Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **44** (2018), 243–302

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ROBUST OPTIMIZATION: STABILIZATION METHODS AND WELL-POSEDNESS IN MATHEMATICAL PROGRAMMING AND SADDLE POINT PROBLEMS

Driss Mentagui

Communicated by A. L. Dontchev

Dedicated to the Memory of Andreï Nikolaïevitch Tikhonov and Jacques Salomon Hadamard, the Pionners of the Stabilization Theory and Well-Posedness in Optimization, Mathematical Physics and Nonlinear Analysis

Abstract. In this paper, we provide various characterizations of several well-posedness concepts in mathematical programming and saddle point problems. We introduce a large class of generalized stabilization methods and display variational asymptotic developments of minimum and saddle values of regularization schemes under consideration. The convex and non-convex cases are studied. A class of well-posed problems has been also studied using infimal-convolution, epigraphical analysis and subdifferentiability. Many examples and applications illustrated our investigation. Notably an application to Legendre-Fenchel transform in locally convex spaces is given. A detailed study of Levitin-Polyak well-posedness in mathematical programming as well as the one for saddle point problems have been displayed in metric and normed spaces.

2010 *Mathematics Subject Classification:* 49K40, 49J45, 49L25, 49J52, 90C26, 90C31.

Key words: generalized stabilization methods, well-posed optimization problems, variational asymptotic developments, conjugacy, stability, α-convexity, epi-convergence, convex-concave functions, subdifferentiability, infimal-convolution, well-posed saddle point problems and variational sets, saddle functions.

1. Introduction. Ill-posed problems cover a large field in pure and applied mathematics [6, 36, 72, 73]. They are notably encountered in optimization, variational analysis and mathematical physics [21, 30, 31]. The central feature of these problems we regularize is their instability. This means that the solution fails to be unique or most importantly, small changes in data of the model, which are closely related to the errors of experimental measurements or unexpected phenomena, could lead to uncontrollable errors. In other words, the gap between the solutions (if any) of the perturbed model and the ones of the original problem may be very large relatively to a specified metric; accordingly meaningless interpretations may occur in the course of physical or economical investigations or other fields of experimental sciences. A natural idea is to replace the initial problem by a sequence of well-posed problems guaranteeing robustness and stability of their solutions and providing a large choice of numerical methods for approximating them. Roughly speaking, a model is said to be robust if its solutions and performance results remain relatively unchanged when exposed to perturbations, random phenomena and uncertainties (For instance, see [76] and references therein).

On the other hand the notion of well-posedness in optimization is strongly related to the regularization methods considered as a logistic support for the theory of small parameter and asymptotic analysis. They play a crucial role in the stabilization and approximation of the solutions of a wide class of problems in pure and applied mathematics. Well-posedness has several definitions and characterizations in the literature [9, 10, 11, 21, 79, 80]. The concept of regularization or stabilization goes back to the works of Tikhonov (for instance, see [21, 71, 72, 73] and references therein) and has considerable applications as in variational analysis and optimization [2, 18, 45, 57, 66], partial differential equations and optimal control [19, 21, 32, 40, 41, 42, 58], inverse problems [12, 13, 15, 24, 29, 33, 78], plasticity theory [70], calculus of variations [21, 22, 27], variational and hemivariational inequalities [1, 14, 28, 43, 44, 51, 59], fixed point theory and inclusion problems [25, 37, 39, 61, 74], minimax and saddle-value problems [46, 60, 62, 67, 68].

The concepts of stability and instability is also extended to optimization problems. If we deal for instance with unstable minimization problems of the kind

$$(P) : \min_{x \in C} f(x)$$

with nonempty solution sets denoted by $\operatorname{argmin}(f, C)$, (i.e. there exists a sequence $(x_n)_n$ in C without any cluster point such that $f(x_n) \to \min_{x \in C} f(x)$) where $f : X \to]-\infty, +\infty]$ is a proper convex lower semicontinuous function defined on a reflexive Banach space X renormed by an equivalent norm $\|\cdot\|$ making it an E-space [21] and C is a weakly compact and convex; we can stabilize it by a sequence of well-posed problems in the Tikhonov sense

$$(P_n) : \min_{x \in C}(f(x) + \epsilon_n g(x)),$$

where $g : X \to [0, +\infty[$ is any lower semicontinuous uniformly convex function; that is for each $\epsilon_n > 0$, $f + \epsilon_n g$ has a unique minimizer x_n on C and every minimizing sequence of (P_n) converges to x_n in the norm topology [21]. Moreover, if $\epsilon_n \to 0$, $\|x_n - \overline{x}\| \to 0$, where $\overline{x}$ is a solution of (P) satisfying remarkable properties [21, 38] (see also [54, 55] if C is the whole space X and g is a specified function). Consequently, every numerical method generating a minimizing sequence for (P_n) leads to an approximation of x_n and so to $\overline{x}$ for a suitable choice of ϵ_n. In the last example, stability and instability characters may be interpreted in terms of special multifunctions as follows: If we set for each fixed $\epsilon > 0$ the multifunction:

$$\begin{aligned} \alpha \in [0, +\infty[\rightrightarrows R^\epsilon(\alpha) &= \alpha - \operatorname{argmin}(f + \epsilon g, C) \\ &= \{t \in C / f(t) + \epsilon g(t) \leq \min_{x \in C}(f(x) + \epsilon g(x)) + \alpha\} \end{aligned}$$

and

$$R^\epsilon(0) = \operatorname{argmin}(f + \epsilon g, C) = \{x_\epsilon\},$$

we see that $R^\epsilon(0) \subset R^\epsilon(\alpha)$ and R^ϵ is stable at 0, that is, $\forall\, y_\alpha \in R^\epsilon(\alpha), (y_\alpha)_\alpha$ converges to x_ϵ if $\alpha \to 0$. Now, if we consider the multifunction

$$\epsilon \in [0, +\infty[\rightrightarrows D(\epsilon) = \epsilon - \operatorname{argmin}(f, C)$$

and $D(0) = \operatorname{argmin}(f, C)$, we observe that D is unstable at 0 because there exists a minimizing sequence $(z_\epsilon)_\epsilon, z_\epsilon \in D(\epsilon)$ without any subsequence converging to a point in $D(0)$; in other words (P) is not well-posed in the generalized sense of Tikhonov [21]. Other types of well-posedness can be found in the literature as Levitin-Polyak well-posedness, Hadamard well-posedness, etc. [11, 21, 49, 50]. It is worth noting that the class of well-posed minimization problems enjoys many interesting generic properties expressing in general that most problems are well-posed or may be approximated in a certain sense by a sequence of well-posed problems involving specified regularization functions [10, 21, 64]. Also, it should be pointed out that the regularization methods with their diversity and rich properties provide flexible tools for characterizing classes of variational convergences for functions and operators [2, 7, 56] in approximation theory and optimization.

The goal of this paper is to investigate the well- posedness in several senses of a class of optimization problems. First we introduce a new generalized regularization method in optimization and variational analysis in a general Hausdorff topological space. We show that this wide class of regularization functions includes most classical regularizations existing in the literature and more. We prove a central theorem (Theorem 3.2) from which we derive various types of variational asymptotic developments either in the convex and nonconvex case; notably an application is given to the Legendre-Fenchel transform for convex functions defined on a locally convex space. Well-posedness of such regularizations is also investigated when the functions under consideration are convex proper lower semicontinuous and defined on a reflexive Banach space. A stability result involving a class of variational convergences of operators has been also displayed within the framework of variational asymptotic developments. Afterwards, we provide various characterizations of well-posedness in terms of infimal-convolution and subdifferentiability in the sense of convex analysis. Finally we state again a central theorem (Theorem 10.2) concerning the regularizations of saddle functions in the Tikhonov sense and their variational asymptotic developments. Well-posedness of saddle point problems is also studied in metric spaces by introducing a variety of variational sets which will characterize various notions of well-posedness under consideration. Many examples illustrate our investigation.

This paper is organized as follows: In Section 2 we consider some notations and recall some results and definitions. In Section 3 we introduce a generalized stabilization method in a general topological space and prove a central theorem for a large class of minimization problems under suitable hypotheses. Afterwards, we observe through special cases of regularization functions that our assumptions are not restrictive and include most classical regularizations and more. Section 4 is devoted to the study of the stability of variational asymptotic developments by epi-convergence. Indeed, if the initial minimization problem is not easy to deal with and can be approximated in a variational sense by a sequence of simple problems $(P_n)_n$, we apply the regularization technique in Theorem 3.2 to each (P_n) and derive variational asymptotic developments for the last problem; so by a diagonalization method established in [2] and concerning double indexed sequences, we prove uniform asymptotic developments for a subsequence $(P_{n_k})_k$ and deduce the stability of the minimum of sum of functions under consideration without having necessarily the stability of this sum by epi-convergence, even in the nonconvex case. Section 5 provides some applications of the previous results to the convex case. Notably well-posedness of generalized regularizations is studied in reflexive Banach spaces. In Section 6 we give an application to Legendre-Fenchel transform in the convex case. Section 7 is devoted to new characterizations of a class of well-posed problems via some operations involving infimal-convolution and special regularization functions in normed spaces. In Section 8 we provide various characterizations of Levitin-Polyak and strong well-posedness. In Section 9 we give a geometrical interpretation of some notions of well-posedness in terms of epigraphical analysis and subdifferentiability. In Section 10 we introduce the generalized regularizations of saddle functions and state again a fundamental theorem in which we provide some approximation results and variational asymptotic developments for the regularizations of bivariate functions in a Hausdorff topological space. Well-posedness of such regularizations is investigated in Section 11. Section 12 is devoted to the investigation of well-posedness of saddle functions in metric spaces. Several examples illustrate various notions of well-posedness under consideration.

