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VIETORIS-TYPE TOPOLOGIES ON HYPERSPACES*

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Dedicated to the memory of Prof. Stoyan Nedev

Abstract. We study the $(\mathcal{F},\mathcal{G})$-*hit-and-miss topology* introduced by Clementino and Tholen [4]. The same topology was introduced independently in the preliminary version [12] of this paper under the name of *Vietoris-type hypertopology* (at that time we were not aware of the paper [4]). We show that the Vietoris-type hypertopology is, in general, different from the Vietoris topology. Also, some of the results of E. Michael [13] about hyperspaces with Vietoris topology are extended to analogous results for hyperspaces with Vietoris-type topology. We obtain as well some results about hyperspaces with Vietoris-type topology which concern some problems analogous to those regarded by H.-J. Schmidt in [14].

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Key words: hyperspace, Vietoris (hyper)topology, Vietoris-type (lower-Vietoris-type, upper-Vietoris-type, Tychonoff-type) (hyper)topology, $(\mathcal{F},\mathcal{G})$-hit-and-miss topology.

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Introduction. In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space for obtaining a generalization of the famous Kolmogoroff Theorem on operations on sets. This new topology is similar to the upper Vietoris topology but is weaker than it. In 1997, M. M. Clementino and W. Tholen [4] introduced, for any families $\mathcal{F}$ and $\mathcal{G}$ of subsets of a set X, three new topologies on the power set $\mathcal{P}(X)$ (and its subsets), called by them, respectively, *upper $\mathcal{F}$-topology, lower $\mathcal{F}$-topology* and $(\mathcal{F},\mathcal{G})$*-hit-and-miss topology.* The upper $\mathcal{F}$-topology turned out to be a generalized version of the Choban topology. In 1998, G. Dimov and D. Vakarelov [9] introduced independently the notion of upper $\mathcal{F}$-topology, which was called by them *Tychonoff-type hypertopology* (later on, in [11], it was called also *upper-Vietoris-type hypertopology* because "Tychonoff hypertopology" and "upper Vietoris hypertopology" are two names of one and same notion); they used it for proving an isomorphism theorem for the category of all Tarski consequence systems. The Tychonoff-type hypertopology was studied in details in [8]. Further, in [11], the notion of lower $\mathcal{F}$-topology was introduced independently and was called *lower-Vietoris-type hypertopology.* In the preliminary version [12] of this paper, the notion of *Vietoris-type hypertopology* was introduced. As we learned later on, it is equivalent to the notion of $(\mathcal{F},\mathcal{G})$*-hit-and-miss topology* introduced by M. M. Clementino and W. Tholen in [4]. In this paper we study the Vietoris-type hypertopology and show that, in general, it is different from the Vietoris topology. Also, some of the results of E. Michael [13] about the hyperspaces with Vietoris topology are extended to analogous results for the hyperspaces with Vietoris-type topology. We obtain as well some results about the hyperspaces with Vietoris-type topology which concern some problems analogous to those regarded by H.-J. Schmidt in [14].

1. Preliminaries. We denote by $\mathbb{N}$ the set of all natural numbers (hence, $0 \notin \mathbb{N}$), by $\mathbb{R}$ the real line (with its natural topology) and by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, \infty\}$.

Let X be a set. We denote by $|X|$ the cardinality of X and by $\mathcal{P}(X)$ (resp., by $\mathcal{P}'(X)$) the set of all (non-empty) subsets of X. Let $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. We set:

- $A^+_{\mathcal{M}} := \{M \in \mathcal{M} \mid M \subseteq A\}$;
- $\mathcal{A}^+_{\mathcal{M}} := \{A^+_{\mathcal{M}} \mid A \in \mathcal{A}\}$;
- $A^-_{\mathcal{M}} := \{M \in \mathcal{M} \mid M \cap A \neq \emptyset\}$
- $\mathcal{A}^-_{\mathcal{M}} := \{A^-_{\mathcal{M}} \mid A \in \mathcal{A}\}$;

- $Fin(X) := \{M \subseteq X \mid 0 < |M| < \aleph_0\}$;
- $Fin_n(X) := \{M \subseteq X \mid 0 < |M| \leq n\}$, where $n \in \mathbb{N}$;
- $\mathcal{A}^{\cap} := \left\{ \bigcap_{i=1}^{k} A_i \mid k \in \mathbb{N}, A_i \in \mathcal{A} \right\}$.
- $\mathcal{A}^{\cup} := \left\{ \bigcup_{i=1}^{k} A_i \mid k \in \mathbb{N}, A_i \in \mathcal{A} \right\}$.

Let $(X, \mathcal{T})$ be a topological space. We put

- $CL(X) := \{M \subseteq X \mid M \text{ is closed in } X,\ M \neq \emptyset\}$.
- $Comp(X) := \{M \in CL(X) \mid M \text{ is compact}\}$.

When $\mathcal{M} = CL(X)$, we will simply write A^+ and A^- instead of $A^+_{\mathcal{M}}$ and $A^-_{\mathcal{M}}$; the same for subfamilies $\mathcal{A}$ of $\mathcal{P}(X)$. The closure of a subset A of X in $(X, \mathcal{T})$ will be denoted by $cl_{(X,\mathcal{T})}A$ or $\overline{A}^{(X,\mathcal{T})}$ (we will also write, for short, $cl_X A$ or $\overline{A}^X$ and even $\overline{A}^{\mathcal{T}}$). By "neighborhood" we will mean an "open neighborhood". The regular spaces are not assumed to be T_1-spaces; by a T_3-space we mean a regular T_1-space.

If X and Y are sets and $f : X \longrightarrow Y$ is a function then, as usual, we denote by $f \restriction X$ the function between X and $f(X)$ which is a restriction of f. If $(X, \mathcal{T})$ and $(Y, \mathcal{O})$ are topological spaces and $f : X \longrightarrow Y$ is an injection, then we say that f is an *inversely continuous function* if the function $(f \restriction X)^{-1} : f(X) \longrightarrow X$ is continuous.

Let X be a topological space. Recall that the *upper Vietoris topology* Υ_{+X} on $CL(X)$ (called also *Tychonoff topology on* $CL(X)$) has as a base the family of all sets of the form

$$U^+ = \{F \in CL(X) \mid F \subseteq U\},$$

where U is open in X, and *the lower Vietoris topology* Υ_{-X} on $CL(X)$ has as a subbase all sets of the form

$$U^- = \{F \in CL(X) \mid F \cap U \neq \emptyset\},$$

where U is open in X. The *Vietoris topology* Υ_X on $CL(X)$ is defined as the supremum of Υ_{+X} and Υ_{-X}, i.e., $\Upsilon_{+X} \cup \Upsilon_{-X}$ is a subbase for Υ_X.

Definition 1.1. ([4, 9]) (*a*) *Let* $(X, \mathcal{T})$ *be a topological space and* $\mathcal{M} \subseteq \mathcal{P}'(X)$. *The topology* $\Upsilon_{+\mathcal{M}}$ *on* $\mathcal{M}$ *having as a base the family* $\mathcal{T}^+_{\mathcal{M}}$ *is called a* Tychonoff topology on $\mathcal{M}$ *generated by* $(X, \mathcal{T})$. *When* $\mathcal{M} = CL(X)$, *then* $\Upsilon_{+\mathcal{M}}$ *is just the classical upper Vietoris topology* Υ_{+X} *on* $CL(X)$ *(= Tychonoff topology* $\Upsilon_{+\mathcal{M}}$ *on* $CL(X)$*).*

(*b*) *Let* X *be a set and* $\mathcal{M} \subseteq \mathcal{P}'(X)$. *A topology* $\mathcal{O}$ *on the set* $\mathcal{M}$ *is called a* Tychonoff-type topology on $\mathcal{M}$ *if the family* $\mathcal{O} \cap \mathcal{P}(X)^+_{\mathcal{M}}$ *is a base for* $\mathcal{O}$.

A Tychonoff topology on $\mathcal{M}$ is always a Tychonoff-type topology on $\mathcal{M}$, but not viceversa (see [8]).

Fact 1.2. ([9]) *Let* X *be a set,* $\mathcal{M} \subseteq \mathcal{P}'(X)$ *and* $\mathcal{O}$ *be a topology on* $\mathcal{M}$. *Then the family*

$$\mathcal{B}_{\mathcal{O}} := \{A \subseteq X \mid A^+_{\mathcal{M}} \in \mathcal{O}\}$$

contains X *and is closed under finite intersections; hence, it can serve as a base for a topology*

$$\mathcal{T}_{+\mathcal{O}}$$

on X. *When* $\mathcal{O}$ *is a Tychonoff-type topology, the family* $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}}$ *is a base for* $\mathcal{O}$.

Definition 1.3. ([9]) *Let* X *be a set,* $\mathcal{M} \subseteq \mathcal{P}'(X)$ *and* $\mathcal{O}$ *be a topology on* $\mathcal{M}$. *Then we say that the topology* $\mathcal{T}_{+\mathcal{O}}$ *on* X, *introduced in Fact 1.2, is* the plus-topology on X induced by the topological space $(\mathcal{M}, \mathcal{O})$.

