Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **43** (2017), 211–220

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON A GENERALIZATION OF MARKOWITZ PREFERENCE RELATION*

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Communicated by V. Drensky

Abstract. Given two families $u = (u_p)_{p\in I}$ and $v = (v_q)_{q\in J}$ of real continuous functions on a topological space X, we define a preorder $R = R(u, v)$ on X by the condition that any member of u is an R-increasing and any member of v is an R-decreasing function. It turns out that if the topological space X is quasi-compact and sequentially compact, then any element $x \in X$ is R-dominated by an R-maximal element $m \in X$: xRm. In particular, since the $(n-1)$-dimensional simplex is a compact subset of $\mathbb{R}^n$, then considering its members as portfolios consisting of n financial assets, we obtain the classical 1952 result of Harry Markowitz that any portfolio is dominated by an efficient portfolio. Moreover, several other examples of possible application of this general setup are presented.

1. Markowitz optimization.

1.1. Return of a portfolio. Let

$$\Delta_{n-1} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1\right\}$$

be the $(n-1)$-dimensional simplex and let $[n] = \{1, \ldots, n\}$. The ordered pairs

2010 *Mathematics Subject Classification:* 06A99, 54A20, 91G10.
Key words: mean-variance theory, efficient financial portfolio, Markowitz preference relation, topological space.

*Partially supported by Grand I 02/18 of the Bulgarian Science Fund.

$([n], x)$, $x \in \Delta_{n-1}$, are sample spaces with set of outcomes $[n]$ and probability assignment $x\colon [n] \to \mathbb{R}$, $x(i) = x_i$, $i = 1, \dots, n$. The set of all sample spaces of this form can be identified with the $(n-1)$-dimensional simplex Δ_{n-1} and also are said to be $(n-1)$-*dimensional lotteries* or $(n-1)$-*dimensional portfolios*.

Given a sample space S, let $s_1, \dots, s_n$ be random variables on S with expected values $\mu_1, \dots, \mu_n$, respectively. For any portfolio $x \in \Delta_{n-1}$ the weighted sum $s(x) = x_1 s_1 + \cdots + x_n s_n$ is a random variable with expected value $u(x) = E(s(x)) = x_1\mu_1 + \cdots + x_n\mu_n$ and the variance $v(x) = \mathrm{Var}(s(x))$ is a non-negative quadratic form in $x_1, \dots, x_n$.

Remark 1.1.1. Below we interpret $i \in [n]$ as *financial assets*, the sample space S as a *financial market*, the random variables s_i on S as *returns on asset* i, $i = 1, \dots, n$, in the end of a fixed time period, and $s(x)$ as the *return of the portfolio* x. Then $u(x) = E(s(x))$ is the *expected return* and $v(x) = \mathrm{Var}(s(x))$ is the *risk* (or, the *volatility*) of the portfolio x — see, for example, [2, 2.1].

1.2. Markowitz preferences. Let $x \in \Delta_{n-1}$ be a portfolio and $u(x) = E(s(x))$ and $v(x) = \mathrm{Var}(s(x))$ be the expected return and the volatility of x. The Markowitz's approach to portfolio selection is based on the following definition of preference R on the set Δ_{n-1} of portfolios: xRy if $u(x) \leq u(y)$ and $v(y) \leq v(x)$. Non-formally, xRy means that the portfolio y is at least as good as x. The symmetric part E of the preorder R is

$$E = \{(x, y) \in \Delta_{n-1}^2 \mid u(x) = u(y) \text{ and } v(y) = v(x)\}$$

and the asymmetric part F of R is $F = R \backslash E$. Thus, xFy if and only if either $u(x) < u(y)$ and $v(y) \leq v(x)$ or $u(x) \leq u(y)$ and $v(y) < v(x)$. Non-formally, xFy means that the portfolio y is definitely better than the portfolio x.

In [1, p. 82] H. Markowitz gives (up to notation) the following definition:

The portfolio x is said to be *efficient* if

$$u(x) = \max_{y \in \Delta_{n-1}, v(y) \leq v(x)} u(y) \text{ and } v(x) = \min_{y \in \Delta_{n-1}, u(y) \geq u(x)} v(y). \tag{1.2.1}$$

In other words, for any portfolio $y \in \Delta_{n-1}$ the inequality $v(y) \leq v(x)$ implies the inequality $u(x) \geq u(y)$ and the inequality $u(y) \geq u(x)$ implies the inequality $v(x) \leq v(y)$. The negation of the last statement is: There exists $y \in \Delta_{n-1}$ such that xFy, that is, the portfolio x is not R-maximal.