2. Notations and preliminaries. Let X be a nonempty set and $f : X \longrightarrow [-\infty, +\infty]$ be a function. The effective domain of f is the set $\operatorname{Dom} f = \{x \in X \mid f(x) < +\infty\}$ and its epigraph is denoted by $\operatorname{epi} f = \{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \leq \lambda\}$. If $K \subseteq X$ we denote the epigraph of f relatively to K by $\operatorname{epi} f_K = \{(x, \lambda) \in K \times \mathbb{R} \mid f(x) \leq \lambda\}$. We say that f is proper on a nonempty subset C of X if $f(x) > -\infty$ for every $x \in C$ and $f(z)$ is finite for some z in C. The characteristic function δ_C of C is the function defined on X by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$, otherwise. The minimization problem of f on C is denoted by $(f, C) : \min_{x \in C} f(x)$ and the solution set of (f, C) is $S = \operatorname{argmin}(f, C) = \{\overline{x} \in C \mid f(\overline{x}) = \min_{x \in C} f(x)\}$. Along the paper we will suppose always that $\upsilon(f, K) = \inf_{x \in K} f(x)$ is finite whenever we are concerned by a minimization problem (f, K). Set

$$\epsilon - \operatorname{argmin}(f, C) = \{x \in C \mid f(x) \leq \upsilon(f, C) + \epsilon\}$$

which is always a nonempty set. When $C = X$, the last set is denoted simply by $\epsilon - \operatorname{argmin} f$. In the case where $\sup_X f$ is finite we denote by

$$\epsilon - \operatorname{argmax} f = \{x' \in X \mid \sup_{x \in X} f(x) - \epsilon \leq f(x')\}.$$

If (X, d) is a metric space and $\epsilon > 0$, define the sets

$$\begin{aligned} L(f, C, \epsilon) &= \{x \in X \mid d(x, C) \leq \epsilon, f(x) \leq \upsilon(f, C) + \epsilon\}, \\ L'(f, C, \epsilon) &= \{x \in X \mid d(x, C) \leq \epsilon, |f(x) - \upsilon(f, C)| \leq \epsilon\}, \\ C_\epsilon &= \{x \in X \mid d(x, C) \leq \epsilon\}. \end{aligned}$$

The Hausdorff distance [21] between two subsets A, B of X is denoted by $d_H(A, B) = \max(e(A, B), e(B, A))$ where $e(A, B) = \sup_{x \in A} d(x, B)$. We say that the minimization problem (f, C) is well-posed in the sense of Levitin-Polyak, if it has a unique minimizer $x' \in C$ and every sequence $(x_n)_n$ of X verifying $d(x_n, C) \to 0$ and $f(x_n) \to f(x')$ converges to x'. The sequence $(x_n)_n$ is called Levitin-Polyak generalized minimizing sequence (see [21] and references therein). Such sequences can be found in some numerical optimization methods as the exterior penalty technique. (f, C) is called well-posed in the Tikhonov sense if it has a unique minimizer $t' \in C$ and every sequence $(x_n)_n$ of C such that $f(x_n) \to f(t')$ converges to t'. For the interest of these two notions in theoretical and algorithmic optimization see for instance [21] and references therein. We say that (f, C) is strongly well-posed if it has a unique minimizer $z' \in C$ and every sequence $(z_n)_n$ of X satisfying $d(z_n, C) \to 0$ and $\overline{\lim}_n f(z_n) \leq f(z')$ converges to z'. Such sequence is called generalized minimizing sequence [9, 10, 11, 21]. In fact (f, C) may have many minimizers, so we need a generalized definition of well-posedness. Then (f, C) is called well-posed in the Tikhonov generalized sense [21] if $\operatorname{argmin}(f, C)$ is nonempty and every minimizing sequence $(x_n)_n$ of C has a subsequence converging to an element of $\operatorname{argmin}(f, C)$. We say that (f, C) is well-posed in the generalized sense of Levitin-Polyak (resp. well-posed in the strong generalized sense) (see [9, 11, 21] and references therein), if $\operatorname{argmin}(f, C)$ is nonempty and every sequence $(x_n)_n$ of X verifying $d(x_n, C) \to 0$ and $f(x_n) \to \min_C f$ (resp. if $\operatorname{argmin}(f, C)$ is nonempty and every sequence $(z_n)_n$ of X satisfying $d(z_n, C) \to 0$ and $\overline{\lim}_n f(z_n) \leq \min_C f$) has a subsequence converging to an element of $\operatorname{argmin}(f, C)$. A function $c : D \to [0, +\infty[$, $D \subseteq \mathbb{R}$ is called a forcing function [21] if $0 \in D$, $c(0) = 0$ and $a_n \in D$, $c(a_n) \to 0 \Longrightarrow a_n \to 0$. If X is a normed space with norm $\|\cdot\|$ and $d(x, y) = \|x - y\|$ is its associated metric, consider two functions $f : K \to]-\infty, +\infty]$, $g : K' \to]-\infty, +\infty]$ where K, K' are two subsets of X. The function denoted by $f\nabla g$ called the infimal-convolution (or epi-sum) of f and g is defined on $K + K'$ by

$$(f\nabla g)(z) = \inf\{f(x) + g(y) \mid (x, y) \in K \times K' \text{ and } z = x + y\}.$$

If $K = K' = X$, we can also write $(f\nabla g)(z) = \inf_{x\in X}\{f(x) + g(z - x)\}$. This notion of convolution plays a crucial role in optimization and variational analysis (see for instance [3, 35, 51, 52, 54] and references therein). We say that $f\nabla g$ is exact at a point $z \in K + K'$ if there exists $(x, y) \in K \times K'$ such that $z = x + y$ and $(f\nabla g)(z) = f(x) + f(y)$. If (f, K) is well-posed in the Tikhonov sense (or in the Tikhonov generalized sense), we can associate to it a forcing c (in general not unique) such that $f(x) \geq \min_K f + c(d(x, \operatorname{argmin}(f, K)))$ for every $x \in K$ [21, 80].

Let X^* be the topological dual of the normed space X and $B_{X^*}(0, 1)$ be the closed ball of origin 0 and radius 1. For a subset $B \subset X$, $\operatorname{conv}(B)$ denotes the convex generated by B. We say that an element $x' \in X^*$, $x' \neq 0$ exposes strongly x on K if $x \in K$ and for every sequence $(x_n)_n$ of K satisfying

$$x'(x_n) = \langle x_n, x' \rangle \to \sup_{t \in K} \langle t, x' \rangle,$$

then $\|x_n - x\| \to 0$, $n \to +\infty$ [21, 26, 63] i.e $(-x', K)$ is Tikhonov well-posed with solution x. If $f : X \longrightarrow [-\infty, +\infty]$ is a function, the generalized regularization of f of parameters $\lambda > 0$, $p \geq 1$ is the function

$$f_{\lambda,p}(x) = \inf_{u \in X} \left\{ f(u) + \frac{1}{p\lambda} \|x - u\|^p \right\}.$$

In the case where $\lambda > 0$ and $p = 2$, $f_\lambda = f_{\lambda,2}$ is called the Moreau-Yoshida regularization of f of parameter $\lambda > 0$.

If $\lim_{\lambda \to 0^+} \dfrac{f(y + \lambda w) - f(y)}{\lambda}$ exists for some $(y, w) \in X \times X$,

$$f'(y; w) = \lim_{\lambda \to 0^+} \frac{f(y + \lambda w) - f(y)}{\lambda}$$

denotes the directional derivative of f at y along the direction w. If $\alpha : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a function, f is said to be α-convex if the function $x \in X \to (\alpha \circ f)(x) = \alpha(f(x))$ is convex with the conventions considered in [35, p. 326]. The Legendre-Fenchel transform of f is the function f^* defined for every $y \in X^*$ by $f^*(y) = \sup_{x \in X} \{\langle x, y \rangle - f(x)\}$ [22, 35]. The subdifferential of f at a point x in the sense of convex analysis is defined by

$$\partial f(x) = \{x' \in X^* \mid f^*(x') + f(x) = \langle x, x' \rangle\}, \quad T(x) = \partial \left(\frac{1}{2} \| \cdot \|^2 \right)(x).$$

For some crucial properties of $\partial f(x)$ and its role in optimization and variational and convex analysis see [21, 35].

By an application of Hahn-Banach theorem, it is well known that $T(x)$ is nonempty for every $x \in X$ and $T(x) = \{x' \in X^* \mid \|x'\| = \|x\| \text{ and } \langle x, x' \rangle = \|x\|^2\}$. When X is reflexive, by renorming [2, 75] we can assume without loss of generality that $\| \cdot \|_X$ and $\| \cdot \|_{X^*}$ are strictly convex; consequently $T(x) = \{\theta(x)\}$ where $\theta(x)$ denotes the Fréchet derivative of $n(x) = \frac{1}{2}\|x\|^2$. If X is a Hilbert space, it is easy to see that $\theta(x) = x$ if X^* is identified to X.

Definition 2.1 ([2]). *Let X be a topological Hausdorff space. Let $f_n, f : X \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ be a sequence of functions. We say that $(f_n)_n$ epi-converges sequentially to f on X and we write $f_n \xrightarrow{\text{epi-seq}} f$ if:*

(1) $\forall\, x, \forall\, x_n \to x$, $f(x) \leq \underline{\lim}_n f_n(x_n)$;
(2) $\forall\, x, \exists x_n \to x$, *such that* $f_n(x_n) \to f(x)$.

Convergence in this sense has remarkable properties in the literature. One of the crucial properties in a general topological setting is the following stability result:

Theorem 2.2 ([2]). *Assume that* $f_n \xrightarrow{\text{epi-seq}} f$ *and* $(x_n)_n$ *be a sequence in* X *such that* $f_n(x_n) \leq \inf_X f_n + \gamma_n$, $\gamma_n \to 0$ *with* $\gamma_n \geqslant 0$ *and* $\inf_X f_n \in \mathbb{R}$. *Then for every converging subsequence* $(x_{n_k})_k$ *to an element* $\overline{x}$, *we have* $\overline{x} \in \operatorname{argmin}(f, X)$ *and* $f_{n_k}(x_{n_k}) \to \min_X f$ *when* $k \to +\infty$.

Definition 2.3 ([2]). *Let* X *be a metric space and* $(j_n)_n$ *be a sequence of functions from* X *into* $[-\infty, +\infty]$. *The epi-limit inferior of the sequence* $(j_n)_n$ *is the function denoted by* $\mathrm{e}-\underline{\lim}\, j_n(x) = \inf\{\underline{\lim}\, j_k(x_k) : (x_k)_k \mid x_k \to x\}$.