Definition 1.4. ([11]) *Let* $(X, \mathcal{T})$ *be a topological space and* $\mathcal{M} \subseteq \mathcal{P}'(X)$. *The topology* $\Upsilon_{-\mathcal{M}}$ *on* $\mathcal{M}$ *having as a subbase the family* $\mathcal{T}^-_{\mathcal{M}}$ *is called a* lower Vietoris topology on $\mathcal{M}$ *generated by* $(X, \mathcal{T})$. *When* $\mathcal{M} = CL(X)$, *then* $\Upsilon_{-\mathcal{M}}$ *is just the classical lower Vietoris topology* Υ_{-X} *on* $CL(X)$.

Definition 1.5. ([4, 11]) *Let* X *be a set,* $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{O}$ *be a topology on* $\mathcal{M}$. *We say that* $\mathcal{O}$ *is a* lower-Vietoris-type topology on $\mathcal{M}$, *if* $\mathcal{O} \cap \{A^-_{\mathcal{M}} \mid A \subseteq X\}$ *is a subbase for* $\mathcal{O}$.

A lower Vietoris topology on $\mathcal{M}$ is always a lower-Vietoris-type topology on $\mathcal{M}$, but not viceversa (see [11]).

Fact 1.6. ([11]) *Let* X *be a set,* $\mathcal{M} \subseteq \mathcal{P}'(X)$ *and* $\mathcal{O}$ *be a topology on* $\mathcal{M}$. *Then the family*

$$\mathcal{P}_{\mathcal{O}} := \{A \subseteq X \mid A^-_{\mathcal{M}} \in \mathcal{O}\}$$

contains X, *and, hence, can serve as a subbase for a topology*

$$\mathcal{T}_{-\mathcal{O}}$$

on X. The family $\mathcal{P}_{\mathcal{O}}$ is closed under arbitrary unions. When $\mathcal{O}$ is a lower-Vietoris-type topology, the family $(\mathcal{P}_{\mathcal{O}})^-_{\mathcal{M}}$ is a subbase for $\mathcal{O}$.

Definition 1.7. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Then we say that the topology $\mathcal{T}_{-\mathcal{O}}$ on X, introduced in Fact* 1.6, *is* the minus-topology on X induced by the topological space $(\mathcal{M}, \mathcal{O})$.

All undefined here notions and notation can be found in [1, 10].

2. The Vietoris-type topologies on hyperspaces.

Definition 2.1. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{M} \subseteq \mathcal{P}'(X)$. The topology $\Upsilon_{\mathcal{M}}$ on $\mathcal{M}$ having as a subbase the family $\mathcal{T}^+_{\mathcal{M}} \cup \mathcal{T}^-_{\mathcal{M}}$ is called* the Vietoris topology on $\mathcal{M}$. *When $\mathcal{M} = CL(X)$ then $\Upsilon_{\mathcal{M}} \equiv \Upsilon_X$.*

Definition 2.2. ([4, 12]) *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. The topology $\mathcal{O}$ is called a* Vietoris-type topology on $\mathcal{M}$ *if the family $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}} \cup (\mathcal{P}_{\mathcal{O}})^-_{\mathcal{M}}$ is a subbase for $\mathcal{O}$.*

Remark 2.3. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{M} \subseteq \mathcal{P}'(X)$. Then, clearly, $\Upsilon_{\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$. As we will see in Example 2.12 below, the converse is not always true even when $\mathcal{M} = CL(X)$. Note that when $\mathcal{M} \subseteq CL(X)$, then $\Upsilon_{\mathcal{M}} \equiv (\Upsilon_X)|_{\mathcal{M}}$.

Notation 2.4. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. We denote by*

$$\mathcal{T}_{\mathcal{O}}$$

the topology on X having $\mathcal{B}_{\mathcal{O}} \cup \mathcal{P}_{\mathcal{O}}$ as a subbase and we say that $\mathcal{T}_{\mathcal{O}}$ is the V-topology on X induced by the topological space $(\mathcal{M}, \mathcal{O})$. *We denote by*

$$\mathcal{O}_u$$

the topology on $\mathcal{M}$ having $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}}$ as a base, and by

$$\mathcal{O}_l$$

the topology on $\mathcal{M}$ having $(\mathcal{P}_{\mathcal{O}})^-_{\mathcal{M}}$ as a subbase.

Definition 2.5. *Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a* natural family in X *if $\{x\} \in \mathcal{M}$ for any $x \in X$.*

The following assertion can be easily proved.

Fact 2.6. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Then $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{+\mathcal{O}} \vee \mathcal{T}_{-\mathcal{O}}$, $\mathcal{O}_u$ is a Tychonoff-type topology on $\mathcal{M}$ and $\mathcal{O}_l$ is a lower-Vietoris-type topology on $\mathcal{M}$. $\mathcal{O}$ is a Vietoris-type topology on $\mathcal{M}$ iff $\mathcal{O} = \mathcal{O}_u \vee \mathcal{O}_l$.*

Fact 2.7. *Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{M} \subseteq CL(X)$, then $(\Upsilon_{+X})|_{\mathcal{M}}$ and $(\Upsilon_{-X})|_{\mathcal{M}}$ are Vietoris-type topologies on $\mathcal{M}$. Moreover, if $\mathcal{M} \subseteq \mathcal{P}'(X)$, then $\Upsilon_{+\mathcal{M}}$ and $\Upsilon_{-\mathcal{M}}$ are Vietoris-type topologies on $\mathcal{M}$.*

Proof. Let $\mathcal{M} \subseteq \mathcal{P}'(X)$. Set $\mathcal{O} = \Upsilon_{+\mathcal{M}}$. Then $\mathcal{T}^+_{\mathcal{M}}$ is a base for $\mathcal{O}$. Since $\mathcal{T}^+_{\mathcal{M}} \subseteq (\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}} \subseteq (\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}} \cup (\mathcal{P}_{\mathcal{O}})^-_{\mathcal{M}} \subseteq \mathcal{O}$, we get that $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}} \cup (\mathcal{P}_{\mathcal{O}})^-_{\mathcal{M}}$ is a (sub)base for $\mathcal{O}$. Hence $\Upsilon_{+\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$. Analogously, we get that $\Upsilon_{-\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$.

Let now $\mathcal{M} \subseteq CL(X, \mathcal{T})$. $\mathcal{T}^+$ is a base for Υ_{+X}. Hence $(\mathcal{T}^+)|_{\mathcal{M}} = \{U^+ \cap \mathcal{M} \mid U \in \mathcal{T}\}$ is a base for $(\Upsilon_{+X})|_{\mathcal{M}}$. Since $U^+ \cap \mathcal{M} = U^+_{\mathcal{M}}$, we get that $\mathcal{T}^+_{\mathcal{M}}$ is a base for $(\Upsilon_{+X})|_{\mathcal{M}}$. Hence $\Upsilon_{+\mathcal{M}} \equiv (\Upsilon_{+X})|_{\mathcal{M}}$ and thus $(\Upsilon_{+X})|_{\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$. Analogously, we get that $(\Upsilon_{-X})|_{\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$ (and $(\Upsilon_{-X})|_{\mathcal{M}} \equiv \Upsilon_{-\mathcal{M}}$). $\square$

Definition 2.8. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$. Then $\mathcal{O}$ is called a* strong Vietoris-type topology on $\mathcal{M}$ *if $\mathcal{T}_{+\mathcal{O}} \equiv \mathcal{T}_{-\mathcal{O}}$.*

Proposition 2.9. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq CL(X)$ and $\mathcal{M}$ be a natural family. Let $\mathcal{O} = (\Upsilon_X)|_{\mathcal{M}}$. Then $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T} \equiv \mathcal{T}_{\mathcal{O}}$. In particular, for every T_1-space X, Υ_X is a strong Vietoris-type topology on $CL(X)$.*

Proof. By Remark 2.3, $\mathcal{O} = (\Upsilon_X)|_{\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$. Obviously, $\mathcal{T} \subseteq \mathcal{B}_{\mathcal{O}} \cap \mathcal{P}_{\mathcal{O}}$. We will show that $\mathcal{B}_{\mathcal{O}} \cup \mathcal{P}_{\mathcal{O}} \subseteq \mathcal{T}$. Clearly, this will imply that $\mathcal{T}_{+\mathcal{O}} = \mathcal{T}_{-\mathcal{O}} = \mathcal{T} = \mathcal{T}_{\mathcal{O}}$. Let $A \subseteq X$ and $A^+_{\mathcal{M}} \in \mathcal{O}$. It is well-known that the family $\{< U_1, \ldots, U_n > \mid n \in \mathbb{N},\ U_i \in \mathcal{T} \text{ for } i \in \{1, \ldots, n\}\}$, where $< U_1, \ldots, U_n >= (\bigcup_{i=1}^{n} U_i)^+ \cap \bigcap_{i=1}^{n} (U_i)^-$, is a base for Υ_X. Let $x \in A$. Since $\mathcal{M}$ is a natural family, we get that $\{x\} \in A^+_{\mathcal{M}}$. Thus there exist $U_1, \ldots, U_n \in \mathcal{T}$ such that $\{x\} \in (\bigcup_{i=1}^{n} U_i)^+_{\mathcal{M}} \cap \bigcap_{i=1}^{n} (U_i)^-_{\mathcal{M}} \subseteq A^+_{\mathcal{M}}$. Then $x \in U = \bigcap_{i=1}^{n} U_i$. Using again naturality of $\mathcal{M}$, we obtain that $U \subseteq A$. Hence, $A \in \mathcal{T}$. So, $\mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$. Let now $A \subseteq X$ and $A^-_{\mathcal{M}} \in \mathcal{O}$. Then, arguing as above, we get that $A \in \mathcal{T}$. Thus, $\mathcal{P}_{\mathcal{O}} \subseteq \mathcal{T}$. Hence $\mathcal{B}_{\mathcal{O}} \cup \mathcal{P}_{\mathcal{O}} \subseteq \mathcal{T}$. $\square$

Example 2.10. There exists a T_0-space $(X, \mathcal{T})$ such that $\mathcal{O} = \Upsilon_X$ is not a strong Vietoris-type topology on $CL(X)$. Also, we have that $\mathcal{T} \neq \mathcal{T}_{-\mathcal{O}}$, $\mathcal{T} \neq \mathcal{T}_{+\mathcal{O}}$, $\mathcal{T} \neq \mathcal{T}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{-\mathcal{O}}$.