Thus, we see that x is Markowitz's efficient portfolio if and only if x is R-maximal – this is our setup.

2. Generalization. In this section we present a wide generalization of Markowitz's preference relation, defined in 1.2. Using Kuratowski-Zorn Theorem (equivalent to the Axiom of Choice), we show that any member of this preference structure is dominated by a maximal element (*generalized efficient portfolio*). In particular, the set of generalized efficient portfolios is not empty.

2.1. A preorder on a topological space. Let X be a topological space and let $u = (u_p)_{p \in I}$ and $v = (v_q)_{q \in J}$ be two families of continuous real functions on X. We define a preorder $R = R(u, v)$ on X in the following way:

(2.1.1)
$$R = \{(x, y) \in X^2 \mid u_p(x) \le u_p(y) \text{ and } v_q(x) \ge v_q(y) \text{ for all } p \in I, q \in J\}.$$

Then for the symmetric part E of R (an equivalence relation) one has

$$E = \{(x, y) \in X^2 \mid u_p(x) = u_p(y) \text{ and } v_q(x) = v_q(y) \text{ for all } p \in I, q \in J\}$$

and for the asymmetric part F of R (an asymmetric and transitive relation) one has $F = R \backslash E$. Thus, xFy means xRy and either there exists index $p_0 \in I$ with $u_{p_0}(x) < u_{p_0}(y)$ or there exists index $q_0 \in J$ with $v_{q_0}(x) > v_{q_0}(y)$.

On the account of repetitions of functions within one family and adding the negatives of functions from one family to the other, we can assume that both families have the same set of indices, $u = (u_p)_{p \in I}$, $v = (v_p)_{p \in I}$, without changing the corresponding preorder on X. Moreover, on the account of adding a third countable family of continuous functions on X to both families, the corresponding preorder can be defined by two systems of inequalities and a system of equalities.

Below, if the opposite is in not stated, the families $u = (u_p)_{p \in I}$ and $v = (v_p)_{p \in I}$ have the same index set.

2.2. Maximal elements. In order to fix the terminology, we remind several definitions. A topological space X is called *quasi-compact* if every open covering of X contains a finite open covering. The space X is called *compact* if it is quasi-compact and Hausdorff, and *sequentially compact* if any infinite sequence of elements of X has a converging subsequence.

It is well known (see, for example, [3, Sec. 1]) that any compact and first countable space is sequentially compact and that every Lindelöf, sequentially compact (and Hausdorf) space is quasi-compact (compact).

Given a prerder R on the set X, a subset $C \subset X$ is said to be *chain* in X if the induced preorder on C is complete. A preordered set X is called *inductive* if every chain in X has an upper bound.

Below, if the opposite is not stated, we suppose that the topological space X is furnished with the preorder R produced by the families of continuous functions $u = (u_p)_{p \in I}$ and $v = (v_p)_{p \in I}$.

The sequence $(x_\iota)_{\iota=1}^{\infty}$, $x_\iota \in X$, is said to be *R-increasing* (respectively, *strictly R-increasing*) if $x_\iota R x_{\iota+1}$ (respectively, $x_\iota F x_{\iota+1}$) for all $\iota \geq 1$. By analogy, we define *R-decreasing* (respectively, *strictly R-decreasing*) sequences.

Given an R-chain $C \subset X$, for any $p \in I$ and any real number $r \in \mathbb{R}$ we set:

$$M_p = \sup_{x \in C} u_p(x),\ m_p = \inf_{x \in C} v_p(x),$$

$$C_p = \{x \in C \mid u_p(x) = M_p\},\ C_p^{(-)} = \{x \in C \mid u_p(x) < M_p\},$$

$$c_p = \{x \in C \mid v_p(x) = m_p\},\ c_p^{(+)}(r) = \{x \in C \mid v_p(x) > m_p\}.$$

Finally, we denote $C_p^* = \{x \in X \mid u_p(x) = M_p\}$, $c_p^* = \{x \in X \mid v_p(x) = m_p\}$, so $C_p \subset C_p^*$ and $c_p \subset c_p^*$. Note that $C = C_p \cup C_p^{(-)} = c_p \cup c_p^{(+)}$.

