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EQUIVARIANT ABSOLUTE EXTENSORS FOR FREE ACTIONS OF COMPACT GROUPS

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Dedicated to the memory of Professor Stoyan Nedev

ABSTRACT. For every compact metrizable group G there is a free universal G-action on the Hilbert space ℓ_2 which makes ℓ_2 a G-equivariant absolute extensor for the class of free G-spaces.

1. Introduction. For a compact Lie group G Milnor constructed a universal G-space EG such that for every free G-action on a topological space X there is a map $f : X/G \to BG = EG/G$ such that X is the pullback of the free G-action on EG [7]. His construction can be modified to assume that EG is homeomorphic to the Hilbert space and his universality result can be stated as follows: *For any compact Lie group G there is a free G-action on the Hilbert space ℓ_2 such that for any free G-action on a compact metric space X there is an equivariant embedding $X \to \ell_2$.*

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In [6] this theorem was extended to all compact metrizable groups G. In the current paper we obtainthat that results as a corollary of the Main Theorem which states that the Hilbert space in our theorem is an equivariant absolute extensor for free G-spaces.

We recall that a space $\mathbb{L}$ with a free action of a group G on it is an *equivariant absolute extensor for free G-spaces* if for any G-equivariant pair $(\mathbb{X}, \mathbb{A})$ of completely regular spaces with free G-action on $\mathbb{X}$ and closed invariant subset $\mathbb{A}$ for any equivariant continuous map $f : \mathbb{A} \to \mathbb{L}$ there is an equivariant continuous extension $\bar{f} : \mathbb{X} \to \mathbb{L}$. We are using the notation $\mathbb{L} \in G\text{-AE}_{free}$ for this condition. When we want to narrow down the class of all completely regular spaces to a subclass $\mathcal{C}$ we write $\mathbb{L} \in G\text{-AE}_{free}(\mathcal{C})$.

2. Preliminaries.

2.1. Compact metrizable groups. It is well known that any compact metrizable topological group G is the inverse limit of an inverse sequence of compact Lie groups

$$G_1 \leftarrow G_2 \leftarrow G_2 \leftarrow \dots$$

with bonding homomorphism ϕ^k_{k-1} (see [9, Theorem 68] or [8, Theorem 2.6] as classic references). Suppose that G acts on a compact metric space X. Using the above, X can be presented as the limit space of the inverse sequence

$$(*) \qquad Y_0 \xleftarrow{q_0^1} Y_1 \xleftarrow{q_1^2} Y_2 \xleftarrow{q_2^3} Y_3 \longleftarrow \dots$$

with $Y_0 = X/G$ and each space Y_k equals the orbit space X/H_k of the action of the subgroup $H_k = \ker\{\phi_k^\infty = \lim\limits_{n\to\infty} \phi_k^{k+n} : G \to G_k\}$. All the bonding maps q_k^{k+1} are the projection to the orbit space of an F_k-action with $F_k = \ker \phi_k^{k+1}$. The compositions

$$q_k^{k+i} = q_k^{k+1} \circ q_{k+1}^{k+2} \cdots \circ q_{k+i-1}^{k+i} : Y_{k+i} \to Y_k$$

are the projections onto the orbit space of an action of the quotient group $F_k^i = \ker \phi_k^{k+i}$. In particular, $F_0^i = G_i$.

2.2. Borel construction. Let a group G act on spaces X and E with the projections onto the orbit spaces $q_X : X \to X/G$ and $q_E : E \to E/G$. Let $q_{X\times E} : X \times E \to X \times_G X = (X \times E)/G$ denote the projection to the orbit space of the diagonal action of G on $X \times E$. Then there is a commutative diagram called the *Borel construction* [4]:

$$\begin{array}{ccccc} X & \xleftarrow{pr_X} & X \times E & \xrightarrow{pr_E} & E \\ {\scriptstyle q_X}\downarrow & & {\scriptstyle q_{X\times E}}\downarrow & & {\scriptstyle q_E}\downarrow \\ X/G & \xleftarrow{p_E} & X \times_G E & \xrightarrow{p_X} & E/G. \end{array}$$

If G is compact and the actions are free, then all projections in the diagram are Hurewicz fibrations. Moreover, if q_E is locally trivial, then so is p_X. The fiber $p_X^{-1}(y)$ is homeomorphic to X/I_z where $I_z = \{g \in G \mid g(z) = z\}$ is the isotropy group of $z \in q_E^{-1}(y)$.