P r o o f. Let $X = \{0,1,2\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,2\}\}$. Then $(X, \mathcal{T})$ is a T_0-space and
$$CL(X) = \{X, \{1\}, \{1,2\}\}.$$
The topology Υ_{-X} has as a subbase the family
$$\{\emptyset^-, X^-, \{0\}^-, \{0,2\}^-\} = \{\emptyset, CL(X), \{X\}, \{\{1,2\}, X\}\}.$$
Thus $\Upsilon_{-X} = \{\emptyset, CL(X), \{X\}, \{\{1,2\}, X\}\}$. The topology Υ_{+X} has as a base the family $\{\emptyset^+, X^+, \{0\}^+, \{0,2\}^+\} = \{\emptyset, CL(X)\}$. Thus $\Upsilon_{+X} = \{\emptyset, CL(X)\}$. Hence $\Upsilon_X = \Upsilon_{+X} \vee \Upsilon_{-X} = \Upsilon_{-X}$. Set $\mathcal{O} = \Upsilon_X$. Then $\mathcal{B}_{\mathcal{O}} = \{A \subseteq X \mid A^+ \in \mathcal{O}\}$ and $\mathcal{P}_{\mathcal{O}} = \{A \subseteq X \mid A^- \in \mathcal{O}\}$. We have that $\emptyset^+ = \emptyset^- = \emptyset$, $X^+ = X^- = CL(X)$, $\{0\}^- = \{X\}$, $\{0\}^+ = \emptyset$, $\{1\}^+ = \{\{1\}\}$, $\{1\}^- = CL(X)$, $\{2\}^+ = \emptyset$, $\{2\}^- = \{X, \{1,2\}\}$, $\{0,1\}^+ = \{\{1\}\}$, $\{0,1\}^- = CL(X)$, $\{0,2\}^+ = \emptyset$, $\{0,2\}^- = \{X, \{1,2\}\}$, $\{1,2\}^+ = \{\{1\}, \{1,2\}\}$, $\{1,2\}^- = CL(X)$. Hence $\mathcal{B}_{\mathcal{O}} = \{\emptyset, X, \{0\}, \{2\}, \{0,2\}\}$ and $\mathcal{P}_{\mathcal{O}} = \{\emptyset, X, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$. Thus $\mathcal{T}_{+\mathcal{O}} = \mathcal{B}_{\mathcal{O}}$ and $\mathcal{T}_{-\mathcal{O}} = \mathcal{P}_{\mathcal{O}}$. Therefore $\mathcal{T}_{+\mathcal{O}} \neq \mathcal{T}_{-\mathcal{O}}$. Hence Υ_X is not a strong Vietoris-type topology on $CL(X)$. Note that $\mathcal{T} \neq \mathcal{T}_{+\mathcal{O}}$, $\mathcal{T} \neq \mathcal{T}_{-\mathcal{O}}$ and $\mathcal{T} \neq \mathcal{T}_{\mathcal{O}}$. Since $\mathcal{B}_{\mathcal{O}} \subseteq \mathcal{P}_{\mathcal{O}}$, we get that $\mathcal{T}_{+\mathcal{O}} \subseteq \mathcal{T}_{-\mathcal{O}}$ and thus $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{-\mathcal{O}}$. □

Example 2.11. There exists a T_0-space $(X, \mathcal{T})$ such that $\mathcal{O} = \Upsilon_X$ is not a strong Vietoris-type topology on $CL(X)$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{-\mathcal{O}} \supsetneqq \mathcal{T}_{+\mathcal{O}} = \mathcal{T}$.

P r o o f. Let $X = \{0,1,2\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}, \{0,2\}, \{1,2\}, \{2\}\}$. Then $(X, \mathcal{T})$ is a T_0-space and $CL(X) = \{X, \{1,2\}, \{1\}, \{0\}, \{0,1\}\}$. The topology Υ_{-X} has as a subbase the family
$$\begin{aligned}&\{\emptyset^-, X^-, \{0\}^-, \{0,2\}^-, \{1,2\}^-, \{2\}^-\} = \\ &\{\emptyset, CL(X), \{X, \{0\}, \{0,1\}\}, \{X, \{0\}, \{0,1\}, \{1,2\}\}, \{\{1\}, \{0,1\}, X, \{1,2\}\}, \\ &\{X, \{1,2\}\}\}.\end{aligned}$$
Then $\Upsilon_{-X} = \{\emptyset, CL(X), \{X, \{0\}, \{0,1\}\}, \{X, \{0\}, \{0,1\}, \{1,2\}\}$, $\{X, \{1\}, \{0,1\}, \{1,2\}\}, \{X, \{1,2\}\}, \{X, \{0,1\}\}, \{X\}, \{X, \{0,1\}, \{1,2\}\}\}$. The topology Υ_{+X} has as a base the family
$$\{\emptyset^+, X^+, \{0\}^+, \{0,2\}^+, \{1,2\}^+, \{2\}^+\} = \{\emptyset, CL(X), \{\{0\}\}, \{\{1\}, \{1,2\}\}\}.$$
Hence $\Upsilon_{+X} = \{\emptyset, CL(X), \{\{0\}\}, \{\{1\}, \{1,2\}\}, \{\{0\}, \{1\}, \{1,2\}\}\}$ and $\mathcal{O} = \Upsilon_X = \Upsilon_{-X} \vee \Upsilon_{+X} = \Upsilon_{-X} \cup \Upsilon_{+X} \cup \{\{1,2\}\} \cup \{\{0\}, \{1,2\}\} \cup \{X, \{0\}, \{1,2\}\} \cup \{X, \{0\}\} \cup$

$\{X, \{1\}, \{1,2\}\} \cup \{X, \{0\}, \{1\}, \{1,2\}\}$. We have that $\emptyset^+ = \emptyset^- = \emptyset$, $X^+ = X^- = CL(X)$, $\{0\}^- = \{X, \{0\}, \{0,1\}\}$, $\{0\}^+ = \{\{0\}\}$, $\{1\}^- = \{X, \{1\}, \{0,1\}, \{1,2\}\}$, $\{1\}^+ = \{\{1\}\}$, $\{2\}^- = \{X, \{1,2\}\}$, $\{2\}^+ = \emptyset$, $\{0,1\}^- = CL(X)$, $\{0,1\}^+ = \{\{0\}, \{1\}, \{0,1\}\}$, $\{0,2\}^- = \{X, \{0\}, \{0,1\}, \{1,2\}\}$, $\{0,2\}^+ = \{\{0\}\}$, $\{1,2\}^- = \{X, \{1\}, \{0,1\}, \{1,2\}\}$, $\{1,2\}^+ = \{\{1\}, \{1,2\}\}$. Hence

$$\mathcal{B}_{\mathcal{O}} = \{\emptyset, X, \{0\}, \{2\}, \{0,2\}, \{1,2\}\} = \mathcal{T}_{+\mathcal{O}} = \mathcal{T}$$

and $\mathcal{P}_{\mathcal{O}} = \{\emptyset, X, \{0\}, \{2\}, \{1,2\}, \{0,2\}, \{1\}, \{0,1\}\} = \mathcal{T}_{-\mathcal{O}} \neq \mathcal{T}$. Thus

$$\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{-\mathcal{O}} \supsetneqq \mathcal{T}_{+\mathcal{O}} \equiv \mathcal{T}. \qquad \square$$

Example 2.12. Let $\mathcal{U} = \{(\alpha, \beta) \mid \alpha, \beta \in \overline{\mathbb{R}}\}$ and $\mathbb{R}$ be regarded with its natural topology. Then the topology $\mathcal{O}$ on $CL(\mathbb{R})$ having as a subbase the family $\mathcal{U}^- \cup \mathcal{U}^+$ is a strong Vietoris-type topology on $CL(\mathbb{R})$ different from the Vietoris topology $\Upsilon_{\mathbb{R}}$ on $CL(\mathbb{R})$. Also, $\mathcal{T}_{\mathcal{O}}$ coincides with the natural topology $\mathcal{T}$ on $\mathbb{R}$.

