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A NOTE ON ALMOST η-RICCI SOLITONS IN EUCLIDEAN HYPERSURFACES

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Abstract. Conditions for existence an almost η-Ricci soliton in an orientable hypersurface M which is isometrically embedded in an Euclidian space are determined and in the existence case, under certain conditions, conclusions on the principal curvatures are formulated. Some rigidity results in the compact case are also obtained.

1. Introduction. Let g be a Riemannian metric on the n-dimensional manifold M, Ric its Ricci curvature tensor field, $\pounds_V$ the Lie derivative in the direction of the vector field V, η a 1-form and λ and μ are smooth functions on M. Then the data (V,λ,μ) which satisfy the equation

$$\frac{1}{2}\pounds_V g+\mathrm{Ric}=\lambda g+\mu\eta\otimes\eta \tag{1}$$

is said to be an *almost η-Ricci soliton* on (M,g) [1]; in particular, if λ and μ are constants, then (V,λ,μ) is an *η-Ricci soliton* [4], if $\mu=0$, (V,λ) is an *almost*

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Ricci soliton [7], respectively a *Ricci soliton* [6] if λ is a function, respectively a constant. The soliton is called *shrinking*, *steady* or *expanding* according as λ is positive, zero or negative, respectively [5]. If the potential vector field V is of gradient type, $V = \text{grad}(f)$, for f a smooth function on M, then (V, λ, μ) is called *gradient almost η-Ricci soliton.*

2. Almost η-Ricci solitons on hypersurfaces of $(\mathbb{R}^n, g)$. Let $M \hookrightarrow \mathbb{R}^n$ be an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$. Denote by $\varphi : M \to \mathbb{R}^n$ the immersion and by N the unit normal vector field to M. Therefore, we can express φ (the position vector field of a point of M in $\mathbb{R}^n$) as $\varphi = T + \rho N$, where $T \in \mathfrak{X}(M)$ and $\rho = g(\varphi, N)$ is the support function of the hypersurface M.

Denoting also by g the Riemannian metric induced on M, by ∇^M and ∇ the Levi-Civita connections on (M, g) and $(\mathbb{R}^n, g)$ respectively, the Gauss and Weingarten formulas corresponding to M are given by:

$$\nabla_X Y = \nabla^M_X Y + h(X, Y), \tag{2}$$

$$\nabla_X N = -A_N X, \tag{3}$$

where h is the (symmetric) second fundamental tensor corresponding to N and A_N is the shape operator (or the Weingarten map) in the direction of the normal vector field N defined by $g(A_N X, Y) = g(h(X, Y), N)$, for X, $Y \in \mathfrak{X}(M)$.

Similarly to [8], we will provide a necessary and sufficient condition to exist an almost η-Ricci soliton on an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$.

Theorem 2.1. *Let M be an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$, V a vector field on M and η the g-dual of V. If η is closed (in particular, if V is a gradient vector field), then M admits an almost η-Ricci soliton with the potential vector field V if and only if there exist two smooth functions λ and μ on M such that:*

$$A_N^2 - (n-1)\alpha A_N - \nabla^M V + \lambda I_{\mathfrak{X}(M)} + \mu \eta \otimes V = 0, \tag{4}$$

where α is the mean curvature function.

Proof. From (2), taking into account that the Ricci curvature of $\mathbb{R}^n$ is zero, we obtain the Ricci curvature tensor field of M given by:

$$\text{Ric}_M(X, Y) = \text{trace}(A_N) g(A_N X, Y) - g(A_N^2 X, Y), \tag{5}$$

for any X, $Y \in \mathfrak{X}(M)$.

Also, from $d\eta = 0$ we get $g(\nabla_X V, Y) = g(\nabla_Y V, X)$, for any $X, Y \in \mathfrak{X}(M)$, hence

$$\frac{1}{2}\pounds_V g(X,Y) = g(\nabla^M_X V, Y),$$

for any X, $Y \in \mathfrak{X}(M)$.