Definition 2.4 ([35]). *Let* X *be a topological Hausdorff space and* $C \subseteq X$. *We say that a function* $f : X \to \overline{\mathbb{R}}$ *is inf-compact on* C *if for every* $\lambda \in \mathbb{R}$ *the set* $L_\lambda = \{x \in C \mid f(x) \leq \lambda\}$ *is compact. If* $C = X$, f *is called inf-compact.*

Now consider two metric spaces (X, d), (Y, d') and $X \times Y$ is the metric space endowed with the product topology associated to the metric

$$d((x, y), (x', y')) = \max(d(x, x'), d'(y, y')).$$

Consider a function $F : X \times Y \to \overline{\mathbb{R}}$ and assume that the following hypotheses hold

(H_1): $\forall\, x \in X$, $\exists y \in Y$ such that $F(x, y) > -\infty$ and

(H_2): $\forall\, y \in Y$, $\exists x \in X$ such that $F(x, y) < +\infty$.

Set $G(x) = \sup_{y \in Y} F(x, y)$, $H(y) = \inf_{x \in X} F(x, y)$. It is clear that $\forall\, x \in X$ $G(x) > -\infty$, $\forall\, y \in Y$ $H(y) < +\infty$ and the function $W(x, y) = G(x) - H(y)$ is well defined on $X \times Y$ with $W \geq 0$. We have

$$\begin{aligned} \epsilon - \operatorname{argmin} W &= \left\{ (x, y) \in X \times Y \mid W(x, y) \leq \inf_{X \times Y} W + \epsilon \right\} \\ &= \left\{ (x, y) \in X \times Y \mid W(x, y) \leq \inf_X \sup_Y F - \sup_Y \inf_X F + \epsilon \right\}. \end{aligned}$$

If $(\overline{x}, \overline{y})$ is a saddle point of F on $X \times Y$, then

$$\begin{aligned} \max_y \min_x F(x, y) &= \min_x F(x, \overline{y}) = \max_y F(\overline{x}, y) \\ &= \min_x \max_y F(x, y) = G(\overline{x}) = H(\overline{y}) = F(\overline{x}, \overline{y}) \end{aligned}$$

are finite.

Definition 2.5 ([16]). *A sequence* $(x_n, y_n)_n$ *in* $X \times Y$ *is called minimaximizing sequence of* F *if* $W(x_n, y_n) \to 0$ *when* $n \to +\infty$.

The last definition is equivalent to the existence of $\epsilon_n \geq 0$, $\epsilon_n \to 0$ such that $F(x_n, y) \leq \epsilon_n + F(x, y_n)$ $\forall (x, y) \in X \times Y$. Note that a function does not always possess a minimaximising sequence [16].

Theorem 2.6 ([16])**.** *The following are equivalent:*
(a) F has a minimaximizing sequence on $X \times Y$;
(b) $\inf_x \sup_y F(x, y) = \sup_y \inf_x F(x, y)$;
(c) $\inf_{(x,y)} W(x, y) = 0$.

Definition 2.7 ([16])**.** *We say that F has well-posed saddle problem on $X \times Y$ or briefly $(F, X \times Y)$ is well-posed if F has a unique saddle point $\overline{z} = (\overline{x}, \overline{y})$ on $X \times Y$ and every minimaximizing sequence of F converges to $\overline{z}$.*

Definition 2.8 ([51])**.** *We say that F has strongly well-posed saddle problem on $X \times Y$ or briefly $(F, X \times Y)$ is strongly well-posed if F has a unique saddle point $\overline{z} = (\overline{x}, \overline{y})$ on $X \times Y$ and every sequence $((x_n, y_n))_n$ such that $F(x_n, y_n) \to F(\overline{z})$ converges to $\overline{z}$.*

Remark 2.9. If $(F, X \times Y)$ is strongly well-posed, then it is well-posed because every minimaximizing sequence $((x_n, y_n))_n$ verifies

$$F(x_n, y_n) \to \sup_{y \in Y} \inf_{x \in X} F(x, y) = \inf_{x \in X} \sup_{y \in Y} F(x, y) \quad \text{(see [16])}.$$

Assume that $\sup_{y \in Y} \inf_{x \in X} F(x, y)$, $\inf_{x \in X} \sup_{y \in Y} F(x, y)$ are finite and the function $W(x, y) = G(x) - H(y)$ is well defined. If $\rho = \inf_X \sup_Y F - \sup_Y \inf_X F$, we have always

$$\begin{aligned}
\epsilon - \operatorname{argmin} W &\subseteq (\epsilon + \rho) - \operatorname{argminmax} F \\
&= \{(x, y) \in X \times Y \mid G(x) - \epsilon - \rho \leq F(x, y) \leq H(y) + \epsilon + \rho\} \\
&\subseteq (2\epsilon + \rho) - \operatorname{argmin} W.
\end{aligned}$$

The set $\epsilon - \operatorname{argminmax} F = \{(x, y) \in X \times Y / G(x) - \epsilon \leq F(x, y) \leq H(y) + \epsilon\}$ is considered in many references as in [62] and references therein. If $\epsilon = 0$ this set is reduced to the set of saddle points of F on $X \times Y$. If $\rho = 0$, it is clear that $\epsilon - \operatorname{argminmax} F$ is nonempty for every $\epsilon > 0$. If $\inf_X \sup_Y F \neq \sup_Y \inf_X F$, it is easy to see that $\epsilon - \operatorname{argminmax} F$ is empty for some $\epsilon_0 > 0$, and then for all $0 \leq \epsilon \leq \epsilon_0$. It is shown in [62] that the following condition holds: if $(x_0, y_0) \in (\epsilon - \operatorname{argmin} G) \times (\epsilon - \operatorname{argmax} H)$ and $\inf_X G - \sup_Y H \leq \epsilon_1$, then $(x_0, y_0) \in (2\epsilon + \epsilon_1) - \operatorname{argminmax} F$; so if $\inf_X \sup_Y F = \sup_Y \inf_X F$, then we find by another mean that $\epsilon - \operatorname{argminmax} F$ is nonempty for every $\epsilon > 0$.

Consider for every $\epsilon > 0$ the following variational sets:

$$\Delta(F, \epsilon) = \{(x, y) \in X \times Y / \sup_{y \in Y} H(y) - \epsilon \leq F(x, y) \leq \inf_{x \in X} G(x) + \epsilon\},$$

$$\Delta_1(F,\epsilon) = \{(x,y) \in X \times Y / \inf_X G - \epsilon \le F(x,y) \le \inf_X G + \epsilon\} \text{ and}$$
$$\Delta_2(F,\epsilon) = \{(x,y) \in X \times Y / \sup_Y H - \epsilon \le F(x,y) \le \sup_Y H + \epsilon\}.$$

If X_1, Y_1 are respectively two subsets of X, Y and $\sup_{y\in Y_1}\inf_{x\in X_1} F(x,y)$, $\inf_{x\in X_1}\sup_{y\in Y_1} F(x,y)$ are finite and the function $Z(x,y) = J(x) - K(y)$ is well defined with $J(x) = \sup_{y\in Y_1} F(x,y)$, $x \in X$ and $K(y) = \inf_{x\in X_1} F(x,y)$, $y \in Y$, we define the sets:

$$\begin{aligned} A(F,\epsilon) &= \{(a,b) \in X \times Y \mid d(a,X_1) \le \epsilon, d'(b,Y_1) \le \epsilon \\ &\qquad \text{and } F(a,y) - \epsilon \le F(x,b) \forall\, (x,y) \in X_1 \times Y_1\}, \\ B(F,\epsilon) &= \{(a,b) \in X \times Y \mid d(a,X_1) \le \epsilon, d'(b,Y_1) \le \epsilon \text{ and } |Z(a,b)| \le \epsilon\} \text{ and} \\ \Delta'(F,\epsilon) &= \{(a,b) \in X \times Y \mid d(a,X_1) \le \epsilon, d'(b,Y_1) \le \epsilon \\ &\qquad \text{and } \sup_{y\in Y_1} K(y) - \epsilon \le F(a,b) \le \inf_{x\in X_1} J(x) + \epsilon\}. \end{aligned}$$

Definition 2.10. *We say that the saddle point problem $(F, X_1 \times Y_1)$ is Levitin-Polyak well-posed if it has a unique saddle point $(\overline{x},\overline{y}) \in X_1 \times Y_1$ and every sequence $(x_n,y_n) \in X \times Y$ such that $d(x_n,X_1) \to 0$, $d'(y_n,Y_1) \to 0$ and $Z(x_n,y_n) = J(x_n) - K(y_n) \to 0$ converges to $(\overline{x},\overline{y})$; that is the minimization problem $(Z, X_1 \times Y_1)$ is Levitin-Polyak well-posed and* $\min_{X_1\times Y_1} Z = Z(\overline{x},\overline{y}) = 0$.

Definition 2.11. *We say that the saddle point problem $(F, X_1 \times Y_1)$ is strongly Levitin-Polyak well-posed if it has a unique saddle point $(\overline{x},\overline{y}) \in X_1 \times Y_1$ and every sequence $(x_n,y_n) \in X \times Y$ such that $d(x_n,X_1) \to 0$, $d'(y_n,Y_1) \to 0$ and $F(x_n,y_n) \to F(\overline{x},\overline{y})$ converges to $(\overline{x},\overline{y})$.*

Definition 2.12 ([21]). *Let A be a bounded subset of a metric space X, the noncompctness degree of A is the Kuratowski number of A, defined by*

$$\alpha(A) = \inf\left\{\epsilon > 0 \;:\; \exists (A_i)_{i=1,2,\ldots,n}\ A \subseteq \bigcup_{i=1}^{n} A_i \quad \textit{and} \quad \operatorname{diam} A_i \le \epsilon\right\}.$$

We can verify easily that $\alpha(A) \le \alpha(B)$ if $A \subseteq B$, and $\alpha(A) = 0$ if and only if A is relatively compact.

3. New generalized regularizations in the Tikhonov sense.

The goal of this section is to introduce a new generalized regularization method in the Tikhonov sense in a general topological Hausdorff space and generalize a result established in reflexive spaces for convex functions [54, 55] and concerns the classical Tikhonov regularization method used in variational analysis and its related topics.