Proof. Clearly, $\mathcal{U} \subseteq \mathcal{B}_{\mathcal{O}} \cap \mathcal{P}_{\mathcal{O}}$ and thus $(\mathcal{B}_{\mathcal{O}})^+ \cup (\mathcal{P}_{\mathcal{O}})^-$ is a subbase for $\mathcal{O}$. Hence $\mathcal{O}$ is a Vietoris-type topology on $CL(\mathbb{R})$. We will show that $\mathcal{T}_{+\mathcal{O}} \equiv \mathcal{T}_{-\mathcal{O}} \equiv \mathcal{T}$. Then, in particular, we will obtain that $\mathcal{O}$ is a strong Vietoris-type topology on $CL(\mathbb{R})$. Note first that $\mathcal{U}^{\cap} = \mathcal{U}$ and $(\mathcal{U}^+)^{\cap} = \mathcal{U}^+$. Fufther, it is not difficult to see (using the fact that $CL(\mathbb{R})$ is a natural family) that $\mathcal{P}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$ (see Fact 1.6 and Fact 1.2 for the notation $\mathcal{P}_{\mathcal{O}}$ and $\mathcal{B}_{\mathcal{O}}$). Since $\mathcal{U}$ is a base for $\mathcal{T}$ and $\mathcal{U} \subseteq \mathcal{B}_{\mathcal{O}} \cap \mathcal{P}_{\mathcal{O}}$, we get that $\mathcal{T}_{+\mathcal{O}} \equiv \mathcal{T}_{-\mathcal{O}} \equiv \mathcal{T}$. So, $\mathcal{O}$ is a strong Vietoris-type topology on $CL(\mathbb{R})$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$.

For showing that $\mathcal{O} \neq \Upsilon_{\mathbb{R}}$, we will prove that $((0,1) \cup (1,2))^+ \notin \mathcal{O}$. Let $F = \{\frac{1}{2}, \frac{3}{2}\}$. Then $F \in ((0,1) \cup (1,2))^+$. We will show that if $n \in \mathbb{N}$, $U_1, \ldots, U_n, V \in \mathcal{U}$ and $F \in V^+ \cap \bigcap_{i=1}^{n} U_i^- = O$, then $O \not\subseteq ((0,1) \cup (1,2))^+$. Indeed, there exist $\alpha, \beta \in \overline{\mathbb{R}}$ such that $\alpha < \beta$ and $V = (\alpha, \beta)$. Then $F \subset (\alpha, \beta)$ and $F \cap U_i \neq \emptyset$, for $i = 1, \ldots, n$. Clearly, $\alpha < \frac{1}{2} < 1 < \frac{3}{2} < \beta$. Let $G = [\frac{1}{2}, \frac{3}{2}]$. Then $G \in CL(\mathbb{R})$, $F \subset G$, $G \subset (\alpha, \beta)$ and $G \cap U_i \neq \emptyset$, for all $i = 1, \ldots, n$. Therefore $G \in O$, but $G \notin ((0,1) \cup (1,2))^+$ since $1 \in G$ and $1 \notin (0,1) \cup (1,2)$. So, $((0,1) \cup (1,2))^+ \in \Upsilon_{\mathbb{R}}$ but $((0,1) \cup (1,2))^+ \notin \mathcal{O}$. Hence $\Upsilon_{\mathbb{R}} \neq \mathcal{O}$ (and, clearly, $\mathcal{O} \subset \Upsilon_{\mathbb{R}}$). $\square$

3. Some properties of the hyperspaces with Vietoris-type topologies. In this section, some of the results of E. Michael [13] concerning hyperspaces with Vietoris topology will be extended to analogous results for the hyperspaces with Vietoris-type topology.

Proposition 3.1. *Let* $(X,\mathcal{T})$ *be a space,* $\mathcal{M} \subseteq \mathcal{P}'(X)$*,* $n \in \mathbb{N}$*,* $Fin_n(X) \subseteq \mathcal{M}$*,* $\mathcal{O}$ *be a Vietoris-type topology on* $\mathcal{M}$ *and* $\mathcal{T}_{\mathcal{O}} \subseteq \mathcal{T}$*. Let* $J_n(X)$ *be the subspace of* $(\mathcal{M},\mathcal{O})$ *consisting of all sets of cardinality* $\leq n$*. Then the map* $j_n : X^n \longrightarrow J_n(X)$*, where* $j_n(x_1,\ldots,x_n) = \{x_1,\ldots,x_n\}$*, is continuous.*

Proof. Let $x = (x_1,\ldots,x_n) \in X^n$ and $j_n(x) \in U^+_{\mathcal{M}} \cap \bigcap\limits_{i=1}^{k}(U_i)^-_{\mathcal{M}} = O$, where $U \in \mathcal{B}_{\mathcal{O}}$, $k \in \mathbb{N}$ and $U_i \in \mathcal{P}_{\mathcal{O}}$, $\forall i = 1,\ldots,k$. Then $\{x_1,\ldots,x_n\} \subseteq U$ and $\{x_1,\ldots,x_n\} \cap U_i \neq \emptyset$, $\forall i = 1,\ldots,k$. Hence, $\forall i \in \{1,\ldots,k\}$ $\exists s(i) \in \{1,\ldots,n\}$ such that $x_{s(i)} \in U_i$. Set

$$V = U^n \cap \prod_{t=1}^{n} V_t,\ \text{where } V_t = \begin{cases} U_i, & \text{if } t = s(i) \text{ for some } i \in \{1,\ldots,k\} \\ X, & \text{otherwise} \end{cases}.$$

Then, clearly, $x \in V$ and $j_n(V) \subseteq O$. So, j_n is a continuous function. □

Proposition 3.2. *Let* $(X,\mathcal{T})$ *be a space,* $\mathcal{M} \subseteq \mathcal{P}'(X)$ *be a natural family,* $\mathcal{O}$ *be a Vietoris-type topology on* $\mathcal{M}$ *and* $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$*. Then* $j_1 : (X,\mathcal{T}) \longrightarrow J_1(X)$ *is a homeomorphism.*

Proof. Using Proposition 3.1, we get that j_1 is a continuous bijection. We will prove that j_1^{-1} is continuous. Let $\{x\} \in J_1(X)$, $U \in \mathcal{B}_{\mathcal{O}}$, $k \in \mathbb{N}$, $U_1,\ldots,U_k \in \mathcal{P}_{\mathcal{O}}$ and $x \in V = U \cap \bigcap\limits_{i=1}^{k} U_i$. Then $\{x\} \in O = U^+_{\mathcal{M}} \cap \bigcap\limits_{i=1}^{k}(U_i)^-_{\mathcal{M}}$ and, clearly, $(j_1)^{-1}(O) \subseteq V$. □

Proposition 3.3. *If* $(X,\mathcal{T})$ *is a* T_2*-space,* $\mathcal{M} \subseteq \mathcal{P}'(X)$*,* $\mathcal{M}$ *is a natural family,* $\mathcal{O}$ *is a Vietoris-type topology on* $\mathcal{M}$ *and* $\mathcal{T}_{-\mathcal{O}} \supseteq \mathcal{T}$*, then* $J_1(X)$ *is closed in* $(\mathcal{M},\mathcal{O})$*.*

Proof. Let $M \in \mathcal{M} \setminus J_1(X)$. Then there exist $x, y \in M$ such that $x \neq y$. Since $(X,\mathcal{T})$ is a Hausdorff space, there exist $U_1,\ldots,U_k,V_1,\ldots,V_l \in \mathcal{P}_{\mathcal{O}}$ (where $k,l \in \mathbb{N}$) such that $x \in U = \bigcap\limits_{i=1}^{k} U_i$, $y \in V = \bigcap\limits_{j=1}^{l} V_j$ and $U \cap V = \emptyset$. Set $O = \bigcap\limits_{i=1}^{k}(U_i)^-_{\mathcal{M}} \cap \bigcap\limits_{j=1}^{l}(V_j)^-_{\mathcal{M}}$. Then $M \in O$ and $O \cap J_1(X) = \emptyset$. Hence $J_1(X)$ is a closed subset of $(\mathcal{M},\mathcal{O})$. □

Proposition 3.4. *Let $(X,\mathcal{T})$ be a space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $Fin_2(X) \subseteq \mathcal{M}$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$, $\mathcal{T}_{\mathcal{O}} \subseteq \mathcal{T}$ and $J_1(X)$ be closed in $(\mathcal{M},\mathcal{O})$. Then X is a T_2-space.*