We conclude that on M there exists an almost η-Ricci soliton with the potential vector field V and η the g-dual of V if there exist two smooth functions λ and μ on M such that:

$$\frac{1}{2}\pounds_V g(X,Y) + \mathrm{Ric}_M(X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y) = 0,$$

for any X, $Y \in \mathfrak{X}(M)$, which is equivalent to

$$g(\nabla^M_X V + (n-1)\alpha A_N X - A_N^2 X - \lambda X - \mu\eta(X)V, Y) = 0,$$

for any X, $Y \in \mathfrak{X}(M)$ from where we get the conclusion. □

Corollary 2.2. *Let M be an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$, V a vector field on M and η the g-dual of V. If V is a concircular vector field on M (i.e. $\nabla^M V = aI_{\mathfrak{X}(M)}$, for $a \in C^\infty(M)$), then M admits an almost η-Ricci soliton with the potential vector field V if and only if there exist two smooth functions λ and μ on M such that:*

$$A_N^2 - (n-1)\alpha A_N - (a-\lambda)I_{\mathfrak{X}(M)} + \mu\eta\otimes V = 0. \tag{6}$$

Also from Theorem 2.1 we deduce:

Corollary 2.3. *If M is an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$ admitting an almost η-Ricci soliton (V, λ, μ) with $V = \mathrm{grad}(f)$, for $f \in C^\infty(M)$ and η the g-dual of V, then:*

$$\mathrm{trace}(A_N^2) - (n-1)^2\alpha^2 = \Delta(f) - (n-1)\lambda - \mu|\,\mathrm{grad}(f)|^2. \tag{7}$$

Proposition 2.4. *Let M be an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$, V a vector field on M and η the g-dual of V. If $V = \mathrm{grad}(\rho)$ with respect to the induced metric on M, then M admits an almost η-Ricci soliton with the potential vector field V if and only if there exist two smooth functions λ and μ on M such that:*

$$(\nabla^M_X A_N)T + (\rho+1)A_N^2 X + [1-(n-1)\alpha]A_N X + \lambda X + \mu X(\rho)\,\mathrm{grad}(\rho) = 0, \tag{8}$$

for any $X \in \mathfrak{X}(M)$.

Proof. Since

$$\frac{1}{2}(\pounds_V g)(X,Y) = \mathrm{Hess}(\rho)(X,Y) := g(\nabla^M_X \operatorname{grad}(\rho), Y) = -g(\nabla^M_X(A_N T), Y) =$$

$$= -g((\nabla^M_X A_N)T + A_N(\nabla^M_X T), Y) = -g((\nabla^M_X A_N)T + A_N X + \rho A_N^2 X, Y),$$

using (5) we deduce that there exists an almost η-Ricci soliton on M given by (V, λ, μ) if and only if (8) holds. □

Assume now that U is a *concircular* vector field on $\mathbb{R}^n$ i.e. $\nabla U = aI_{\mathfrak{X}(\mathbb{R}^n)}$, for $a \in C^\infty(\mathbb{R}^n)$ (in particular, if $a = 1$, we call U *concurrent* vector field). Then for any $X \in \mathfrak{X}(M)$:

$$aX = \nabla_X(U^T + U^\perp) = \nabla^M_X U^T - g(U^T, N)A_N X + h(X, U^T) + X(g(U^T, N))N$$

which implies

$$\nabla^M_X U^T = aX + g(U^T, N)A_N X, \tag{9}$$

for any $X \in \mathfrak{X}(M)$.

A similar result as in the Ricci soliton case [2], [3] can be proved for almost η-Ricci solitons.