Lemma 2.2.1. *Let $p, q \in I$.*

(i) *One has $c_p \subset C_p$ or $C_p \subset c_p$.*

(ii) *One has $c_p \cap C_p \subset c_q \cap C_q$ or $c_q \cap C_q \subset c_p \cap C_p$.*

Proof. (i) If $v_p(x) = m_p$ for all $x \in C_p$, then $C_p \subset c_p$. Otherwise, there exists $x \in C_p$ with $v_p(x) > m_p$ and, hence, $v_p(y) < v_p(x)$ for all $y \in c_p$. Since any $y \in c_p$ is R-comparable with x, we have $u_p(y) \geq u_p(x) = M_p$, that is, $y \in C_p$. In other words, $c_p \subset C_p$.

(ii) If $v_q(x) = m_q$ and $u_q(x) = M_q$ for all $x \in c_p \cap C_p$, then $c_p \cap C_p \subset c_q \cap C_q$. Otherwise, there exists $x \in c_p \cap C_p$ with $v_q(x) > m_q$ or $u_q(x) < M_q$. If $v_q(x) > m_q$ (respectively, $u_q(x) < M_q$), then $v_q(y) < v_q(x)$ (respectively, $u_q(x) < u_q(y)$) for all $y \in c_q \cap C_q$. Since x and y are R-comparable, in both cases we have $u_p(y) \geq u_p(x) = M_p$ and $m_p = v_p(x) \geq v_p(y)$. In other words, $y \in c_p \cap C_p$ for all $y \in c_q \cap C_q$. □

Let us fix a positive integer s and a finite subset $\{p_1, \ldots, p_s\} \subset I$.

Using Lemma 2.2.1, (i), (ii), and induction, we obtain immediately the following:

Corollary 2.2.2. *The intersection $c_{p_1} \cap C_{p_1} \cap \ldots \cap c_{p_k} \cap C_{p_k}$ is equal to one of the sets c_{p_1}, C_{p_1}, ..., c_{p_k}, C_{p_k} for all $k \leq s$.*

Given an $s \geq 1$, in accord with Lemma 2.2.1, (i), (ii), and eventual renumbering of the pairs of functions u_{p_k}, v_{p_k}, we order the intersections $c_{p_k} \cap C_{p_k}$, $k \leq s$, with respect to inclusion from smallest to largest:

$$c_{p_1} \cap C_{p_1} \subset \cdots \subset c_{p_\ell} \cap C_{p_\ell} \subset c_{p_{\ell+1}} \cap C_{p_{\ell+1}} \subset \cdots \subset c_{p_s} \cap C_{p_s}, \tag{2.2.1}$$

where $c_{p_i} = \emptyset$ or $C_{p_i} = \emptyset$, $1 \leq i \leq \ell$, and $c_{p_{\ell+1}} \cap C_{p_{\ell+1}} \neq \emptyset$. Below, if the opposite is not stated, after fixing $\{p_1, \ldots, p_s\} \subset I$, we assume that (2.2.1) holds.

Thus, the existence of $k \leq s$ with $c_{p_k} = \emptyset$ or $C_{p_k} = \emptyset$ after renumbering implies $\ell \geq 1$, that is, $c_{p_1} = \emptyset$ or $C_{p_1} = \emptyset$.

Lemma 2.2.3. *Let X be a sequentially compact space and let $C_{p_1} = \emptyset$ (respectively, $c_{p_1} = \emptyset$).*

(i) *There exists a strictly R-increasing and divergent sequence*

$$(x_\iota)_{\iota=1}^{\infty}, \tag{2.2.2}$$

with $x_\iota \in C$ and limit $x^* \in X$, such that the sequence of real numbers $(u_{p_1}(x_\iota))_{\iota=1}^{\infty}$ is strictly increasing and diverges to $u_{p_1}(x^*) = M_{p_1}$ and every sequence of real numbers $(v_q(x_\iota))_{\iota=1}^{\infty}$, $q \in I$, is decreasing and diverges to $v_q(x^*) = m_q$ (respectively, the sequence of real numbers $(v_{p_1}(x_\iota))_{\iota=1}^{\infty}$ is strictly decreasing and diverges to $v_{p_1}(x^*) = m_{p_1}$ and every sequence of real numbers $(u_q(x_\iota))_{\iota=1}^{\infty}$, $q \in I$, is increasing and diverges to $u_q(x^*) = M_q$).