Proof. Let $x_1, x_2 \in X$, $x_1 \neq x_2$. Then $\{x_1,x_2\} \in \mathcal{M}$ and $\{x_1,x_2\} \notin J_1(X)$. Since $J_1(X)$ is closed, $\exists V \in \mathcal{B}_{\mathcal{O}}$ and $U_1,\ldots,U_k \in \mathcal{P}_{\mathcal{O}}$ such that $\{x_1,x_2\} \in V^+_{\mathcal{M}} \cap \bigcap\limits_{i=1}^{n}(U_i)^-_{\mathcal{M}} = O$ and $O \cap J_1(X) = \emptyset$. Then $\{x_1,x_2\} \subset V$ and $\{x_1,x_2\} \cap U_i \neq \emptyset$ $\forall i \in \{1,\ldots,k\}$. Let $\mathcal{U} = \{U_1,\ldots,U_k\}$, $V_1 = \bigcap\{U \in \mathcal{U} \mid x_1 \in U\}$ and $V_2 = \bigcap\{U \in \mathcal{U} \mid x_2 \in U\}$. Then $V_1, V_2 \in \mathcal{T}$. We will show that $x_1 \in V_1$, $x_2 \in V_2$ and $V \cap V_1 \cap V_2 = \emptyset$. Indeed, since $\{x_1\} \notin O$, $\exists U \in \mathcal{U}$ such that $x_1 \notin U$. Then $x_2 \in U$ (because $U \cap \{x_1,x_2\} \neq \emptyset$). Hence $V_2 \neq \emptyset$ and, obviously, $x_2 \in V_2$. Analogously, we get that $x_1 \in V_1$. Suppose that $\exists x \in V \cap V_1 \cap V_2$. Then $x \in U_i$ for every $i = 1,\ldots,k$. (Indeed, for every $U \in \mathcal{U}$, we have that either $x_1 \in U$ or $x_2 \in U$.) Since $x \in V$, we get that $\{x\} \in O \cap J_1(X)$– a contradiction. Hence $x_1 \in V \cap V_1$, $x_2 \in V \cap V_2$ and $(V \cap V_1) \cap (V \cap V_2) = \emptyset$. □

Proposition 3.5. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $Fin(X) \subseteq \mathcal{M}$ and $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$. Then $J(X) = \bigcup\limits_{i \in \mathbb{N}} J_i(X)$ is dense in $(\mathcal{M},\mathcal{O})$.*

Proof. Let $n \in \mathbb{N}$, $U \in \mathcal{B}_{\mathcal{O}} \setminus \{\emptyset\}$, $U_i \in \mathcal{P}_{\mathcal{O}} \setminus \{\emptyset\}$ for $i = 1,\ldots,n$ and $O = U^+_{\mathcal{M}} \cap \bigcap\limits_{i=1}^{n}(U_i)^-_{\mathcal{M}} \neq \emptyset$. Then O is a basic open subset of $(\mathcal{M},\mathcal{O})$ and it is enough to show that $O \cap J(X) \neq \emptyset$. Since $O \neq \emptyset$, we have that $U \cap U_i \neq \emptyset$ for every $i \in \{1,\ldots,n\}$. Indeed, if $M \in O$ then $M \subseteq U$ and $M \cap U_i \neq \emptyset$ $\forall i = 1,\ldots,n$. So, $\forall i = 1,\ldots,n$, $\exists x_i \in U \cap U_i$. Then $\{x_1,\ldots,x_n\} \in J(X) \cap O$. Hence, $J(X)$ is dense in $(\mathcal{M},\mathcal{O})$. □

Definition 3.6. Let X be a set and $\mathcal{P} \subseteq \mathcal{P}(X)$. Set $w(X,\mathcal{P}) = \min\{|\mathcal{P}'| \mid (\mathcal{P}' \subseteq \mathcal{P}) \wedge (\forall U \in \mathcal{P}$ and $\forall x \in U\ \exists V \in \mathcal{P}'$ such that $x \in V \subseteq U)\}$.

Clearly, $w(X,\mathcal{P}) \leq |\mathcal{P}|$; also, when $\mathcal{P}$ is a topology on X, then $w(X,\mathcal{P})$ is just the weight of the topological space $(X,\mathcal{P})$.

Fact 3.7. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{P}$ be a subbase for $(X,\mathcal{T})$. Then:*

(a) the families $\mathcal{P}'$ from Definition 3.6 are also subbases for $(X,\mathcal{T})$;

(b) if $w(X,\mathcal{T}) \geq \aleph_0$ then $w(X,\mathcal{P}) \geq w(X,\mathcal{T})$.

Remark 3.8. Let X be a set and $\mathcal{P} \subseteq \mathcal{P}(X)$. Then, clearly, there exists a unique topology $\mathcal{T}(\mathcal{P})$ on X for which $\mathcal{P} \cup \{X\}$ is a subbase. Obviously, if $\bigcup \mathcal{P} = X$ then $\mathcal{P}$ is a subbase for $\mathcal{T}(\mathcal{P})$. Hence, in Definition 3.6 we can always assume that X is a topological space and $\mathcal{P} \cup \{X\}$ is a subbase for X.

In connection with Fact 3.7, note that the following assertion holds (it should be well-known):

Lemma 3.9. *Let X be a space, $w(X) = \tau \geq \aleph_0$ and $\mathcal{P}$ be a subbase for X. Then there exists a $\mathcal{P}' \subseteq \mathcal{P}$, such that $|\mathcal{P}'| = \tau$ and $\mathcal{P}'$ is a subbase for X.*

Proof. Let $\mathcal{B} = \mathcal{P}^{\cap}$. Then there exists a base $\mathcal{B}' \subseteq \mathcal{B}$ for X such that $|\mathcal{B}'| = \tau$. For every element U of $\mathcal{B}'$ fix a finite subfamily $\mathcal{U}(U)$ of $\mathcal{P}$ such that $U = \bigcap \mathcal{U}(U)$. Set $\mathcal{P}' = \{U' \in \mathcal{P} \mid \exists U \in \mathcal{B}' \text{ such that } U' \in \mathcal{U}(U)\}$. Then $\mathcal{P}'$ is a subbase for X (because $\mathcal{B}' \subseteq (\mathcal{P}')^{\cap}$ and $\mathcal{B}'$ is a base for X) and $|\mathcal{P}'| \leq |\mathcal{B}'| \cdot \aleph_0 = |\mathcal{B}'| = \tau$. Since $|(\mathcal{P}')^{\cap}| = |\mathcal{P}'|$ and $(\mathcal{P}')^{\cap}$ is a base for X, we get that $|\mathcal{P}'| \geq \tau$. Hence $|\mathcal{P}'| = \tau$. □

Proposition 3.10. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Let $\tau \geq \aleph_0$, $\mathcal{P}' \subseteq \mathcal{P}(X)$ and $(\mathcal{P}')^-_{\mathcal{M}}$ be a subbase for $\mathcal{O}$. If $w(X, \mathcal{P}') \leq \tau$, then $w(\mathcal{M}, \mathcal{O}) \leq \tau$. In particular, if $\tau \geq \aleph_0$ and $w(X, \mathcal{P}_{\mathcal{O}}) \leq \tau$ then $w(\mathcal{M}, \mathcal{O}) \leq \tau$.*

Proof. If $w(\mathcal{M}, \mathcal{O}) < \aleph_0$ then $w(\mathcal{M}, \mathcal{O}) \leq \tau$. Hence, we can suppose that $w(\mathcal{M}, \mathcal{O}) \geq \aleph_0$. Let $\mathcal{P} \subseteq \mathcal{P}'$, $|\mathcal{P}| \leq \tau$ and for every $U \in \mathcal{P}'$ and for every $x \in U$ there exists a $V \in \mathcal{P}$, such that $x \in V \subseteq U$. We will prove that $(\mathcal{P}^-_{\mathcal{M}})^{\cap}$ is a base for $(\mathcal{M}, \mathcal{O})$. Indeed, let $M \in \mathcal{M}$, $O \in \mathcal{O}$ and $M \in O$. Then there exist $U_1, \ldots, U_n \in \mathcal{P}'$ such that $M \in \bigcap_{i=1}^{n} (U_i)^-_{\mathcal{M}} \subset O$. Hence $M \cap U_i \neq \emptyset$ for every $i = 1, \ldots, n$. Let $x_i \in U_i \cap M$. Then, for every $i = 1, \ldots, n$, there exists a $V_i \in \mathcal{P}$ such that $x_i \in V_i \subseteq U_i$. Hence $M \in \bigcap_{i=1}^{n} (V_i)^-_{\mathcal{M}} \subset \bigcap_{i=1}^{n} (U_i)^-_{\mathcal{M}} \subset O$. So, $(\mathcal{P}^-_{\mathcal{M}})^{\cap}$ is a base for $(\mathcal{M}, \mathcal{O})$ and thus $|(\mathcal{P}^-_{\mathcal{M}})^{\cap}| \geq \aleph_0$. Therefore $|\mathcal{P}^-_{\mathcal{M}}| = |(\mathcal{P}^-_{\mathcal{M}})^{\cap}| \geq \aleph_0$ and we obtain that $w(\mathcal{M}, \mathcal{O}) \leq |(\mathcal{P}^-_{\mathcal{M}})^{\cap}| \leq |\mathcal{P}| \leq \tau$. □