Proposition 2.5. *Let M be an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$. If U is a concircular vector field on $\mathbb{R}^n$ with $\nabla U = aI_{\mathfrak{X}(\mathbb{R}^n)}$, for $a \in C^\infty(\mathbb{R}^n)$, then M admits an almost η-Ricci soliton with the potential vector field U^T, for η the g-dual of U^T, if and only if there exist two smooth functions λ and μ on M such that:*

$$\begin{aligned}\mathrm{Ric}_M(X,Y) = -(a-\lambda)g(X,Y) + \mu g(X,U^T)g(Y,U^T) \\ - g(U^\perp, N)g(A_N X, Y),\end{aligned} \tag{10}$$

for any X, $Y \in \mathfrak{X}(M)$.

Proof. Computing $\frac{1}{2}(\pounds_{U^T} g)(X,Y) = \frac{1}{2}[g(\nabla^M_X U^T, Y) + g(\nabla^M_Y U^T, X)]$ and using (9) we deduce that there exists an almost η-Ricci soliton on M given by (U^T, λ, μ) if and only if (10) holds. □

From (5) and Proposition 2.5 we conclude:

Corollary 2.6. *Under the hypotheses of Proposition 2.5, M admits an almost η-Ricci soliton with the potential vector field U^T, for η the g-dual of U^T, if and only if there exist two smooth functions λ and μ on M such that:*

$$A_N^2 X - [(n-1)\alpha + g(U^\perp, N)]A_N X - (a-\lambda)X + \mu g(X, U^T)U^T = 0, \tag{11}$$

for any $X \in \mathfrak{X}(M)$.

Theorem 2.7. *Let M be an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$ and U a concircular vector field on $\mathbb{R}^n$ with $\nabla U = aI_{\mathfrak{X}(\mathbb{R}^n)}$, for $a \in C^\infty(\mathbb{R}^n)$. Assume that M admits an almost η-Ricci soliton (U^T, λ, μ), for η a 1-form on M. If there exists $i_0 \in \{1, \dots, n-1\}$ such that:*

$$\lambda + \mu\eta(e_{i_0}) < \frac{[(n-1)\alpha + g(U,N)]^2}{4} + a, \tag{12}$$

for $\{e_1, \dots, e_{n-1}\}$ a local orthonormal frame field on M, N the unit normal vector field to M and α the mean curvature function, then M has exactly two distinct principal curvatures given by:

$$k_{1,2} = \frac{(n-1)\alpha + g(U,N) \pm \sqrt{[(n-1)\alpha + g(U,N)]^2 + 4[a - \lambda - \mu(\eta(e_{i_0}))^2]}}{2}. \tag{13}$$

Proof. Writing the Gauss-Codazzi equation

$$g(R_M(X,Y)Z,W) = g(R(X,Y)Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))$$

for $X := e_j$ and $W := e_j$ with $\{e_1, \dots, e_{n-1}\}$ a local orthonormal frame field on M and summing over j we get:

$$\mathrm{Ric}_M(Y,Z) = (n-1)\alpha g(h(Y,Z),N) - \sum_{j=1}^{n-1} g(h(e_j,Y),h(e_j,Z)),$$

where $\frac{1}{n-1}\sum_{j=1}^{n-1} h(e_j,e_j) = \alpha N$ with α the mean curvature function. Denoting by k_i the eigenvalues of A_N corresponding to e_i, by replacing Ric_M from the previous relation into the soliton equation computed in (e_i, e_i), we obtain:

$$k_i^2 - (n-1)\alpha k_i + \lambda + \mu(\eta(e_i))^2 - g(\nabla_{e_i} U^T, e_i) = 0$$

and using the Gauss and Weingarten formulas (2) and (3) we get that the principal curvatures satisfy the equation:

$$k_i^2 - [(n-1)\alpha + g(U,N)]k_i - a + \lambda + \mu(\eta(e_i))^2 = 0. \qquad \square$$

Remark 2.8. If M is an orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$, V a conformal Killing vector field on M (i.e. $\frac{1}{2}\pounds_V g = fg$, for f a smooth function on M) and η the g-dual of V, then M admits an almost η-Ricci soliton with the potential vector field V if and only if M is quasi-Einstein manifold.