We will refer to G-equivariant maps as to a G-maps.

3. Main Theorem. We denote by $\mathcal{S}$ the class of metrizable separable spaces. We prove our main result for this class though the same proof works for the class of paracompact spaces.

Theorem 3.1. *Let G be a compact metrizable group. Then there exists a free G-space $\mathbb{L}$ homeomorphic to the separable infinite dimensional Hilbert space ℓ_2 such that $\mathbb{L} \in G\text{-AE}_{free}(\mathcal{S})$.*

A G-action on a space $\mathbb{L}$ is called *universal* for a class of free G-spaces $\mathcal{C}$ if for any $X \in \mathcal{C}$ there is an equivariant topological embedding $X \to \mathbb{L}$. Theorem 3.1 implies in particular the main result of [6].

Corollary 3.2. *For every compact metrizable group G there is a free G-action on the Hilbert space $\mathbb{L} \times \ell_2$ which is universal for free G-actions on metric separable spaces.*

P r o o f. Let $\mathbb{X}$ be a separable metric space with a free G-action. By Theorem 3.1 there is G-equivariant map $f : \mathbb{X} \to \mathbb{L}$. This map induces a map of the orbit spaces $\bar{f} : \mathbb{X}/G \to \mathbb{L}/G$. Since the orbit space $\mathbb{X}/G$ is separable metrizable, $\mathbb{X}/G$ admits an topological embedding $j : \mathbb{X}/G \to \ell_2$. Then the map

$\phi : \mathbb{X} \to \mathbb{L} \times \ell_2$ defined as $\phi(x) = (f(x), j[x])$ is an equivariant embedding where $[x] = Gx$ is the orbit of x. $\square$

4. Proof of Main Theorem.

4.1. The case of compact Lie group. Let X be a topologically complete, metric, separable space with at least two points. A function $f : [0,1] \to X$ is measurable if $f^{-1}(U)$ is a Borel subset for every open $U \subset X$. Measurable functions $f, g : [0,1] \to X$ are equivalent if the set $\{t \in [0,1] \mid f(t) \neq g(t)\}$ has measure 0. Let $\mathcal{M}([0,1], X)$ denote the space of equivalence classes of all measurable functions $f : [0,1] \to X$ supplied with the metric

$$\rho(f,g) = \left(\int_0^1 d(f(t), g(t))^2 dt \right)^{1/2}$$

where d is a metric on X. In a general setting, a result of Bessaga and Pełczyński [3] says that $\mathcal{M}([0,1], X) \approx \ell_2$ which we now state.

Theorem 4.1 (Bessaga-Pełczyński)**.** $\mathcal{M}([0,1], X)$ *is homeomorphic to separable infinite dimensional Hilbert space.*

The following proposition is well-known [10]:

Proposition 4.2. *Let $p : E \to B$ be a locally trivial bundle over separable metrizable space with the fiber $F \in AE(\mathcal{S})$ and let $s_0 : A \to E$ be a partial section on a closed subset $A \subset B$. Then p admits a section extending s_0.*

Let G be a compact Lie group. By Theorem 4.1 $\mathcal{M}([0,1], G)$ is a separable infinite dimensional Hilbert space which admits the free G-action $(g \cdot f)(t) = gf(t), t \in I, g \in G$. The fact that $\mathcal{M}([0,1], G) \in G\text{-AE}_{free}$ easily follows from Theorem 4.3.

Theorem 4.3. *Let G be a compact Lie group. A free metric G-space $\mathbb{E}$ is G-$\mathrm{AE}_{free}(\mathcal{S})$ if and only if $\mathbb{E}$ is an* $\mathrm{AE}(\mathcal{S})$*-space.*

P r o o f o f T h e o r e m 4.3. Let G act freely on a metric space $\mathbb{X}$ and let