Proposition 3.11. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$. Let $\tau \geq \aleph_0$, $w(X, \mathcal{P}_{\mathcal{O}}) \leq \tau$ and there exists a family $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{O}}$ such that $|\mathcal{B}| \leq \tau$ and $\forall M \in \mathcal{M}$, $\forall U \in \mathcal{B}_{\mathcal{O}}$ with $M \subseteq U$, $\exists V \in \mathcal{B}$ with $M \subseteq V \subseteq U$. Then $w(\mathcal{M}, \mathcal{O}) \leq \tau$.*

P r o o f. Clearly, Proposition 3.10 implies that $w(\mathcal{M}, \mathcal{O}_l) \leq \tau$. We will show that $\mathcal{B}^+_{\mathcal{M}}$ is a base for $(\mathcal{M}, \mathcal{O}_u)$. Indeed, let $M \in \mathcal{M}$, $\mathcal{U} \in \mathcal{O}_u$ and $M \in \mathcal{U}$. Then $\exists U \in \mathcal{B}_{\mathcal{O}}$ such that $M \in U^+_{\mathcal{M}} \subseteq \mathcal{U}$. Thus $M \subseteq U$ and hence $\exists V \in \mathcal{B}$ such that $M \subseteq V \subseteq U$. This implies that $M \in V^+_{\mathcal{M}} \subseteq U^+_{\mathcal{M}}$. Therefore, $M \in V^+_{\mathcal{M}} \subseteq \mathcal{U}$ and $V^+_{\mathcal{M}} \in \mathcal{B}^+_{\mathcal{M}}$. Hence, $\mathcal{B}^+_{\mathcal{M}}$ is a base for $(\mathcal{M}, \mathcal{O}_u)$. Since $|\mathcal{B}^+_{\mathcal{M}}| \leq |\mathcal{B}| \leq \tau$, we get that $w(\mathcal{M}, \mathcal{O}_u) \leq \tau$. Having in mind that $\mathcal{O} = \mathcal{O}_u \vee \mathcal{O}_l$ (see Fact 2.6), we obtain that $w(\mathcal{M}, \mathcal{O}) \leq \tau$. $\square$

The space $(Comp(X), (\Upsilon_X)|_{Comp(X)})$ will be denoted by $\mathcal{Z}(X)$.

Corollary 3.12. ([13]) *Let $(X, \mathcal{T})$ be a T_1-space and $w(X) \geq \aleph_0$. Then $w(\mathcal{Z}(X)) = w(X)$.*

P r o o f. Let $\mathcal{O}$ be the restriction of the Vietoris topology Υ_X on $Comp(X)$. Then $\mathcal{Z}(X) = (Comp(X), \mathcal{O})$. Using Proposition 2.9, we get that $\mathcal{O}$ is a strong Vietoris-type topology on $Comp(X)$ and $\mathcal{T} = \mathcal{T}_{\mathcal{O}}$. Hence $\mathcal{P}_{\mathcal{O}} = \mathcal{T} = \mathcal{B}_{\mathcal{O}}$. Then, clearly, $w(X, \mathcal{P}_{\mathcal{O}}) = w(X)$. Let $\mathcal{B}_0$ be a base for $(X, \mathcal{T})$ such that $|\mathcal{B}_0| = w(X)$. Then $\mathcal{B} = (\mathcal{B}_0)^{\cup}$ satisfies the hypothesis of Proposition 3.11 for $\tau = |\mathcal{B}| = w(X)$. Hence, by Proposition 3.11, $w(\mathcal{Z}(X)) \leq w(X)$. Since X can be embedded in $\mathcal{Z}(X)$ (by Proposition 3.2), we have that $w(X) \leq w(\mathcal{Z}(X))$. Therefore, $w(X) = w(\mathcal{Z}(X))$. $\square$

Proposition 3.13. *Let $(X, \mathcal{T})$ be a space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $Fin(X) \subseteq \mathcal{M}$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. If $d(X) \geq \aleph_0$ then $d(X) \geq d(\mathcal{M}, \mathcal{O})$.*

P r o o f. Let $d(X) = \tau \geq \aleph_0$. Then there exists an $A \subset X$ such that $\overline{A} = X$ and $|A| = \tau$. Hence $|Fin(A)| = \tau$. We will prove that $Fin(A)$ is dense in $(\mathcal{M}, \mathcal{O})$. Indeed, for every $U^+_{\mathcal{M}} \cap \bigcap_{i=1}^{n} (V_i)^-_{\mathcal{M}} = O \neq \emptyset$, where $U \in \mathcal{B}_{\mathcal{O}}$ and $V_i \in \mathcal{P}_{\mathcal{O}}$ for $i = 1, \ldots, n$, there exist $x_i \in A \cap V_i \cap U$. Hence $\{x_i \mid i = 1, \ldots, n\} \in Fin(A) \cap O$. Thus $d(\mathcal{M}, \mathcal{O}) \leq \tau$. Therefore $d(\mathcal{M}, \mathcal{O}) \leq d(X)$. $\square$

Proposition 3.14. *Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. If $\mathcal{M} \subseteq \{X \setminus A \mid A \in \mathcal{P}_{\mathcal{O}}\}$ or $\mathcal{M} \subseteq \mathcal{B}_{\mathcal{O}}$ then $(\mathcal{M}, \mathcal{O})$ is a T_0-space.*

P r o o f. Let $M, M_1 \in \mathcal{M}$ and $M \neq M_1$. Then $M \setminus M_1 \neq \emptyset$ or $M_1 \setminus M \neq \emptyset$. Let $M_1 \setminus M \neq \emptyset$. If $\mathcal{M} \subseteq \{X \setminus A \mid A \in \mathcal{P}_{\mathcal{O}}\}$ then, setting $O = (X \setminus M)^-_{\mathcal{M}}$, we get that $O \in \mathcal{O}$, $M_1 \in O$ and $M \notin O$. If $\mathcal{M} \subseteq \mathcal{B}_{\mathcal{O}}$ then, setting $O' = M^+_{\mathcal{M}}$, we get that $O' \in \mathcal{O}$, $M \in O'$ and $M_1 \notin O'$. We argue analogously if $M \setminus M_1 \neq \emptyset$. $\square$

Corollary 3.15. ([13]) *If $(X, \mathcal{T})$ is a topological space then $(CL(X), \Upsilon_X)$ is a T_0-space.*

Proof. Set $\mathcal{O} = \Upsilon_X$. Then $\mathcal{T} \subseteq \mathcal{P}_{\mathcal{O}}$. Hence $\mathcal{M} = CL(X) \subseteq \{X \setminus A \mid A \in \mathcal{P}_{\mathcal{O}}\}$. Thus, by Proposition 3.14, $(\mathcal{M}, \mathcal{O})$ is a T_0-space. $\square$

Proposition 3.16. *Let* $(X, \mathcal{T})$ *be a topological space,* $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{P}$ *be a subbase for* $(X, \mathcal{T})$ *and* $\mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$. *Let* $\mathcal{M}$ *be a natural family and* $\mathcal{O}$ *be the topology on* $\mathcal{M}$ *having as a subbase the family* $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$. *Then* $\mathcal{O}$ *is a strong Vietoris-type topology on* $\mathcal{M}$, $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$ *and* $(\mathcal{M}, \mathcal{O})$ *is a* T_1*-space.*

Proof. Since $\bigcup \mathcal{P} = X$, we get easily that $\bigcup \mathcal{P}^-_{\mathcal{M}} = \mathcal{M}$. Thus $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$ can serve as a subbase for a topology on $\mathcal{M}$. Clearly, $\mathcal{O}$ is a Vietoris-type topology on $\mathcal{M}$, $\mathcal{P} \subseteq \mathcal{P}_{\mathcal{O}}$ and $\mathcal{P}^{\cap} \subseteq \mathcal{B}_{\mathcal{O}}$. We will show that $\mathcal{P}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$. Indeed, let $A \in \mathcal{B}_{\mathcal{O}}$. Let $x \in A$. Then $\{x\} \in A^+_{\mathcal{M}} \in \mathcal{O}$ and hence $\exists U \in \mathcal{P}^{\cap}$ and $\exists V_1, \ldots, V_n \in \mathcal{P}$ such that $\{x\} \in U^+_{\mathcal{M}} \cap \bigcap_{i=1}^{n} (V_i)^-_{\mathcal{M}} = O \subseteq A^+_{\mathcal{M}}$. Let $V = \bigcap_{i=1}^{n} V_i$. Then $x \in U \cap V \in \mathcal{P}^{\cap} \subseteq \mathcal{T}$. We will show that $U \cap V \subseteq A$. Indeed, let $y \in U \cap V$. Then $\{y\} \in O \subseteq A^+_{\mathcal{M}}$ and thus $y \in A$. Therefore, $x \in U \cap V \subseteq A$ and $U \cap V \in \mathcal{T}$. Hence, $A \in \mathcal{T}$. Analogously, we get that if $A \in \mathcal{P}_{\mathcal{O}}$ then $A \in \mathcal{T}$. Hence $\mathcal{P} \subseteq \mathcal{P}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$. This implies that $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$.

Since we also have that $\mathcal{P} \subseteq \mathcal{P}_{\mathcal{O}} \subseteq \mathcal{T}$ and $\mathcal{P} \subseteq \mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$, we get that $\mathcal{T}_{-\mathcal{O}} = \mathcal{T} = \mathcal{T}_{+\mathcal{O}}$. Thus $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$.