Definition 3.1. *Let X be a nonempty set. For $\epsilon > 0$ a function F_ϵ is called a generalized regularization in the Tikhonov sense of a function $f : X \to [-\infty, +\infty]$ if $F_\epsilon(x) = f(x) + \epsilon g(x) + h_\epsilon(x)$, where $g, h_\epsilon : X \to \mathbb{R}$ are two functions.*

Throughout, unless otherwise stated, X stands for a general Hausdorff topological space, $f : X \to \mathbb{R} \bigcup \{+\infty\}$, $g : X \to \mathbb{R}$ be two lower semicontinuous (lsc) functions with f is proper and C be a nonempty closed subset of X. Along the paper we are concerned by the minimization problem

$$(f, C) : \min_{x \in C} f(x).$$

The solution set $S = \operatorname{argmin}(f, C)$ is assumed to be nonempty and $\min_C f$ is finite. Now consider a sequence $h_k : X \to \mathbb{R}$ of functions such that $r_k = \inf_{x \in C} h_k(x)$ is finite for all k. To (f, C) we associate the following generalized regularization problems

$$(P_k) : \min_{x \in C} F_k(x),$$

where $F_k(x) = f(x) + \epsilon_k g(x) + h_k(x)$, $\epsilon_k > 0$ and we suppose that $\epsilon_k \to 0$ if $k \to +\infty$.

Theorem 3.2. *Assume that the following conditions hold*
(A): $i_k = \inf_C F_k$ *is finite for every k and $(z_k)_k$ is a sequence of C relatively compact satisfying*

$$\frac{F_k(z_k) - i_k}{\epsilon_k} \to 0, \qquad k \to +\infty \tag{3.1}$$

Suppose also that

$$\frac{h_k(s) - r_k}{\epsilon_k} \to 0, \qquad k \to +\infty, \ \forall s \in S \tag{3.2}$$

Then:

(1) *any cluster point $\overline{z} \in C$ of $(z_k)_k$ verifies $\overline{z} \in \operatorname{argmin}(g, S)$.*

(2) *$f(z_k) \to f(\overline{z})$ and $g(z_k) \to g(\overline{z})$ when $k \to +\infty$.*

(3) *there exist sequences $(\delta_k)_k$, $(\delta'_k)_k$, $(\theta_k)_k$, $(\theta'_k)_k$ of scalars converging to 0 such that we have the following asymptotic developments:*

$$\begin{aligned} F_k(z_k) &= \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in C} h_k(x) + \epsilon_k \delta_k \\ &= \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in S} h_k(x) + \epsilon_k \delta'_k \end{aligned}$$

and

$$\begin{aligned}\inf_C F_k &= \min_{x\in C} f(x) + \epsilon_k \min_{x\in S} g(x) + \inf_{x\in C} h_k(x) + \epsilon_k\theta_k \\ &= \min_{x\in C} f(x) + \epsilon_k \min_{x\in S} g(x) + \inf_{x\in S} h_k(x) + \epsilon_k\theta'_k.\end{aligned}$$

Moreover

$$\frac{f(z_k) - \min_C f}{\epsilon_k} \to 0 \text{ when } k \to +\infty.$$

Remark 3.3. In Theorem 3.2, g is not necessarily positive as it is always supposed in the literature. On the other hand, it is clear that we can find always a sequence $(z_k)_k$ in C such that $F_k(z_k) \leq i_k + \epsilon_k^2$ if i_k is finite, so $\lim_{k\to+\infty} \dfrac{F_k(z_k) - i_k}{\epsilon_k} = 0$, but in general $(z_k)_k$ is not relatively compact. If C is compact and h_k is lsc, (P_k) has a solution x_k for every k and if we take $z_k = x_k$, then (A) and (3.1) are straightforward satisfied, so from (3) there exists a sequence $(\alpha_k)_k$ of numbers converging to 0 such that

$$\begin{aligned}F_k(x_k) &= \min_{x\in C}(f(x) + \epsilon_k g(x) + h_k(x)) \\ &= \min_{x\in C} f(x) + \epsilon_k \min_{x\in S} g(x) + \inf_{x\in C} h_k(x) + \epsilon_k\alpha_k.\end{aligned}$$

Proof. Pick $\epsilon > 0$. By (3.1) we have for all k large enough,

$$F_k(z_k) = f(z_k) + \epsilon_k g(z_k) + h_k(z_k) \leq \inf_C F_k + \epsilon_k\epsilon \leq f(s) + \epsilon_k g(s) + h_k(s) + \epsilon_k\epsilon$$

$\forall\, s \in S$. Then,

$$0 \leq f(z_k) - f(s) \leq \epsilon_k(g(s) - g(z_k)) + h_k(s) - r_k + \epsilon_k\epsilon,\ \forall\, s \in S. \tag{3.3}$$

Hence we deduce that

$$g(z_k) \leq g(s) + \frac{h_k(s) - r_k}{\epsilon_k} + \epsilon,\ \forall\, s \in S \tag{3.4}$$

and $\underline{\lim}_k\, g(z_k) \leq \overline{\lim}_k\, g(z_k) \leq g(s) + \epsilon$ by (3.2) for every $\epsilon > 0$ and $s \in S$. So,

$$\underline{\lim}_k\, g(z_k) \leq \overline{\lim}_k\, g(z_k) \leq g(s),\ \forall\, s \in S \tag{3.5}$$

Now by lower semicontinuity of g and relative compactness of $(z_k)_k$ we can find $m \in \mathbb{R}$ such that $m \leq g(z_k)\ \forall\, k$. We derive from (3.5) that $(g(z_k))_k$ is bounded and $f(z_k) \to \min_{x\in C} f(x)$ by (3.2) and (3.3); so by lower semicontinuity of f we get $f(\overline{z}) = \min_{x\in C} f(x)$ for any cluster point $\overline{z} \in C$ of $(z_k)_k$ i.e $\overline{z} \in S$. Since g is lower semicontinuous, (3.4) and (3.2) imply that $\overline{z} \in \operatorname{argmin}(g, S)$. Again by lower semicontinuity of g, boundedness of $(g(z_k))_k$ and (3.5) we may check easily that $(g(z_k))_k$ has the unique cluster point $g(\overline{z}) = \min_{s\in S} g(s)$ to which the sequence converges. This ends the proof of (1) and (2).

Now take $s = \overline{z}$ in (3.3); we have

$$0 \leq \frac{f(z_k) - \min_C f}{\epsilon_k} \leq g(\overline{z}) - g(z_k) + \frac{h_k(\overline{z}) - r_k}{\epsilon_k} + \epsilon,$$

hence $\alpha_k = \dfrac{f(z_k) - \min_C f}{\epsilon_k} \to 0$ when $k \to +\infty$. Keeping in mind that $h_k(z_k) \leq h_k(\overline{z}) + \epsilon_k(g(\overline{z}) - g(z_k)) + \epsilon_k\epsilon$, we derive that

$$0 \leq \frac{h_k(z_k) - r_k}{\epsilon_k} \leq \frac{h_k(\overline{z}) - r_k}{\epsilon_k} + g(\overline{z}) - g(z_k) + \epsilon,$$

and then $\dfrac{h_k(z_k) - r_k}{\epsilon_k} \to 0$ when $k \to +\infty$ by (3.2) and (2). On the other hand,

$$\frac{F_k(z_k) - f(z_k) - r_k}{\epsilon_k} = \frac{\epsilon_k g(z_k) + h_k(z_k) - r_k}{\epsilon_k} = g(z_k) + \frac{h_k(z_k) - r_k}{\epsilon_k}.$$

Set $\vartheta_k = \dfrac{F_k(z_k) - f(z_k) - r_k}{\epsilon_k} - g(\overline{z})$. It is clear that ϑ_k and $\delta_k = \vartheta_k + \alpha_k = \dfrac{F_k(z_k) - f(\overline{z}) - r_k}{\epsilon_k} - g(\overline{z})$ converge to 0 if $k \to +\infty$ and

$$F_k(z_k) = f(\overline{z}) + \epsilon_k g(\overline{z}) + r_k + \epsilon_k\delta_k = \min_{x\in C} f(x) + \epsilon_k \min_{x\in S} g(x) + \inf_{x\in C} h_k(x) + \epsilon_k\delta_k.$$

From (3.2) it is easy to see that $\dfrac{\inf_{s\in S} h_k(s) - r_k}{\epsilon_k} = d_k \to 0$, $k \to +\infty$, so $r_k = \inf_{s\in S} h_k(s) - \epsilon_k d_k$ and

$$F_k(z_k) = \min_{x\in C} f(x) + \epsilon_k \min_{x\in S} g(x) + \inf_S h_k + \epsilon_k\delta'_k$$

with $\delta'_k = \delta_k - d_k$. The asymptotic development in $\inf_C F_k$ is an immediate consequence of the last developments and (3.1) which completes the proof of (3). □

Remark 3.4. Hypothesis (3.2) in the above theorem is not restrictive. Indeed, consider the wide class of functions h_k given by $h_k(x) = \sum_{i=1}^p \beta_{k,i} g_i(x)$ where $g_i : X \to \mathbb{R}$ is any function bounded below by a scalar m_i on C, $\beta_{k,i} \geq 0$ for every $k \in \mathbb{N}$, $1 \leq i \leq p$ and $\dfrac{\beta_{k,i}}{\epsilon_k} \to 0$ when $k \to +\infty$, $\forall\, i$. It is easy to see that $r_k \geq \displaystyle\sum_{i=1}^p \beta_{k,i} m_i$ and $0 \leq \dfrac{h_k(x) - r_k}{\epsilon_k} \leq \displaystyle\sum_{i=1}^p \frac{\beta_{k,i}}{\epsilon_k}(g_i(x) - m_i)$ which goes to 0 when $k \to +\infty$ for every $x \in C$, so (3.2) is satisfied. The regularization functions become $F_k(x) = f(x) + \epsilon_k g(x) + \sum_{i=1}^p \beta_{k,i} g_i(x)$. In particular one may consider the functions of the kind $h_k(x) = \epsilon_k^q h(x)$, $q > 1$ with $h : X \to \mathbb{R}$ is any function such that $\inf_C h > -\infty$. If X is a normed space we can use also the regularization functions

$$F_k(x) = f(x) + \epsilon_k \|x - x_1\|^{p_1} + \sum_{i=2}^{p} \epsilon_k^i \|x - x_i\|^{p_i}$$