(ii) *Let for the sequence* (2.2.2) *from part* (i) *one has $u_{p_1}(x^*) = M_{p_1}$, $u_{p_2}(x^*) = M_{p_2}, \ldots, u_{p_k}(x^*) = M_{p_k}$ (respectively, $v_{p_1}(x^*) = m_{p_1}$, $v_{p_2}(x^*) = m_{p_2}, \ldots, v_{p_k}(x^*) = m_{p_k}$), for some $k < s$. Then either there exists $y \in \cap_{\lambda \in I} c_\lambda \cap C_{p_1} \cap \ldots \cap C_{p_k} \cap C_{p_{k+1}}$ (respectively, $y \in c_{p_1} \cap \ldots \cap c_{p_k} \cap c_{p_{k+1}} \cap_{\lambda \in I} C_\lambda$), or there exists a strictly R-increasing and divergent sequence $(y_\kappa)_{\kappa=1}^{\infty}$, with $y_\kappa \in C$ and limit $y^* \in X$, such that $u_{p_1}(y^*) = M_{p_1}$, $u_{p_2}(y^*) = M_{p_2}, \ldots, u_{p_k}(y^*) = M_{p_k}$, and $v_q(y^*) = m_q$, $q \in I$ (respectively, $v_{p_1}(y^*) = m_{p_1}$, $v_{p_2}(y^*) = m_{p_2}, \ldots, v_{p_k}(y^*) = m_{p_k}$, and $u_q(x^*) = M_q$, $q \in I$), the sequence of real numbers $(u_{p_{k+1}}(y_\kappa))_{\kappa=1}^{\infty}$ is strictly increasing and diverges to $u_{p_{k+1}}(y^*) = M_{p_{k+1}}$ and every sequence of real numbers $(v_q(y_\kappa))_{\kappa=1}^{\infty}$, $q \in I$, is decreasing and diverges to $v_q(y^*) = m_q$ (respectively, the sequence of real numbers $(v_{p_{k+1}}(y_\kappa))_{\kappa=1}^{\infty}$ is strictly decreasing and diverges to $v_{p_{k+1}}(y^*) = m_{p_{k+1}}$ and every sequence of real numbers $(u_q(y_\kappa))_{\kappa=1}^{\infty}$, $q \in I$, is increasing and diverges to $u_q(y^*) = M_q$).*

Proof. Below, when $c_{p_1} = \emptyset$, we replace u_q with $-v_q$, v_q with $-u_q$, and use the corresponding proofs in case $C_{p_1} = \emptyset$.

(i) Let $C_{p_1} = \emptyset$. Then $M_{p_1} = \sup\limits_{x \in C_{p_1}^{(-)}} u_{p_1}(x)$ and we choose $(x_\iota)_{\iota=1}^{\infty}$ to be a sequence of members of $C = C_{p_1}^{(-)}$ such that the sequence of real numbers $(u_{p_1}(x_\iota))_{\iota=1}^{\infty}$ is strictly increasing with $\lim\limits_{\iota \to \infty} u_{p_1}(x_\iota) = M_{p_1}$. Since the elements x_ι $\iota \geq 1$, are pairwise R-comparable, it turns out that the sequences of real numbers $(u_q(x_\iota))_{\iota=1}^{\infty}$, $q \in I$, $q \neq p_1$, are increasing and $(v_q(x_\iota))_{\iota=1}^{\infty}$, $q \in I$, are decreasing. Thus, the sequence $(x_\iota)_{\iota=1}^{\infty}$ is strictly R-increasing. In accord with the sequential compactness of the topological space X, we can suppose that $(x_\iota)_{\iota=1}^{\infty}$ diverges to a point $x^* \in X$. Thus, $u_{p_1}(x^*) = M_{p_1}$. For any $q \in I$ we set $m'_q = \lim\limits_{\iota \to \infty} v_q(x_\iota)$. Let us suppose $m_{q_0} < m'_{q_0}$ for some $q_0 \in I$ and let $y \in C$ be such that $v_{q_0}(y) < m'_{q_0}$. In particular, $v_{q_0}(y) < v_{q_0}(x_\iota)$, hence $u_{p_1}(y) \geq u_{p_1}(x_\iota)$ for all $\iota \geq 1$. Taking the limit we obtain $u_{p_1}(y) \geq M_{p_1}$, that is, $y \in C_{p_1}$, which is a contradiction. Therefore $m_q = m'_q$ and $v_q(x^*) = m_q$ for all $q \in I$.