Let $M \in \mathcal{M}$. We will prove that $\overline{\{M\}}^{\mathcal{O}} = \{M\}$. Indeed, let $M_1 \in \mathcal{M}$ and $M_1 \neq M$. Then $M_1 \setminus M \neq \emptyset$ or $M \setminus M_1 \neq \emptyset$. Let $M_1 \setminus M \neq \emptyset$. Then $U = X \setminus M \in \mathcal{P}$, $M_1 \in U^-_{\mathcal{M}}$ and $M \notin U^-_{\mathcal{M}}$. Hence $M_1 \notin \overline{\{M\}}^{\mathcal{O}}$. Let now $M \setminus M_1 \neq \emptyset$. Let $x \in M \setminus M_1$ and $U = X \setminus \{x\}$. Then $U \in \mathcal{P}$, $M_1 \in U^+_{\mathcal{M}}$ and $M \notin U^+_{\mathcal{M}}$. Hence $M_1 \notin \overline{\{M\}}^{\mathcal{O}}$. Therefore $\overline{\{M\}}^{\mathcal{O}} = \{M\}$. So, $(\mathcal{M}, \mathcal{O})$ is a T_1-space. $\square$

Remark 3.17. The above proof shows that the condition "$\mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$" in Proposition 3.16 can be replaced by the following one:

(*) if $M, M' \in \mathcal{M}$ and $M \setminus M' \neq \emptyset$ then $\exists U, V \in \mathcal{P}$ such that $M' \subseteq U$, $M \not\subseteq U$, $M \cap V \neq \emptyset$ and $M' \cap V = \emptyset$.

Note that condition (*) is fulfilled if the following condition holds:

(**) if $M \in \mathcal{M}$ and $x \notin M$ then $\exists U, V \in \mathcal{P}$ such that $M \subseteq U \subseteq X \setminus \{x\}$ and $x \in V \subseteq X \setminus M$.

Corollary 3.18. ([13]) *Let* $(X, \mathcal{T})$ *be a* T_1*-space. Then* $(CL(X), \Upsilon_X)$ *is a* T_1*-space.*

Proof. Set $\mathcal{P} = \mathcal{T}$ and $\mathcal{M} = CL(X)$ in Proposition 3.16. $\square$

Definition 3.19. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{P}$ be a subbase for $(X, \mathcal{T})$. $(X, \mathcal{T})$ is said to be* $\mathcal{P}$-regular, *if for every $x \in X$ and for every $U \in \mathcal{P}$ such that $x \in U$, there exist $V, W \in \mathcal{P}$ with $x \in V \subset X \setminus W \subset U$.*

Clearly, a topology space $(X, \mathcal{T})$ is $\mathcal{T}$-regular iff it is regular. Also, every $\mathcal{P}$-regular space is regular.

Example 3.20. Let $(\mathbb{R}, \mathcal{T})$ be the real line with its natural topology and $\mathcal{P} = \{(\alpha, \beta) \setminus F \mid \alpha, \beta \in \overline{\mathbb{R}},\ \alpha < \beta,\ F \subset \mathbb{R},\ |F| < \aleph_0\}$ or $\mathcal{P} = \{(\alpha, \beta) \mid \alpha, \beta \in \overline{\mathbb{R}},\ \alpha < \beta\}$. Then $\mathcal{P}$ is a base for $\mathcal{T}$, $\mathcal{P}^{\cap} = \mathcal{P}$, $(\mathbb{R}, \mathcal{T})$ is regular but it is not $\mathcal{P}$-regular.

Proposition 3.21. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{P}$ be a subbase for $(X, \mathcal{T})$ and $\mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$. Let $\mathcal{M}$ be a natural family, X be $\mathcal{P}$-regular and $\mathcal{O}$ be the topology on $\mathcal{M}$ having as a subbase the family $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$. Then $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$, $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$ and $(\mathcal{M}, \mathcal{O})$ is a T_2-space.*

Proof. From Proposition 3.16, we get that $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. Let us prove that $(\mathcal{M}, \mathcal{O})$ is a T_2-space. Let $M_1 \neq M_2$. Then $M_1 \setminus M_2 \neq \emptyset$ or $M_2 \setminus M_1 \neq \emptyset$. Let, for example, $M_1 \setminus M_2 \neq \emptyset$. Let us fix $x \in M_1 \setminus M_2$. Then $U = X \setminus M_2 \in \mathcal{P}$ and $x \in U$. Hence, there exist $V, W \in \mathcal{P}$, such that $x \in V \subset X \setminus W \subset U$. Then $V \cap W = \emptyset$ and $W \supset X \setminus U = M_2$. Therefore $M_1 \in V^-_{\mathcal{M}}$, $M_2 \in W^+_{\mathcal{M}}$ and $V^-_{\mathcal{M}} \cap W^+_{\mathcal{M}} = \emptyset$. If $M_2 \setminus M_1 \neq \emptyset$ then we argue analogously. $\square$

Proposition 3.22. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{P}$ be a base for $(X, \mathcal{T})$, $\mathcal{M} = \{X \setminus U \mid U \in \mathcal{P}\}$ and $\mathcal{P} = \mathcal{P}^{\cap}$. Let $\mathcal{M}$ be a natural family, $\mathcal{O}$ be the topology on $\mathcal{M}$ having as a subbase the family $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$ and $(\mathcal{M}, \mathcal{O})$ be a T_2-space. Then X is $\mathcal{P}$-regular.*

Proof. Note that $Fin(X) \subseteq \mathcal{M}$ and $\mathcal{M} = \mathcal{M}^{\cup}$. Let now $U \in \mathcal{P}$ and $x \in U$. Then $\{x\} \in \mathcal{M}$, $X \setminus U \in \mathcal{M}$, $\{x\} \cup (X \setminus U) \in \mathcal{M}$ and $\{x\} \cup (X \setminus U) \neq X \setminus U$. Since $(\mathcal{M}, \mathcal{O})$ is a T_2-space, there exist $V, W, V_1, \dots, V_n, W_1, \dots, W_k \in \mathcal{P}$ such that

$$\{x\} \cup (X \setminus U) \in V^+_{\mathcal{M}} \cap \bigcap_{i=1}^{n} (V_i)^-_{\mathcal{M}} = O,\ X \setminus U \in W^+_{\mathcal{M}} \cap \bigcap_{j=1}^{k} (W_j)^-_{\mathcal{M}} = O' \text{ and } O \cap O' = \emptyset.$$

Then $\{x\} \cup (X \setminus U) \subseteq V$, $(\{x\} \cup (X \setminus U)) \cap V_i \neq \emptyset$ for $i = 1, \dots, n$, $X \setminus U \subseteq W$ and $(X \setminus U) \cap W_j \neq \emptyset$ for $j = 1, \dots, k$. Fix $x_j \in (X \setminus U) \cap W_j$ for every $j = 1, \dots, k$. Set $F = \{x_j \mid j = 1, \dots, k\}$. If $(X \setminus U) \cap V_i \neq \emptyset$ for $i = 1, \dots, n$ then for each $i \in \{1, \dots, n\}$ there exists $y_i \in (X \setminus U) \cap V_i$. Then $F \cup \{y_i \mid i = 1, \dots, n\} \in O \cap O'$, a contradiction. Hence, there exists $i_0 \in \{1, \dots, n\}$ such that $(X \setminus U) \cap V_{i_0} = \emptyset$. Then $x \in V_{i_0}$. Let $G = \{i \in \{1, \dots, n\} \mid (X \setminus U) \cap V_i = \emptyset\}$. Then $G \neq \emptyset$

and $x \in \bigcap_{i\in G} V_i = V'$. Let $H = \{1,\dots,n\} \setminus G$. If $H \neq \emptyset$ then, for every $i \in H$, fix $z_i \in (X \setminus U) \cap V_i$ and set $F' = \{z_i \mid i \in H\}$. If $H = \emptyset$ then set $F' = \emptyset$. We will show that $V \cap V' \cap W = \emptyset$. Indeed, suppose that there exists $y \in V \cap V' \cap W$ and set $F'' = F \cup F' \cup \{y\}$. Then $F'' \in O \cap O'$, a contradiction. Hence $V'' = V \cap V' \subseteq X \setminus W$. Therefore, we obtain that $V'' \in \mathcal{P}$ and $x \in V'' \subseteq X \setminus W \subseteq U$. Thus, X is $\mathcal{P}$-regular. $\square$

Remark 3.23. The above proof shows that the conditions "$\mathcal{M} = \{X \setminus U \mid U \in \mathcal{P}\}$" and "$\mathcal{M}$ is a natural family" in Proposition 3.22 can be replaced by "$\mathcal{M} \supseteq \{X \setminus U \mid U \in \mathcal{P}\}$, $FinX \subseteq \mathcal{M}$ and if $x \in U \in \mathcal{P}$ then $\{x\} \cup (X \setminus U) \in \mathcal{M}$".