where x_i is any given point in X and $p_i \in \mathbb{N}$. More generally, if we take in a general topological space $h_k(x) = \sum_{i=1}^{p} \beta_{k,i} g_{k,i}(x)$ with $g_{k,i} : X \to \mathbb{R}$, $\beta_{k,i} \geq 0$, $\dfrac{\beta_{k,i}}{\epsilon_k} \to 0$ if $k \to +\infty$, $\forall\, i = 1, 2, \ldots, p$ and we assume that there exist scalars $m_{k,i}$ such that $m_{k,i} \leq g_{k,i}(x)$, $\forall\, x \in C$ with $\lim_{k\to+\infty} \sum_{i=1}^{p} \frac{\beta_{k,i}}{\epsilon_k}(g_{k,i}(x) - m_{k,i}) = 0$ for every x in C, then (3.2) is satisfied. In particular, one may consider the useful regularization functions $h_k(x) = \sum_{i=1}^{p} q_i(\epsilon_k) e^{\varphi_i(\epsilon_k)(A_i x - b_i) + \gamma_i}$, where $\varphi_i(\epsilon_k) \geq 0$, $q_i(\epsilon_k) \geq 0$, $\gamma_i \in \mathbb{R}$ for every $i = 1, \ldots, p$ and $k \in \mathbb{N}$ with $\dfrac{q_i(\epsilon_k)}{\epsilon_k} \to 0$ if $k \to \infty$, $b_i \in \mathbb{R}$ and $A_i : X \to \mathbb{R}$ is any lower semicontinuous operator. In this case the constraint set is defined by $C = \{x \in X \mid A_i x - b_i \leq 0,\ \forall\, i\}$ assumed to be nonempty. For the case $q_i(\epsilon_k) = \epsilon_k^2$, $\varphi_i(\epsilon_k) = \epsilon_k^{-2}$, $\gamma_i = 0$ and its importance in optimization, see for instance [18] where the authors study the special regularizations

$$F_k(x) = \langle c, x\rangle + \epsilon_k^2 \sum_{i=1}^{p} e^{\epsilon_k^{-2}(A_i x - b_i)}$$

for solving the linear program $\min\{\langle c, x\rangle : A_i x - b_i \leq 0,\ i = 1, \ldots, p\}$ in finite dimensional setting with A_i is a linear operator. This kind of regularizations combines the interesting properties of the interior barrier method and of the exterior penalty method. It should be pointed out that we cannot always ensure higher order asymptotic developments in Theorem 3.2 even under strong regularity conditions on functions under consideration. For example consider the sequence of regularization functions $F_\epsilon(x) = x^2 + \epsilon x + \epsilon^2 x^2$, $x \in C = \mathbb{R}$ reaching their minimizers at the points $x_\epsilon = \dfrac{-\epsilon}{2}(1+\epsilon^2)^{-1}$. If we write $F_\epsilon(x_\epsilon) = \dfrac{-\epsilon^2}{4}(1+\epsilon^2)^{-1} = \min\limits_{\mathbb{R}} x^2 + \epsilon \min\limits_{S=\{0\}} x + \epsilon^2 \min\limits_{\mathbb{R}} x^2 + \epsilon^2 \varphi(\epsilon)$, then $\varphi(\epsilon) = \dfrac{-1}{4}(1+\epsilon^2)^{-1} \to -\dfrac{1}{4}$ if $\epsilon \to 0$.

Now take another example: Let $F_\epsilon(x) = f(x) + \epsilon x^2 + \epsilon^2 x^3$, $0 < \epsilon < 1$ with $f(x) = x - 1$ if $x \geq 1$, $f(x) = 0$ if $x \in [-1, 1]$ and $f(x) = +\infty$ otherwise. Then $S = \operatorname{argmin} f = [-1, 1]$ and the minimizer of F_ϵ is attained at $x_\epsilon = 0$. If $F_\epsilon(x_\epsilon) = 0 = \min_{x\geq -1} f + \epsilon \min_{x\in S} x^2 + \epsilon^2 \min_S x^3 + \epsilon^2 \varphi(\epsilon)$, then $\varphi(\epsilon) = 1$.

Now we state two corollaries expressing that under suitable hypotheses, a sequence $(z_k)_k$ in C satisfying (3.1) is relatively compact.

Corollary 3.5. *Let F_k be the functions considered in Theorem* 3.2 *such* *that $i_k = \inf_C F_k$ is finite for every k. Let $(z_k)_k$ be a sequence in C such that* (3.1) *is satisfied. Assume also that* (3.2) *holds and the set $\{x \in C \mid g(x) \leq \lambda_0\}$ is compact for some scalar $\lambda_0 > g(s_0)$ and some $s_0 \in S$. Then $(z_k)_k$ is relatively compact.*

P r o o f. From the proof of Theorem 3.2, we see that $\overline{\lim}_k g(z_k) \leq g(s)$ for every $s \in S$. Let $\lambda_0 > g(s_0)$ for some $s_0 \in S$ such that $L_{\lambda_0} = \{x \in C \mid g(x) \leq \lambda_0\}$ is compact. We have $\overline{\lim}_k g(z_k) < \lambda_0$ and $z_k \in L_{\lambda_0}$ for every k large enough, so $(z_k)_k$ is relatively compact and the conclusions of Theorem 3.2 hold. □

Corollary 3.6. *Let F_k be the functions considered in Theorem* 3.2 *such that $i_k = \inf_C F_k$ is finite for every k and* (3.1) *holds with $\overline{\lim}_k F_k(z_k) < +\infty$ and g is bounded below on C by a scalar m. Consider the following hypotheses*

(A_1): *There exists $\gamma \in \mathbb{R}$ such that $r_k \geq \gamma$, for every k and f is inf-compact on C;*

(A_2): *X is a vector space of finite dimension, $h_k : X \to \mathbb{R}$ are lsc and $\forall \lambda \in \mathbb{R}\ \exists\, k_\lambda \in \mathbb{N}$ such that $\forall\, k \geq k_\lambda$ the set $\{x \in X \mid h_k(x) \leq \lambda\}$ is connected (in particular this is true if h_k is convex) and the function $x \in X \to e - \underline{\lim}\, h_k(x)$ is inf-compact.*

If (A_1) *or* (A_2) *is satisfied, then $(z_k)_k$ is relatively compact; accordingly if hypothesis* (3.2) *is satisfied, then the conclusions of Theorem* 3.2 *hold.*

P r o o f. First we observe that there exist two scalars δ, β such that for all k sufficiently large $F_k(z_k) \leq \delta$ and $F_k(x) \geq f(x) + \beta + h_k(x)\ \forall\, x \in C$, so $F_k(x) \geq f(x) + \beta + \gamma$ if (A_1) is verified, and then $z_k \in \{x \in C \mid f(x) \leq \delta - \beta - \gamma\}$ which is compact. On the other hand, it is clear that there exists a number α such that $F_k(x) \geq \alpha + h_k(x)$ for every $x \in C$ and every k large enough. Now if (A_2) is satisfied, by [77] the functions h_k are uniformly inf-compact, in particular there exist a compact K and $k' \in \mathbb{N}$ such that $\forall\, k \geq k'$, $\{x \in X/h_k(x) \leq \delta - \alpha\} \subseteq K$, so $z_k \in K$. The remainder follows which completes the proof. □

Remark 3.7. Hypothesis (A_2) in Corollary 3.6 is in particular satisfied if X is a vector space of finite dimension and $(h_k)_k$ is a sequence of convex functions from X into $\mathbb{R}$ epi-converging to a proper inf-compact function $h : X \to \mathbb{R} \bigcup \{+\infty\}$ [77].

Corollary 3.8. *Let $f{:}C \to \mathbb{R} \bigcup \{+\infty\}$, $g_i : C \to \mathbb{R}$, $i = 1, 2, \ldots, p$, be lower semicontinuous functions and assume that f is finite at a point of a compact C of X. Then,*

$$\lim_{\epsilon \to 0^+} \frac{\min_{x \in C}(f(x) + \epsilon g_1(x) + \sum_{i=2}^{p} a_{i\epsilon} g_i(x)) - \min_{x \in C} f(x)}{\epsilon} = \min_{x \in S} g_1(x)$$

for all sequences $(a_{i\epsilon})_\epsilon$, $i = 2, \dots, p$ *such that* $a_{i\epsilon} \geq 0$ *and* $\lim_{\epsilon \to 0^+} \dfrac{a_{i\epsilon}}{\epsilon} = 0$.

Proof. The proof is an immediate consequence of Theorem 3.2, Remark 3.4 and the fact that $\lim_{\epsilon \to 0^+} \dfrac{\inf_{x \in C} h_\epsilon(x)}{\epsilon} = 0$ with $h_\epsilon(x) = \sum_{i=2}^p a_{i\epsilon} g_i(x)$. □

It is useful to provide general asymptotic developments with particular regularizations by application of Theorem 3.2. In the sequel we are concerned by the regularizations of the kind $F_k(x) = f(x) + \epsilon_k g(x) + \sum_{i=1}^p \beta_{k,i} g_i(x)$ as it is mentioned above with $\beta_{k,i} > 0$, $\dfrac{\beta_{k,i}}{\epsilon_k} \to 0$ for every i when $k \to +\infty$. We assume furthermore that all functions $g_i : X \to \mathbb{R}$ are lsc with $\dfrac{\beta_{k,i+1}}{\beta_{k,i}} \to 0$ when $k \to +\infty$, $\forall\, i$ and C is compact. Then $S_i = \operatorname{argmin}(g_i, C)$ is a nonempty compact and $g_i(x) \geq m_i$ $\forall\, x \in C$ for a scalar m_i. Our goal is to compute $\min_C F_k$.

Proposition 3.9. *We have the following formulas:*

$$\begin{aligned}(3.6)\quad &\min_C (f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2 + \cdots + \beta_{k,p} g_p)\\ &= \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \cdots + \beta_{k,2i-1} \min_C g_{2i-1}\\ &\qquad + \beta_{k,2i} \min_{S_{2i-1}} g_{2i} + \cdots + \beta_{k,p} \min\nolimits_{S_{p-1}} g_p + \epsilon_k v_{k,p}\end{aligned}$$

if p *is even and*

$$\begin{aligned}(3.7)\quad &\min_C (f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2 + \cdots + \beta_{k,p} g_p)\\ &= \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \cdots + \beta_{k,2i-1} \min_C g_{2i-1}\\ &\qquad + \beta_{k,2i} \min_{S_{2i-1}} g_{2i} + \cdots + \beta_{k,p} \min_C g_p + \epsilon_k w_{k,p}\end{aligned}$$

if p *is odd with* $v_{k,p}$, $w_{k,p}$ *converge to* 0 *when* $k \to +\infty$.