(ii) Let $M'_{p_{k+1}} = \lim_{\iota\to\infty} u_{p_{k+1}}(x_\iota)$. We have $M'_{p_{k+1}} \leq M_{p_{k+1}}$ and if $M'_{p_{k+1}} = M_{p_{k+1}}$, then $u_{p_{k+1}}(x^*) = M_{p_{k+1}}$. In other words, $x^* \in \cap_{\lambda=1}^{\infty} c^*_\lambda \cap C^*_{p_1} \cap \ldots \cap C^*_{p_k} \cap C^*_{p_{k+1}}$. Now, let $M'_{p_{k+1}} < M_{p_{k+1}}$.

In case $C_{p_{k+1}} \neq \emptyset$, we choose $y \in C_{p_{k+1}}$ and since x_ι's and y are R-comparable, the inequalities $u_{p_{k+1}}(x_\iota) \leq M'_{p_{k+1}} < u_{p_{k+1}}(y)$ yield

$$u_q(x_\iota) \leq u_q(y) \tag{2.2.3}$$

for all $q \in I$, $q \neq p_{k+1}$, and

$$v_q(x_\iota) \geq v_q(y) \tag{2.2.4}$$

for all $q \in I$. Taking the limit $\iota \to \infty$ in (2.2.3) for all $q = p_1, \ldots, p_k$ and in (2.2.4) for all $q \in I$, we obtain $y \in \cap_{\lambda=1}^{\infty} c_\lambda \cap C_{p_1} \cap \ldots \cap C_{p_k} \cap C_{p_{k+1}}$.

In case $C_{p_{k+1}} = \emptyset$, there exists a sequence $(y_\kappa)_{\kappa=1}^{\infty}$, $y_\kappa \in C$, such that $M'_{p_{k+1}} < u_{p_{k+1}}(y_\kappa) < M_{p_{k+1}}$, $\kappa \geq 1$, the sequence of real numbers $(u_{p_{k+1}}(y_\kappa))_{\kappa=1}^{\infty}$ is strictly increasing and diverges to $M_{p_{k+1}}$. In particular, $u_{p_{k+1}}(x_\iota) < u_{p_{k+1}}(y_\kappa)$ for all $\iota, \kappa \geq 1$. Since x_ι's and y_κ's are R-comparable, we obtain for all $\iota, \kappa \geq 1$ the inequalities

$$u_q(x_\iota) \leq u_q(y_\kappa) \leq M_q \tag{2.2.5}$$

for all $q \neq p_{k+1}$, and

$$v_q(x_\iota) \geq v_q(y_\kappa) \geq m_q \tag{2.2.6}$$

for all $q \in I$. Since the topological space X is sequentially compact, we can assume that $(y_\kappa)_{\kappa=1}^{\infty}$ diverges with limit $y^* \in X$, so $u_{p_{k+1}}(y^*) = M_{p_{k+1}}$. Taking consecutively the limits $\iota \to \infty$, $\kappa \to \infty$, in (2.2.5) for all $q = p_1, \ldots, p_k$ and in (2.2.6) for all $q \in I$, we obtain $y^* \in \cap_{\lambda=1}^{\infty} c^*_\lambda \cap C^*_{p_1} \cap \ldots \cap C^*_{p_k} \cap C^*_{p_{k+1}}$. □

Proposition 2.2.4. *Let X be a sequentially compact space endowed with the preorder R from* (2.1.1) *and let $C \subset X$ be a chain.*

(i) *For any finite subset $\{p_1, \ldots, p_s\} \subset I$ one has*

$$\cap_{i=1}^{s} C^*_{p_i} \cap c^*_{p_i} \neq \emptyset. \tag{2.2.7}$$

(ii) *If X is, in addition, quasi-compact, then*

$$\cap_{p\in I} C^*_p \cap c^*_p \neq \emptyset. \tag{2.2.8}$$

P r o o f. (i) If C is a finite R-chain, then its largest element is a member of the intersection $\cap_{i=1}^{s} C_i \cap c_i$.

Now, let us suppose that the R-chain C is infinite. In case all sets c_1, C_1, …, c_s, C_s, are nonempty Corollary 2.2.2 implies that their intersection is not empty, hence (2.2.8) holds. Otherwise, using Lemma 2.2.3 and induction with respect to k, we are done.