Corollary 3.24. ([13]) *Let $(X, \mathcal{T})$ be a T_1-space. Then $(X, \mathcal{T})$ is a T_3-space iff $(CL(X), \Upsilon_X)$ is a T_2-space.*

Proof. Set $\mathcal{P} = \mathcal{T}$ and $\mathcal{M} = CL(X)$ in Propositions 3.21 and 3.22. $\square$

Proposition 3.25. *Let $(X, \mathcal{T})$ be a compact T_1-space, $\mathcal{O}$ be a Vietoris-type topology on $CL(X, \mathcal{T})$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. Then $(CL(X, \mathcal{T}), \mathcal{O})$ is a compact space.*

Proof. Clearly, the identity map $i : (CL(X, \mathcal{T}), \Upsilon_X) \longrightarrow (CL(X, \mathcal{T}), \mathcal{O})$ is continuous. By a theorem of E. Michael [13, Theorem 4.2], the space $(CL(X, \mathcal{T}), \Upsilon_X)$ is compact. Hence $(CL(X, \mathcal{T}), \mathcal{O})$ is compact. $\square$

4. Subspaces and hyperspaces.

In this section, we will regard the problem of continuity or inverse continuity of the maps of the form $i_{A,X}$ (see Proposition 4.3 below for the definition of the maps $i_{A,X}$) for the hyperspaces with a srong Vietoris-type topology. This problem was regarded by H.-J.Schmidt [14] for the lower Vietoris topology, by G. Dimov [6, 7] for the (upper) Vietoris topology and by Barov- Dimov- Nedev [2, 3] for the upper Vietoris topology.

We will need the following result from [11]:

Proposition 4.1. ([11]) *Let $(X, \mathcal{T})$ be a space, $\mathcal{P}$ be a subbase for $\mathcal{T}$, $X \in \mathcal{P}$ and $\mathcal{P}^-$ be a subbase for a topology $\mathcal{O}$ on $CL(X)$. Let A be a subspace of X. Set $\mathcal{P}_A = \{U \cap A \mid U \in \mathcal{P}\}$. Let $\mathcal{O}_-^A$ be the topology on $CL(A)$ having $(\mathcal{P}_A)^-_{CL(A)}$ as a subbase. Then $i_{A,X,-} : (CL(A), \mathcal{O}_-^A) \longrightarrow (CL(X), \mathcal{O})$, where $i_{A,X,-}(F) = \overline{F}^X$, is a homeomorphic embedding.*

The next assertion is trivial.

Proposition 4.2. *Let A and X be sets, and $f : A \longrightarrow X$ be a function. Let, for $i = 1, 2$, $\mathcal{T}_i$ (resp., $\mathcal{O}_i$) be a topology on A (resp., X). Let the maps*

$f : (A, \mathcal{T}_1) \longrightarrow (X, \mathcal{O}_1)$ and $f : (A, \mathcal{T}_2) \longrightarrow (X, \mathcal{O}_2)$ be continuous. Then $f : (A, \mathcal{T}_1 \vee \mathcal{T}_2) \longrightarrow (X, \mathcal{O}_1 \vee \mathcal{O}_2)$ is a continuous map.

Proposition 4.3. *Let $(X, \mathcal{T})$ be a T_1-space, $\mathcal{M} = CL(X, \mathcal{T})$, $\mathcal{O}$ be a strong Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. Let A be a subspace of X, $\mathcal{M}_A = CL(A)$, $\mathcal{O}^A_-$ be the topology on $\mathcal{M}_A$ having as a subbase the family $(\mathcal{P}^A_{\mathcal{O}})^-_{CL(A)}$, where $\mathcal{P}^A_{\mathcal{O}} = \{U \cap A \mid U \in \mathcal{P}_{\mathcal{O}}\}$, and let $\mathcal{O}^A_+$ be the topology on $\mathcal{M}_A$ having as a base the family $(\mathcal{B}^A_{\mathcal{O}})^+_{CL(A)}$, where $\mathcal{B}^A_{\mathcal{O}} = \{U \cap A \mid U \in \mathcal{B}_{\mathcal{O}}\}$. Then:*

(a) $\mathcal{O}^A = \mathcal{O}^A_- \vee \mathcal{O}^A_+$ is a strong Vietoris-type topology on $\mathcal{M}_A$, and

(b) the map $i_{A,X} : (\mathcal{M}_A, \mathcal{O}^A) \longrightarrow (\mathcal{M}, \mathcal{O})$, where $i_{A,X}(F) = \overline{F}^X$ for every $F \in \mathcal{M}_A$, is continuous (resp., inversely continuous) if and only if the map $i_{A,X,+} : (\mathcal{M}_A, \mathcal{O}^A_+) \longrightarrow (\mathcal{M}, \mathcal{O}_u)$ is continuous (resp., inversely continuous) (here $i_{A,X,+}(F) = \overline{F}^X$ for every $F \in CL(A)$).

Proof. (a) Since $\mathcal{T}_{+\mathcal{O}} \equiv \mathcal{T}_{-\mathcal{O}} \equiv \mathcal{T}_{\mathcal{O}} \equiv \mathcal{T}$, we get that $\mathcal{P}_{\mathcal{O}}$ is a subbase for $\mathcal{T}$ and $\mathcal{B}_{\mathcal{O}}$ is a base for $\mathcal{T}$. Also, $X \in \mathcal{P}_{\mathcal{O}} \cap \mathcal{B}_{\mathcal{O}}$. Then we easily obtain that $\mathcal{T}_{+\mathcal{O}^A} = \mathcal{T}_{-\mathcal{O}^A} = \mathcal{T}_A$, where $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$. Hence $\mathcal{O}^A$ is a strong Vietoris-type topology on $\mathcal{M}_A$ and $\mathcal{T}_{\mathcal{O}^A} = \mathcal{T}_A$.

(b) Clearly, the map $i_{A,X}$ is an injection.

($\Leftarrow$) It follows from Propositions 4.1 and 4.2.

($\Rightarrow$) Let $i_{A,X} : (\mathcal{M}_A, \mathcal{O}^A) \longrightarrow (\mathcal{M}, \mathcal{O})$ be continuous. We will prove that $i_{A,X,+} : (\mathcal{M}_A, \mathcal{O}^A_+) \longrightarrow (\mathcal{M}, \mathcal{O}_u)$ is continuous. Let $U \in \mathcal{B}_{\mathcal{O}}$, $F \in \mathcal{M}_A$ and $\overline{F}^X \subset U$. Then there exist $V \in \mathcal{B}^A_{\mathcal{O}}$ and $V_1, \ldots, V_n \in \mathcal{P}^A_{\mathcal{O}}$ such that $F \in V^+_{\mathcal{M}_A} \cap \bigcap_{i=1}^{n} (V_i)^-_{\mathcal{M}_A} = O$ and $i_{A,X}(O) \subseteq U^+_{\mathcal{M}}$. Thus $F \cap V_i \neq \emptyset$, $\forall i = 1, \ldots, n$ and $F \subseteq V$. Let $a_i \in F \cap V_i$, for $i = 1, \ldots, n$. We will show that $i_{A,X,+}(V^+_{\mathcal{M}_A}) \subset U^+_{\mathcal{M}}$. Indeed, let $G \in \mathcal{M}_A$ and $G \subset V$. Set $G' = G \cup \{a_i \mid i = 1, \ldots, n\}$. Since X is a T_1-space, we get that $G' \in \mathcal{M}_A$. Obviously, $G' \in O$. Hence $\overline{G'}^X \subset U$. Thus $\overline{G}^X \subset U$. So, $i_{A,X,+}(V^+_{\mathcal{M}_A}) \subseteq U^+_{\mathcal{M}}$. Therefore, the map $i_{A,X,+}$ is continuous.

Let $i_{A,X}$ be inversely continuous. We will show that the map $i_{A,X,+}$ is inversely continuous. Let $U \in \mathcal{B}^A_{\mathcal{O}}$, $F \in \mathcal{M}_A$ and $F \subset U$. Then there exist $V \in \mathcal{B}_{\mathcal{O}}$ and $V_1, \ldots, V_n \in \mathcal{P}_{\mathcal{O}}$ such that $\overline{F}^X \in V^+ \cap \bigcap_{i=1}^{n} V_i^- = O$ and $(i_{A,X})^{-1}(O) \subseteq U^+_{\mathcal{M}_A}$. Thus $\overline{F}^X \subset V$ and $\overline{F}^X \cap V_i \neq \emptyset$, $\forall i = 1, \ldots, n$. Obviously, $\forall i = 1, \ldots, n$, there exists $a_i \in F \cap V_i$. We will show that $(i_{A,X,+})^{-1}(V^+) \subseteq U^+_{\mathcal{M}_A}$. Indeed, let $G \in \mathcal{M}_A$ and $\overline{G}^X \subset V$. Then $G' = G \cup \{a_i \mid i = 1, \ldots, n\} \in \mathcal{M}_A$ and $\overline{G'}^X \in O$. Hence

$G' \subset U$. Thus $G \subset U$. So, $(i_{A,X,+})^{-1}(V^+) \subseteq U^+_{\mathcal{M}_A}$. Therefore, the map $i_{A,X,+}$ is inversely continuous. $\square$

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