Proof. For $p = 1$ we have

$$\min_C (f + \epsilon_k g + \beta_{k,1} g_1) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \epsilon_k \alpha_{k,1}.$$

For $p = 2$,

$$\begin{aligned}&\min_C (f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2)\\ &\quad= \min_C f + \epsilon_k \min_S g + \beta_{k,1} \left(\min_C g_1 + \frac{\beta_{k,2}}{\beta_{k,1}} \min_{S_1} g_2 + \frac{\beta_{k,2}}{\beta_{k,1}} \delta_{k,2} \right) + \epsilon_k \alpha_{k,2}\\ &\quad= \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \beta_{k,2} \delta_{k,2} + \epsilon_k \alpha_{k,2}\end{aligned}$$

with $\lim_{k \to +\infty} \dfrac{\beta_{k,2}\delta_{k,2} + \epsilon_k \alpha_{k,2}}{\epsilon_k} = 0$ because all hypotheses of Theorem 3.2 are satisfied.

Now assume that (3.6) is satisfied for $p = 2, 4, \ldots, 2p$ and show that it is verified for $2p+2$. Consider $F_k(x) = f + \epsilon_k g + \sum_{i=1}^{2p+2} \beta_{k,i} g_i$ which reaches its minimizer at a point x_k because all functions under consideration are lsc on the compact C. The relatively compact sequence $(x_k)_k$ satisfies (3.1) and the function $R_k(x) = \sum_{i=1}^{2p+2} \beta_{k,i} g_i$ verifies (3.2) by Remark 3.4, so all assumptions of Theorem 3.2 hold. Then

$$\min_C F_k = \min_C f + \epsilon_k \min_S g + \min_C R_k + \epsilon_k \rho_{k,2}$$

with $\lim_{k\to+\infty} \rho_{k,2} = 0$. But

$$\min_C R_k = \min_C \beta_{k,1}(g_1 + \frac{\beta_{k,2}}{\beta_{k,1}} g_2 + \sum_{i=3}^{2p+2} \frac{\beta_{k,i}}{\beta_{k,1}} g_i)$$

and the function $D_k(x) = \sum_{i=3}^{2p+2} \frac{\beta_{k,i}}{\beta_{k,1}} g_i$ contains $2p$ terms, so by recurrence hypothesis one has

$$\begin{aligned}
\min_C R_k &= \beta_{k,1} \min_C \left(g_1 + \frac{\beta_{k,2}}{\beta_{k,1}} g_2 + \sum_{i=3}^{2p+2} \frac{\beta_{k,i}}{\beta_{k,1}} g_i \right) \\
&= \beta_{k,1} \left(\min_C g_1 + \frac{\beta_{k,2}}{\beta_{k,1}} \min_{S_1} g_2 + \frac{\beta_{k,3}}{\beta_{k,1}} \min_C g_3 + \frac{\beta_{k,4}}{\beta_{k,1}} \min_{S_3} g_4 + \cdots \right. \\
&\qquad \left. + \frac{\beta_{k,2p+1}}{\beta_{k,1}} \min_C g_{2p+1} + \frac{\beta_{k,2p+2}}{\beta_{k,1}} \min_{S_{2p+1}} g_{2p+2} + \frac{\beta_{k,2}}{\beta_{k,1}} \theta_{k,2p+2} \right)
\end{aligned}$$

with $\lim_{k\to+\infty} \theta_{k,2p+2} = 0$. Then

$$\begin{aligned}
\min_C F_k = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \beta_{k,3} \min_C g_3 + \beta_{k,4} \min_{S_3} g_4 \\
+ \cdots + \beta_{k,2p+1} \min_C g_{2p+1} + \beta_{k,2p+2} \min_{S_{2p+1}} g_{2p+2} + \beta_{k,2}\theta_{k,2p+2} + \epsilon_k \rho_{k,2}
\end{aligned}$$

and $\lim_{k\to+\infty} \dfrac{\beta_{k,2}\theta_{k,2p+2} + \epsilon_k \rho_{k,2}}{\epsilon_k} = 0$. In the same way we show (3.7) which completes the proof. □

Remark 3.10. Under the same hypotheses except that C is only closed and $g_i : X \to \mathbb{R}$ is inf-compact for every $i = 1, \ldots, p$ we have the same developments as in Proposition 3.9.

4. Stability of asymptotic developments by sequential epi-convergence. In this short section we investigate the stability of asymptotic developments under epi-convergence. Now return to problem (f, C) with $C = X$ and assume that the lsc proper functions $f_n : X \to \mathbb{R}\bigcup\{+\infty\}$ epi-converge sequentially to $f : X \to \mathbb{R}\bigcup\{+\infty\}$ and there exists a sequence $(x_n)_n$ in X having a subsequence $x_{n_p} = y_p \to \overline{t}$ and $\min_X f_n = f_n(x_n)\ \forall\, n$. Denote by $F_k^n(x) = f_n(x) + \epsilon_k g_n(x) + h_k(x)$ the regularization functions associated to f_n with $g_n : X \to \mathbb{R}$ is lsc and suppose that $h_k \xrightarrow{\text{epi-seq}} h$ with $h : X \to \mathbb{R}\bigcup\{+\infty\}$ is proper. Assume also that there exists a sequence $(z_k^n)_{k,n}$ relatively compact such that for every k, n, $i_k^n = \inf_X F_k^n$ is finite, $\dfrac{F_k^n(z_k^n) - i_k^n}{\epsilon_k} \to 0$, $\dfrac{h_k(s) - r_k}{\epsilon_k} \to 0$ when $k \to +\infty$, $\forall\, s \in S_n = \operatorname{argmin}(f_n, X)$, $\forall\, n$. Following Theorem 3.2, there exists a sequence $(\delta_k^n)_{k,n}$ of scalars converging to 0 for each fixed n when $k \to +\infty$ such that we have the following asymptotic development $A_k^n = F_k^n(z_k^n) = \min_X f_n + \epsilon_k \min_{S_n} g_n + \inf_X h_k(x) + \epsilon_k \delta_k^n$. The stability result is stated as follows:

Theorem 4.1. *There exists a subsequence $(n'_k)_k$ satisfying*

$$\left(A_k^{n'_k}, \delta_k^{n'_k}, \frac{F_k^{n'_k}(z_k^{n'_k}) - i_k^{n'_k}}{\epsilon_k}\right) \to \left(\min_X f + \min_X h = f(\overline{t}) + h(\overline{t}), 0, 0\right)$$

if $k \to +\infty$, where $\overline{t} \in \operatorname{argmin}(f, X) \bigcap \operatorname{argmin}(h, X)$. In particular if $z_k^n = x_k^n \in \operatorname{argmin}(F_k^n, X)$ we have

$$\min_X (f_{n'_k} + \epsilon_k g_{n'_k} + h_k) \to \min_X (f + h) = \min_X f + \min_X h \quad \textit{when} \quad k \to +\infty.$$

P r o o f. Since $d_k^p = \dfrac{h_k(y_p) - r_k}{\epsilon_k} \to 0$, when $k \to +\infty\ \forall\, p$, the reference [2, Corollary 1.18, p. 37] shows that there exits a subsequence $(p_k)_k$ (which can be computed) such that $d_k^{p_k} \to 0$ if $k \to +\infty$, so for a given $\epsilon > 0$, $h_k(y_{p_k}) \le \inf_X h_k + \epsilon_k \epsilon$ for all k large enough, and by Theorem 2.2 we conclude that $\inf_X h_k \to \min_X h$ and $\overline{t} \in \operatorname{argmin}(f, X) \bigcap \operatorname{argmin}(h, X)$. But for each fixed p,

$$\left(A_k^{n_p}, \delta_k^{n_p}, \frac{F_k^{n_p}(z_k^{n_p}) - i_k^{n_p}}{\epsilon_k}\right) \to \left(\min_X f_{n_p} + \min_X h, 0, 0\right)$$

if $k \to +\infty$ and $\min_X f_{n_p} \to \min_X f$ when $p \to +\infty$; accordingly by [2, Corollary 1.18, p. 37] again, there exists a subsequence $(n_{p_k = n'_k})_k$ satisfying

$$\left(A_k^{n'_k}, \delta_k^{n'_k}, \frac{F_k^{n'_k}(z_k^{n'_k}) - i_k^{n'_k}}{\epsilon_k}\right) \to \left(\min_X f + \min_X h, 0, 0\right)$$

if $k \to +\infty$. The remainder follows, which completes the proof. □

Remark 4.2. It is worth pointing out that the stability result established here holds regardless of the epi-convergence or not of the sequence of sum of functions $(f_{n'_k} + \epsilon_k g_{n'_k} + h_k)_k$ to $f + h$ and without having any information on the behavior of the sequence $(\epsilon_k g_{n'_k})_k$. For the stability concept of sum of functions (and sets) by variational convergences and its crucial role in variational analysis and optimization we refer the reader to [5, 9, 47, 48, 50, 51, 53, 54, 23].

5. The convex case. In this section we apply the above results to convex functions defined on a normed space and derive asymptotic developments for the Legendre-Fenchel transform.

Proposition 5.1. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}\bigcup\{+\infty\}$, g, $h_k : X \to \mathbb{R}$, $k \in \mathbb{N}$ be convex proper lower semicontinuous functions. Let C be a nonempty closed convex set of X such that $S = \operatorname{argmin}(f, C)$ and $C \cap \operatorname{Dom} f$ are nonempty. Assume that $\{x \in C \ : \ g(x) \leq \lambda\}$ is bounded $\forall\, \lambda \in \mathbb{R}$ and (3.2) holds. Let $(z_k)_k$ be a sequence of C such that (3.1) is verified. Then $(z_k)_k$ is weakly relatively compact and the conclusions of Theorem 3.2 hold. If $\operatorname{argmin}(g, S) = \{\overline{a}\}$ (particularly when g is strictly convex), then $(z_k)_k$ has a unique weakly cluster point $\overline{z} = \overline{a}$ and $z_k \rightharpoonup \overline{a}$ where $\rightharpoonup$ denotes the weak convergence. Moreover if the conditions $g(t_k) \to g(t)$ and $t_k \rightharpoonup t$ imply that $\|t_k - t\| \to 0$, $k \to +\infty$ one has $\|z_k - \overline{a}\| \to 0$ when $k \to +\infty$.*

P r o o f. First, we point out that

$$L^k_\lambda = \{x \in C \mid F_k(x) \leq \lambda\} \subset \left\{x \in C \mid g(x) \leq \frac{\lambda - \min_C f - r_k}{\epsilon_k}\right\}$$

for each k and $\lambda \in \mathbb{R}$, so L^k_λ is weakly compact and by a classical argument [35] , $i_k = \min_C F_k = F_k(x_k)$ for some $x_k \in C$. Now by reflexivity, convexity and Corollary 3.5, it is immediate that $(z_k)_k$ is weakly relatively compact. The remainder follows by an obvious verification. □

In the theorem below, we give sufficient conditions ensuring that the minimization problem (F_k, C) is well-posed in the Tikhonov sense for the norm topology. Given two functions $p, q : X \to \mathbb{R}\bigcup\{+\infty\}$ and consider the following hypotheses:

(H_p) : $A_\lambda = \{x \in C \mid p(x) - \lambda\|x\| \leq 0\}$ is bounded for every $\lambda \in \mathbb{R}$.