(ii) Since X is quasi-compact, part (i) implies part (ii). □

Corollary 2.2.5. *If X is a quasi-compact and sequentially compact space, then the preordered set X is inductive.*

P r o o f. Every element $x^* \in \cap_{p \in I} C_p^* \cap c_p^*$ is an upper bound of the R-chain C, hence the preordered set X is inductive. □

Now, Corollary 2.2.5 and Kuratowski-Zorn Theorem yield the following:

Theorem 2.2.6. *Let X be a quasi-compact and sequentially compact space. For any element $x \in X$ there exists an R-maximal element $y \in X$ with xRy.*

2.3. Examples. Since the $(n-1)$-dimensional simplex Δ_{n-1} is a compact set in $\mathbb{R}^n$, it is a quasi-compact and sequentially compact topological space. In case the family u consists of one function $u(x)$ — the expected return of the portfolio x and the family v consists of one function $v(x)$ — its volatility, using Theorem 2.2.6, we obtain the existence of Markowitz efficient portfolios and something more: Any portfolio is R-dominated by a Markowitz efficient portfolio.

Moreover, replacing the simplex Δ_{n-1} with a closed ball B_{n-1} in the affine hyperplane $\sum_{i=1}^{n} x_i = 1$ in $\mathbb{R}^n$, such that $\Delta_{n-1} \subset B_{n-1}$, we admit bounded negative x_i's (that is, constrained *short sales*) and again Theorem 2.2.6 assures existence of Markowitz efficient portfolios which dominate any given portfolio.

Below, we remind some notions from statistics and give examples of application of Theorem 2.2.6.

Given the integer $\ell \geq 2$, the *ℓ-th central moment* of the random variable $s(x)$ is $E((s(x)-E(s(x)))^{\ell})$. The standard variance is the second central moment $v(x) = E((s(x)-E(s(x)))^2)$ of $s(x)$ and it is a quadratic form in $x_1, \ldots, x_n$. The third central moment $E((s(x)-E(s(x)))^3)$ is a cubic form and the fourth central moment $E((s(x)-E(s(x)))^4)$ is a form of degree 4 in $x_1, \ldots, x_n$.

Given $x \in \Delta_{n-1}$ and $t \in \mathbb{R}$, we set $F_x(t) = P(\{m \in S \mid s(x)(m) < t\})$, so $F_x \colon \mathbb{R} \to [0,1]$ is the *cumulative distribution function* of the random variable $s(x)$. We assume that $s(x)$ is a continuous random variable with *density function*

$f_x(t)$, so $F_x(t) = \int_{-\infty}^{t} f_x(\tau)d\tau$ and $F'_x(t) = f_x(t)$. In particular, the functions $F_x(t)$ are continuous.

We define recursively $D_x^{(1)}(t) = F_x(t)$, $D_x^{(2)}(t) = \int_{-\infty}^{t} F_x(\tau)d\tau, \ldots, D_x^{(\ell)}(t) = \int_{-\infty}^{t} D_x^{(\ell-1)}(\tau)d\tau, \ldots$.

The portfolio $x \in \Delta_{n-1}$ is said to be *ℓ-th order stochastically dominated* by portfolio $y \in \Delta_{n-1}$ if $D_y^{(\ell)}(t) \leq D_x^{(\ell)}(t)$ for all $t \in \mathbb{R}$. In case the previous inequalities hold and $D_y^{(\ell)}(t) < D_x^{(\ell)}(t)$ for some $t \in \mathbb{R}$, x is said to be *ℓ-th order strictly stochastically dominated* by y.

We set

$$\mathrm{Skew}(s(x)) = \frac{E((s(x) - E(s(x)))^3)}{\mathrm{Var}(s(x))^{\frac{3}{2}}}$$

to be the *skewness* and

$$\mathrm{Kurt}(s(x)) = \frac{E((s(x) - E(s(x)))^4)}{\mathrm{Var}(s(x))^2} - 3$$

to be the *kurtosis*, or, *excess kurtosis* of the random variable $s(x)$.

If the random variable $s(x)$ is normal, then $\mathrm{Skew}(s(x)) = \mathrm{Kurt}(s(x)) = 0$.

Example 2.3.1. In case $I = \{1\}$, $J = \emptyset$, the function $u = u_1$ can be considered as an utility function on Δ_{n-1} and R is the corresponding preference relation with negatively transitive asymmetric part F.

Example 2.3.2. In case $I = \{1\}$, $J = \{1\}$,

$$u_1(x) = E(s(x)),$$

$$v_1(x) = \mathrm{Var}(s(x)),$$

we obtain the classical Markowitz setup.