3. The compact case. If M is a compact orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$, from Hodge decomposition theorem we know that any vector field V on M can be decomposed into a gradient vector field and a divergence-free vector field i.e.

$$V = \operatorname{grad}(f) + V_0,$$

with f a smooth function on M and $V_0 \in \mathfrak{X}(M)$ with $div(V_0) = 0$.

Theorem 3.1. *Let M be a compact orientable hypersurface isometrically immersed into $(\mathbb{R}^n, g)$, $n \geq 3$, and (V, λ, μ) an almost η-Ricci soliton on M with $V = \operatorname{grad}(f) + V_0$, $div(V_0) = 0$ and $\eta = df$ the g-dual of $\operatorname{grad}(f)$. Then:*

$$\int_M \alpha^2 \geq \frac{1}{n-2} \int_M \lambda + \frac{1}{(n-2)(n-1)} \int_M \mu |\operatorname{grad}(f)|^2, \tag{14}$$

where α is the mean curvature function.

Proof. In this case, the soliton equation for (V, λ, μ) is equivalent to:

$$\begin{aligned} \operatorname{Hess}(f)(X,Y) &+ \frac{1}{2}[g(\nabla^M_X V_0, Y) + g(\nabla^M_Y V_0, X)] \\ &+ \operatorname{Ric}_M(X,Y) - \lambda g(X,Y) - \mu \eta(X)\eta(Y) = 0, \end{aligned} \tag{15}$$

for any X, $Y \in \mathfrak{X}(M)$ which by taking the trace gives:

$$\Delta(f) + \operatorname{scal}_M - (n-1)\lambda - \mu \sum_{i=1}^{n-1} (\eta(e_i))^2 = 0, \tag{16}$$

for $\{e_1, \dots, e_{n-1}\}$ a local orthonormal frame field on M. Since

$$\operatorname{scal}_M = (n-1)^2 \alpha^2 - |A_N|^2$$

in the case when $\eta = df$ is the g-dual of $\operatorname{grad}(f)$, (16) becomes:

$$\Delta(f) + (n-1)^2 \alpha^2 - |A_N|^2 - (n-1)\lambda - \mu |\operatorname{grad}(f)|^2 = 0 \tag{17}$$

and by Schwarz inequality $|A_N|^2 \geq (n-1)\alpha^2$:

$$\Delta(f) + (n-2)(n-1)\alpha^2 \geq (n-1)\lambda + \mu |\operatorname{grad}(f)|^2$$

which by integrating on M gives:

$$\int_M \alpha^2 \geq \frac{1}{n-2} \int_M \lambda + \frac{1}{(n-2)(n-1)} \int_M \mu |\operatorname{grad}(f)|^2. \qquad \square$$

Remark 3.2. i) If α and $|\operatorname{grad}(f)| =: k$ are constant, for the η-Ricci soliton case we get the inequality:

$$\alpha^2 \geq \frac{1}{n-2}\left(\lambda + \frac{k^2\mu}{n-1}\right)$$

which for the steady η-Ricci soliton becomes:

$$\mu \leq \frac{(n-2)(n-1)}{k^2}\alpha^2.$$

ii) If M is totally geodesic submanifold, from (17) we get the Poisson equation:

$$\Delta(f) = (n-1)\lambda + \mu|\operatorname{grad}(f)|^2$$

which by integration gives:

$$\int_M \mu|\operatorname{grad}(f)|^2 = -(n-1)\int_M \lambda$$

and we conclude that there are no steady almost η-Ricci soliton.

Example 3.3. Let M be the sphere $S^{n-1}(1)$ isometrically immersed into $\mathbb{R}^n$ by $\varphi : S^{n-1}(1) \to \mathbb{R}^n$. Taking $f := \frac{1}{2}|\varphi|^2$ and $V := \operatorname{grad}(f)$, the data $(V, 0, \mu)$ with $\mu = n-2$ defines a steady η-Ricci soliton on $S^{n-1}(1)$, where $\eta = df$.

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