(H'_q) : $q(x_n) \to q(x)$ and $x_n \rightharpoonup x$ imply that $\|x_n - x\| \to 0$ when $n \to +\infty$.

Theorem 5.2. *Let X be a reflexive Banach space and suppose that F_k is strictly convex on a closed convex subset C of X and $\epsilon_k > 0$. Assume that (H_p)*

and (H'_q) hold simultaneously at least for two functions p, q (eventually identical) belonging in the set $\{f, g, h_k\}$ where $f : X \to \mathbb{R}\bigcup\{+\infty\}$, g, $h_k : X \to \mathbb{R}$ are convex proper lower semicontinuous functions. Then (F_k, C) is well-posed in the Tikhonov sense for the norm topology.

Proof. Each function f, g, h_k has a continuous affine minorant, so if $p \in \{f, g, h_k\}$, then there exist $\alpha \geq 0$, $\beta \in \mathbb{R}$ such that $F_k(x) \geq -\alpha\|x\| + \beta + \gamma p(x)$ ($\gamma = \epsilon_k$ if $p = g$ and $\gamma = 1$ otherwise). It turns out that

$$L^k_\lambda = \{x \in C \mid F_k(x) \leq \lambda\} \subseteq \left\{x \in C \mid p(x) - \frac{\alpha}{\gamma}\|x\| + \frac{\beta}{\gamma} \leq \frac{\lambda}{\gamma}\right\}$$

which is bounded by (H_p) for every $\lambda \in \mathbb{R}$; so L^k_λ is weakly compact and (F_k, C) has a unique solution x_k. Now let $(x_n)_n$ be a mimizing sequence for (F_k, C). Since $F_k(x_n) \to F_k(x_k) \in \mathbb{R}$ when $n \to +\infty$, $(x_n)_n$ belongs to a sublevel of F_k and $(x_n)_n$ is bounded. By lower semicontinuity of F_k, every cluster point $z \in C$ of $(x_n)_n$ for the weak topology satisfies $F_k(z) = F_k(x_{k)})$ so $z = x_k$ and $x_n \rightharpoonup x_k$. Set $a_n = f(x_n) + \epsilon_k g(x_n) + h_k(x_n) - f(x_k) - \epsilon_k g(x_k) - h_k(x_k) \to 0$ if $n \to +\infty$. We have $f(x_n) - f(x_k) = a_n + \epsilon_k(g(x_k) - g(x_n)) + h_k(x_k) - h_k(x_n)$ and $\overline{\lim}_n(f(x_n) - f(x_k)) \leq \epsilon_k(g(x_k) - \underline{\lim}_n g(x_n)) + h_k(x_k) - \underline{\lim}_n h_k(x_n) \leq 0$; accordingly $\overline{\lim}_n f(x_n) \leq f(x_k) \leq \underline{\lim}_n f(x_n)$ and $f(x_n) \to f(x_k)$. By the same argument one has $g(x_n) \to g(x_k)$ and $h_k(x_n) \to h_k(x_k)$. From (H'_q) we conclude that $\|x_n - x_k\| \to 0$ when $n \to +\infty$. $\square$

Remark 5.3. Note that a reflexive Banach space X may be always renormed by a strictly convex norm $\|\cdot\|$ such that (H'_q) is satisfied with $q(x) = \|\cdot\|$ [2, 20, 21, 75] that is X is an E-space; so one can take for instance in Proposition 5.1 or in Theorem 5.2, $g(x) = \|x - x_0\|^r$, $r \geq 1$ and x_0 is any given point in X. In this case the sequence $(x_k)_k$, where x_k is the minimizer of F_k on C, converges strongly to $\operatorname{proj}_S x_0$ in Proposition 5.1 (or in Theorem 5.2 if (3.2) is satisfied). Also we point out that, even though (H_p) and (H'_q) fail to be satisfied with $p, q \in \{f, g\}$ which can be imposed by an algorithm, a large choice of the "negligible" terms h_k may guarantee the verification of (H_{h_k}) and (H'_{h_k}). It is worth noting that the only role of hypothesis (H_p) is to ensure the boundedness of L^k_λ. It can be replaced for instance by the following hypothesis: two functions of $\{f, g, h_k\}$ are bounded below and the third is weakly inf-compact; for example f, g are bounded below and h_k is weakly inf-compact.

Corollary 5.4. *Let X be a reflexive Banach space renormed by by a strictly convex norm $\|\cdot\|$ making it an E-space and $f : X \to \mathbb{R}\bigcup\{+\infty\}$, $h_k : X \to \mathbb{R}$, $k \in \mathbb{N}$ be convex proper lower semicontinuous functions. Consider the generalized stabilization functions $F_k(x) = f(x) + \epsilon_k\|x - x_0\|^p + h_k(x)$*

with $p > 1$, $x_0 \in X$ and $\epsilon_k > 0$, $\epsilon_k \to 0$. Let C be a nonempty closed convex set of X such that $S = \operatorname{argmin}(f, C)$ and $C \cap \operatorname{Dom} f$ are nonempty. Assume that (3.2) is satisfied. Then:

(1) $\operatorname{argmin}(F_k, C) = \{x_k\}$.

(2) *(F_k, C) is well-posed in the Tikhonov sense for the norm topology.*

(3) *The sequence $(x_k)_k$ converges strongly to $\operatorname{proj}_S x_0$ when $k \to +\infty$.*

(4) $\dfrac{\min_C F_k - \min_C f}{\epsilon_k} \to \|x_0 - \operatorname{proj}_S x_0\|^p$, $\dfrac{f(x_k) - \min_C f}{\epsilon_k} \to 0$ *when* $k \to +\infty$.

Proof. The proof is an immediate consequence of Proposition 5.1 and Theorem 5.2. □

Remark 5.5. If $h_k = 0$ for every k in the above corollary and $p = 2$, then we find this particular case in [54, 55].

6. Application to Legendre-Fenchel transform. In the sequel we are interested by asymptotic developments of the Legendre-Fenchel transform [35] $(f + \epsilon g_1 + \epsilon^2 g_2 + \cdots + \epsilon^n g_n)^*(y)$, $\epsilon > 0$, $\epsilon \to 0$ where y is a fixed point of the topological dual X^* of a locally convex space X and $f : X \to \mathbb{R} \bigcup \{+\infty\}$ be a proper convex lower semicontinuous function, $g_k : X \to \mathbb{R}$, $k = 1, 2, \ldots, n$ are convex continuous functions.

Theorem 6.1. *If $f(\cdot) - \langle \cdot, y\rangle$, $(g_{2i})_{i \geq 1}$ are weakly inf-compact and $(g_{2j+1})_{j \geq 0}$ are bounded below for every i, j satisfying $2i$, $2j + 1 \leq n$, then the following formula holds:*

$$
\begin{aligned}
(6.1)\qquad &(f + \epsilon g_1 + \epsilon^2 g_2 + \cdots + \epsilon^n g_n)^*(y) \\
&= f^*(y) + \epsilon(g_1 + \delta_{\partial f^*(y)})^*(0) + \epsilon^2 g_2^*(0) + \epsilon^3 (g_3 + \delta_{\partial g_2^*(0)})^*(0) \\
&\quad + \epsilon^4 g_4^*(0) + \epsilon^5 (g_5 + \delta_{\partial g_4^*(0)})^*(0) + \cdots + \epsilon^{2i} g_{2i}^*(0) \\
&\quad + \epsilon^{2i+1} (g_{2i+1} + \delta_{\partial g_{2i}^*(0)})^*(0) + \cdots + \epsilon \mu_n(y, \epsilon) \\
&= f^*(y) + \epsilon(g_1^* \nabla (f^*)'(y, \cdot))(0) + \epsilon^2 g_2^*(0) \\
&\quad + \epsilon^3 (g_3^* \nabla (g_2^*)'(0, \cdot))(0) + \cdots \\
&\quad + \epsilon^{2i} g_{2i}^*(0) + \epsilon^{2i+1} (g_{2i+1}^* \nabla (g_{2i}^*)'(0, \cdot))(0) + \cdots + \epsilon \mu_n(y, \epsilon)
\end{aligned}
$$

with $\lim_{\epsilon \to 0} \mu_n(y, \epsilon) = 0$.