$\mathbb{X} \hookleftarrow \mathbb{A} \stackrel{\varphi}{\to} \mathbb{E}$ be a partial G-map. Consider the Borel construction

$$\begin{array}{ccccc} \mathbb{X} & \xleftarrow{pr_{\mathbb{X}}} & \mathbb{X}\times\mathbb{E} & \xrightarrow{pr_{\mathbb{E}}} & \mathbb{E} \\ {\scriptstyle q_{\mathbb{X}}}\downarrow & & {\scriptstyle q_{\mathbb{X}\times\mathbb{E}}}\downarrow & & {\scriptstyle q_{\mathbb{E}}}\downarrow \\ X & \xleftarrow{p_{\mathbb{E}}} & \mathbb{X}\times_G\mathbb{E} & \xrightarrow{p_{\mathbb{X}}} & E \end{array}$$

where we denote by $X = \mathbb{X}/G$, $A = \mathbb{A}/G$, and $E = \mathbb{E}/G$. In view of [5, Theorem 5.4.] $\mathbb{X}$ has a local slice at any point. Hence we derive that $p_{\mathbb{E}}\colon \mathbb{X}\times_G\mathbb{E} \to X$ is a locally trivial bundle. Since the fiber of $p_{\mathbb{E}}$ is homeomorphic to $\mathbb{E} \in \mathrm{AE}(\mathcal{S})$ and the orbit space X is a metric space, Proposition 4.2 implies that there is a section $s : X \to \mathbb{X}\times_G\mathbb{E}$ of $p_{\mathbb{E}}$ extending the partial section $\sigma : A \to \mathbb{X}\times_G\mathbb{E}$ defined by the formula $\sigma([a]) = [(a,\varphi(a))]$ for $a \in \mathbb{A}$. Since the left square in the Borel construction is a pullback diagram, the identity map on $\mathbb{X}$ and the composition $s \circ q_X$ define an equivariant map $i : \mathbb{X} \to \mathbb{X}\times\mathbb{E}$. Then the composition $pr_{\mathbb{E}}\circ i$ defines a G-extension $\hat{\varphi} : \mathbb{X} \to \mathbb{E}$ of φ. Indeed, the pullback space $\mathbb{X}\times\mathbb{E}$ is imbedded into the product $\mathbb{X}\times(\mathbb{X}\times_G\mathbb{E})$ by means of the correspondence $(x,y) \longmapsto x\times([x,y])$. In particular, if $a \in \mathbb{A}$, then $(a,\phi(a)) \longmapsto a\times[a,\phi(a)] = a\times s([a])$. Thus, $i(a) = (a,\phi(a))$ and hence $\hat{\phi}(a) = pr_{\mathbb{E}}(i(a)) = \phi(a)$.

In the other direction, for a partial map $X \supset A \stackrel{\phi}{\to} \mathbb{E}$ we consider the following commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\supset} & A & \xrightarrow{\phi} & \mathbb{E} \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_0}\downarrow & & {\scriptstyle =}\downarrow \\ X\times G & \xleftarrow{\supset} & A\times G & \xrightarrow{f} & \mathbb{E} \end{array}$$

where $i(x) = (x,e)$, e is the unit in G, and $f(a,g) = g\phi(x)$. Clearly, f is equivariant. Since $\mathbb{E}$ is a G-$\mathrm{AE}_{free}(\mathcal{S})$, there is an equivariant extension $\bar{f} : X \to \mathbb{E}$ of f. We define an extension $\bar{\phi}$ of ϕ by the formula $\bar{\phi}(x) = \bar{f}(x,e)$. □

We note that Theorem 4.3 is a generalization of Theorem 4 from [1].

4.2. The general case. It is well-known (see the Preliminaries) that a compact metrizable group G is a closed subgroup of $\prod\{G_n \mid n \in \mathbb{N}\} = H$ where all $G_n, n \in \mathbb{N}$, are compact Lie groups. Consider the separable metric space $\mathcal{M} = \prod\{\mathcal{M}([0,1],G_n) \mid n \in \mathbb{N}\}$ with the product action of H. Note that

$$\mathcal{M} = \prod\{\mathcal{M}/H_n \mid n \in \mathbb{N}\}$$

where $H_n = \prod_{i \neq n} G_i$.

Since $\mathcal{M}([0,1], G_n)$ is a free G_n-space homeomorphic to ℓ_2, $\mathcal{M}$ is a free H-space homeomorphic to ℓ_2. Therefore, it is a free G-space homeomorphic to ℓ_2.

We obtain the Main Theorem (Theorem 3.1) from the following Propositions with $\mathbb{L} = \mathcal{M}$:

Proposition 4.4. $\mathcal{M} \in H\text{-AE}_{free}$.

