\ Q_2=1,\ Q_3=a+r\cos x_1,\ s^3=-\ln(r^2-a^2+x_3^2).$$

The vector valued functions are given by

$$\begin{aligned}G_1(x_1)&=\left(0,0,\frac{a}{\sin x_1(a+r\cos x_1)},0,0\right),\\ G_2(x_2)&=\left(-\frac{a}{r}\cos x_2,-\frac{a}{r}\sin x_2,0,0,0\right),\\ G_3(x_3)&=\left(0,0,0,\frac{ax_3}{r},0\right),\\ G_4(x_4)&=\left(0,0,0,0,\frac{a}{r}\right).\end{aligned}$$

Observe that $m_{213}+m_{314}C^{234}=-\dfrac{r^2\sin x_1}{(a+r\cos x_1)(r+a\cos x_1)}\neq 0$, hence, from Theorem 3.3, the Lie curvature of X is nonconstant.

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