Example 2.3.3. In case

$$u_1(x) = E(s(x)),$$

$$v_1(x) = \mathrm{Var}(s(x)),\ v_2(x) = \mathrm{Skew}^2(s(x)),$$

we simultaneously maximize the expected return $E(s(x))$ and minimize the volatility $\mathrm{Var}(s(x))$ and the absolute value of the skewness $\mathrm{Skew}(s(x))$ of the return $s(x)$ of the portfolio x.

Example 2.3.4. In case

$$u_1(x) = E(s(x)),$$

$$v_1(x) = \mathrm{Var}(s(x)),\ v_2(x) = \mathrm{Kurt}^2(s(x)),$$

we simultaneously maximize the expected return $E(s(x))$ and minimize the volatility $\mathrm{Var}(s(x))$ and the absolute value of the kurtosis $\mathrm{Kurt}(s(x))$ of the return $s(x)$, thus balancing the tails of its distribution.

Example 2.3.5. In case

$$u_1(x) = E(s(x)),$$

$$v_1(x) = \mathrm{Var}(s(x)),\ v_2(x) = \mathrm{Skew}^2(s(x)),\ v_3(x) = \mathrm{Kurt}^2(s(x)),$$

we simultaneously maximize the expected return $E(s(x))$ and minimize the volatility $\mathrm{Var}(s(x))$, the the absolute value of the skewness $\mathrm{Skew}(s(x))$, and the absolute value of the kurtosis $\mathrm{Kurt}(s(x))$ of the return $s(x)$. In this way we balance both the tails of the distribution of $s(x)$ and "round" the maximum of its density function $f_x(t)$.

Example 2.3.6. In case

$$v_t(x) = D_x^{(\ell)}(t),\ t \in \mathbb{R},$$

we maximize the ℓ-th order stochastic dominance, $\ell \geq 1$.

Example 2.3.7. In case

$$u(x) = E(s(x)),$$

$$v(x) = \mathrm{Var}(s(x)),\ v_t(x) = D_x^{(\ell)}(t),\ t \in \mathbb{R},$$

we simultaneously maximize the expected return $E(u(x))$ and the ℓ-th order stochastic dominance, $\ell \geq 1$, and minimize the volatility $\mathrm{Var}(s(x))$.

Example 2.3.8. Let X be a quasi-compact and sequentially compact space and let $f\colon X \times X \to \mathbb{R}$ be a continuous real function. For any $p \in X$ we set

$$u_p(x) = f(x, p),\ x \in X,$$

$$v_p(y) = f(p, y),\ y \in X.$$

Further, for any $x \in X$ we set

$$U_x^{(\geq)} = \{y \in X \mid f\,(y, p) \geq f\,(x, p) \text{ for all } p \in X\},$$

$$V_x^{(\leq)} = \{y \in X \mid f(p,y) \leq f(p,x) \text{ for all } p \in X\},$$

and for any $x, p \in X$ we set

$$U_x^{(\hat{p};\geq)} = \{y \in X \mid f(y,q) \geq f(x,q) \text{ for all } q \in X, q \neq p\},$$

$$V_x^{(\hat{p};\leq)} = \{y \in X \mid f(q,y) \leq f(q,x) \text{ for all } q \in X, q \neq p\}.$$

Note that $U_x^{(\geq)}$, $V_x^{(\leq)}$, $U_x^{(\hat{p};\geq)}$, $V_x^{(\hat{p};\leq)}$, are closed subsets of X and that

$$x \in U_x^{(\geq)} \subset U_x^{(\hat{p};\geq)},\ x \in V_x^{(\leq)} \subset V_x^{(\hat{p};\leq)}$$

for all $x, p \in X$. According to Theorem 2.2.6, there exists an element $m \in X$, such that for any $p \in X$ one has

$$f(m,p) = \max_{y \in U_m^{(\hat{p};\geq)} \cap V_m^{(\leq)}} f(y,p)$$

and

$$f(p,m) = \min_{y \in U_m^{(\geq)} \cap V_m^{(\hat{p};\leq)}} f(p,y).$$

Acknowledgement. The friendship of Prof. Fernando Fernandez Rodrigues, Department of Quantitative Methods on Economics and Management, Las Palmas de Gran Canaria University, is much appreciated and has led to many interesting and good-spirited discussions relating to this research.

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Received June 7, 2017