P r o o f. Let $\mathbb{X} \hookleftarrow \mathbb{A} \overset{\varphi}{\to} \mathcal{M}$ be a partial H-map defined on a free metric H-space $\mathbb{X}$. Since $\mathcal{M} = \prod\{\mathcal{M}([0,1], G_n) \mid n \in \mathbb{N}\}$ is a product It suffices to extend this map followed by the projection for every n. To do this consider the extension problem

$$\mathbb{X}/H_n \hookleftarrow \mathbb{A}/H_n \overset{\varphi_n}{\to} \mathcal{M}/H_n.$$

Note that there is an G_n-equivariant map $\mathcal{M}/H_n \to \mathcal{M}(I, G_n)$ sending the orbit $H_n([f_i]) \to [f_n]$ where $[f_i] \in \mathcal{M}(I, G_i)$. By Theorem 4.3 there is an extension $\overline{\varphi}_n : \mathbb{X}/H_n \to \mathcal{M}(I, G_n)$ of the composition of $\pi_n \circ \varphi$ where

$$\pi_n : \mathcal{M} = \prod\{\mathcal{M}([0,1], G_n) \mid n \in \mathbb{N}\} \to \mathcal{M}(I, G_n)$$

is projection onto the factor. $\square$

Proposition 4.5. *Let $\mathbb{X}$ be a free metrizable G-space for a compact group G which is a subgroup of a metrizable group H. Then $H \times_G \mathbb{X}$ is a free metrizable H-space.*

P r o o f. We consider the G-action $G \times H \to H$ on H given by the formula $g \times h \to hg^{-1}$. We define an H-action on $H \times_G \mathbb{X}$ as follows: $\gamma G(h, x) = G(\gamma h, x)$ where $\gamma \in H$ and $G(h, x) \in H \times_G \mathbb{X}$ is the orbit of $(h, x) \in H \times \mathbb{X}$. The acction is well-defined in view of the equality $\gamma G(hg^{-1}, gx) = G(\gamma hg^{-1}, gx) = G(\gamma h, x) = \gamma G(h, x)$. $\square$

Proposition 4.6. $\mathcal{M} \in G\text{-AE}_{free}$.

P r o o f. Let $\mathbb{X} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathcal{M}$ be a partial G-map. We consider the partial H-map

$$H \times_G \mathbb{X} \hookleftarrow H \times_G \mathbb{A} \xrightarrow{\Phi} H \times_G \mathcal{M},$$

defined as $\Phi[h,a] = [h, \varphi(a)]$. It is well-defined in view of the equality $\Phi([hg^{-1}, ga] = [hg^{-1}, \varphi(g(a))] = g[h, \varphi(a)]$ for all $g \in G$. Note that Φ is H-equivariant: $\Phi(\gamma[h,a]) = \Phi([\gamma h, a]) = [\gamma h, \varphi(a)] = \gamma\Phi([h,a])$.

Note that the map $f\colon H \times_G \mathcal{M} \to \mathcal{M}$ defined by the formula $f([h,m]) = hm$ is well-defined: $f([hg^{-1}, m]) = hg^{-1}gm = hm$. It is an H-map: $f(\gamma[h,m]) = \gamma hm = \gamma f([h,m])$. Since $\mathcal{M} \in H\text{-AE}_{free}(\mathcal{S})$ (see Proposition 4.4), there exists an H-extension $\hat{\Phi} : H \times_G \mathbb{X} \to \mathcal{M}$ of $f \circ \Phi$.

Consider the restriction of $\hat{\Phi}$ to $\mathbb{X} = q(e \times \mathbb{X}) = G \times_G \mathbb{X} \subset H \times_G \mathbb{X}$ where $q : H \times \mathbb{X} \to H \times_G \mathbb{X}$ is the projection to the orbit space and $e \in G$ is the unit. Clearly, $\hat{\Phi}$ is a G-equivariant map. Its restriction to $\mathbb{A} = G \times_G \mathbb{A} \subset H \times_G \mathbb{X}$ coinsides with φ: $\hat{\Phi}([e,a]) = f\Phi([e,a]) = f([e, \varphi(a)]) = e\varphi(a) = \varphi(a)$. $\square$

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