Proof. First, we point out that the set $S_y = \operatorname{argmin}_X (f(\cdot) - \langle \cdot, y\rangle)$ is nonempty, because $f(\cdot) - \langle \cdot, y\rangle$ is weakly inf-compact [35]. Now let $m_i \in \mathbb{R}$ such that $g_i(x) \geq m_i$ for every i and $x \in X$. We have

$$F_\epsilon(x) = f(x) - \langle x, y\rangle + \sum_{i=1}^{n} \epsilon^i g_i(x) \geq f(x) - \langle x, y\rangle + \sum_{i=1}^{n} \epsilon^i m_i \geq f(x) - \langle x, y\rangle - 1$$

for every $\epsilon \leq \epsilon_0$, so F_ϵ is weakly inf-compact and reaches its minimum at a point x_ϵ. Let $a \in \operatorname{Dom} f$; we have

$$f(x_\epsilon) - \langle x_\epsilon, y\rangle + \sum_{i=1}^{n} \epsilon^i m_i \leq f(a) - \langle a, y\rangle + \sum_{i=1}^{n} \epsilon^i g_i(a)$$

and

$$f(x_\epsilon) - \langle x_\epsilon, y\rangle \leq f(a) - \langle a, y\rangle + \sum_{i=1}^{n} \epsilon^i (g_i(a) - m_i) \leq f(a) - \langle a, y\rangle + 1$$

for every ϵ sufficiently small. Then $(x_\epsilon)_\epsilon$ is weakly relatively compact, so all conditions in Theorem 3.2 are fulfilled (here F_ϵ plays the same role as F_k considered in Theorem 3.2), and then

$$\min_X F_\epsilon = \min_{x \in X}(f(x) - \langle x, y\rangle) + \epsilon \min_{x \in S_y} g_1(x) + \inf_X h_\epsilon + \epsilon \varphi_n(y, \epsilon),$$

with $h_\epsilon(x) = \sum_{i=2}^{n} \epsilon^i g_i(x)$ and $\varphi_n(y, \epsilon) \to 0$ if $\epsilon \to 0$. But $z \in S_y$ if and only if $f(z) - \langle z, y\rangle \leq \inf_{x \in X} f(x) - \langle x, y\rangle = -f^*(y)$ or equivalently, $f(z) + f^*(y) \leq \langle z, y\rangle$, i.e $z \in \partial f^*(y)$. It turns out that

$$\alpha = -\min_X F_\epsilon = f^*(y) + \epsilon(g_1 + \delta_{\partial f^*(y)})^*(0) - \inf_X h_\epsilon - \epsilon \varphi_n(y, \epsilon).$$

By [35, Theorem 6.5.8], one has

$$(g_1 + \delta_{\partial f^*(y)})^*(0) = (g_1^* \nabla \delta^*_{\partial f^*(y)})(0) = g_1^*(t) + \delta^*_{\partial f^*(y)}(-t)$$

for some $t \in X^*$. But f^* is finite and $\tau(X^*, X)$ continuous at y by [35, Theorem 6.3.9 and its Corollary, pp 347–348] where $\tau(X^*, X)$ is the Mackey topology on X^*, then $\delta^*_{\partial f^*(y)}(w) = \max_{r \in \partial f^*(y)} \langle w, r\rangle = (f^*)'(y, w)\ \forall\, w$ [35, Theorem 6.4.8]. Accordingly

$$\begin{aligned} \alpha &= f^*(y) + \epsilon(g_1 + \delta_{\partial f^*(y)})^*(0) - \inf_X h_\epsilon - \epsilon \varphi_n(y, \epsilon) \\ &= f^*(y) + \epsilon(g_1^* \nabla (f^*)'(y, \cdot))(0) - \inf_X h_\epsilon - \epsilon \varphi_n(y, \epsilon). \end{aligned}$$

Now by the same argument

$$\begin{aligned} \inf_X h_\epsilon &= \epsilon^2 \inf_{x \in X} (g_2(x) + \epsilon g_3(x) + \cdots + \epsilon^{n-2} g_n(x)) \\ &= \epsilon^2 \min_{x \in X} (g_2(x) + \epsilon g_3(x) + \cdots + \epsilon^{n-2} g_n(x)) \\ &= \epsilon^2 \min_{x \in X} g_2(x) + \epsilon^3 \min_{x \in \operatorname{argmin}(g_2,\, \mathrm{X})} g_3(x) + \epsilon^4 \left\{ \inf_{x \in X} \sum_{i=4}^{n} \epsilon^{i-4} g_i(x) \right\} + \epsilon^3 \varphi_n^1(\epsilon) \end{aligned}$$

with $\lim_{\epsilon\to 0}\varphi_n^1(\epsilon) = 0$ and

$$\begin{aligned} -\inf_X h_\epsilon &= \epsilon^2 g_2^*(0) + \epsilon^3 (g_3 + \delta_{\partial g_2^*(0)})^*(0) - \epsilon^4 \left\{ \inf_{x\in X} \sum_{i=4}^n \epsilon^{i-4} g_i(x) \right\} - \epsilon^3 \varphi_n^1(\epsilon) \\ &= \epsilon^2 g_2^*(0) + \epsilon^3 (g_3^* \nabla (g_2^*)'(0,\cdot))(0) - \epsilon^4 \left\{ \inf_{x\in X} \sum_{i=4}^n \epsilon^{i-4} g_i(x) \right\} - \epsilon^3 \varphi_n^1(\epsilon). \end{aligned}$$

So

$$\begin{aligned} \alpha &= (f + \epsilon g_1 + \epsilon^2 g_2 + \cdots + \epsilon^n g_n)^*(y) \\ &= f^*(y) + \epsilon (g_1 + \delta_{\partial f^*(y)})^*(0) + \epsilon^2 g_2^*(0) + \epsilon^3 (g_3 + \delta_{\partial g_2^*(0)})^*(0) \\ &\quad -\epsilon^4 \left\{ \inf_{x\in X} \sum_{i=4}^n \epsilon^{i-4} g_i(x) \right\} - \epsilon^3 \varphi_n^1(\epsilon) - \epsilon \varphi_n(y,\epsilon) \\ &= f^*(y) + \epsilon (g_1^* \nabla (f^*)'(y,\cdot))(0) + \epsilon^2 g_2^*(0) + \epsilon^3 (g_3^* \nabla (g_2^*)'(0,\cdot))(0) \\ &\quad -\epsilon^4 \left\{ \inf_{x\in X} \sum_{i=4}^n \epsilon^{i-4} g_i(x) \right\} - \epsilon^3 \varphi_n^1(\epsilon) - \epsilon \varphi_n(y,\epsilon); \end{aligned}$$

and step by step using the function $M_\epsilon^n(x) = \epsilon^4 (\sum_{i=4}^n \epsilon^{i-4} g_i(x))$ and its components we derive formula (6.1) by applying Theorem 3.2 and [35, Theorems 6.4.8, 6.5.8] several times; and by the fact that the weak inf-compactness of functions g_{2i} implies that g_{2i}^* are $\tau(X^*, X)$ continuous at 0 [35, Theorem 6.3.9 and its Corollary, pp 347–348] which completes the proof. □

Corollary 6.2. *Let X be a reflexive Banach space renormed by a striclty convex norm $\|\cdot\|$ making it an E-space. Let $f : X \to \mathbb{R}\bigcup\{+\infty\}$ be a convex proper lower semicontinuous function and $g_i(x) = \|x\|^i$ for every $i = 1, 2, \ldots, n$. If f^* is $\|\ \|_{X^*}$-continuous at y, then* (6.1) *holds with $g_{2i}^*(0) = (g_{2i+1} + \delta_{\partial g_{2i}^*(0)})^*(0) = 0$ for every $i \geq 1$ such that $2i$, $2i+1 \leq n$; and then*

$$\left(f + \sum_{i=1}^n \epsilon^i \|\cdot\|^i \right)^* (y) = f^*(y) + \epsilon \min_{t\in B_{X^*}(0,1)} (f^*)'(y,t) + \epsilon \rho_n(y,\epsilon)$$

with $\min_{t\in B_{X^*}(0,1)} (f^*)'(y,t) = -\|\operatorname{proj}_{S_y} 0\|$ and $\lim_{\epsilon\to 0}\rho_n(y,\epsilon) = 0$.

Proof. It is clear that g_i $i = 1, 2, \ldots, n$ satisfy all hypotheses in Theorem 6.1 and the Mackey topology $\tau(X^*, X)$ on X^* is exactly the norm $\|\ \|_{X^*}$ topology, so the $\|\ \|_{X^*}$-continuity of f^* at y is equivalent [35] to the weak inf-compactness of $f(\cdot) - \langle \cdot, y\rangle$. The remainder follows by an obvious verification using classical calcalutions. □

7. New characterization of well-posedness in terms of infimal-convolution operations. This section will be devoted to a new characterization of well-posedness in normed spaces using infimal-convolution operations and a class of regularization functions.

Theorem 7.1. *Let X be a normed space. The following assertions are equivalent:*

(i) (f, K) and (g, K') are well-posed in the Tikhonov sense with x_0, y_0 their unique solutions, respectively.

(ii) $(f\nabla g, K + K')$ is well-posed in the Tikhonov sense with solution $z_0 = x_0 + y_0$ and $f\nabla g$ is exact at z_0.

If (i) or (ii) is satisfied, then there exist a forcing function C_1 associated to (f, K) and a forcing function C_2 associated to (g, K') such that $C_1\nabla C_2$ is forcing function associated to $(f\nabla g, K + K')$.

Proof. $(i) \Longrightarrow (ii)$. For every $x \in K$, $y \in K'$ one has $f(x_0) \leq f(x)$ and $g(y_0) \leq g(y)$, so $(f\nabla g)(x_0 + y_0) \leq f(x_0) + g(y_0) \leq (f\nabla g)(z)\ \forall\, z \in K + K'$; accordingly $z_0 = x_0+y_0 \in \operatorname{argmin}(f\nabla g, K+K')$ and $(f\nabla g)(z_0) = f(x_0)+g(y_0)$ i.e $f\nabla g$ is exact at z_0. Now let $(z_n)_n$ be a sequence of $K+K'$ such that $(f\nabla g)(z_n) \to (f\nabla g)(z_0)$, then there exists a sequence $(u_n, v_n) \in K \times K'$ such that $z_n = u_n + v_n$ and $f(u_n) + f(v_n) \to f(x_0) + g(y_0)$ so $f(u_n) \to f(x_0)$ and $f(v_n) \to f(y_0)$ and $(u_n, v_n) \to (x_0, y_0)$, i.e $z_n \to z_0$.

$(ii) \Longrightarrow (i)$. The exactness of $f\nabla g$ at $z_0 = x_0+y_0$ implies that $(f\nabla g)(z_0) = f(x_0) + g(y_0) \leq (f\nabla g)(z)$ for every $z \in K + K'$, so $f(x_0) + g(y_0) \leq f(x) + g(y)$ $\forall\,(x, y) \in K \times K'$ and $x_0 \in \operatorname{argmin}(f, K)$, $y_0 \in argmin(g, K')$. Now consider a sequence $(x_n)_n$ in K such that $f(x_n) \to f(x_0)$. For every n, we have $f(x_0)+g(y_0) \leq (f\nabla g)(x_n + y_0) \leq f(x_n)+g(y_0)$, so $(f\nabla g)(x_n + y_0) \to (f\nabla g)(z_0)$ and by hypothesis $x_n + y_0 \to x_0 + y_0$, i.e $x_n \to x_0$ and (f, K) is well-posed in the Tikhonov sense with solution x_0. In the same way (g, K') is well-posed in the Tikhonov sense with solution y_0. If (i) or (ii) is satisfied, it is easy to check that the following positive functions defined by

$$C_1(t) = \inf\{f(x) - f(x_0) : t \leq \|x - x_0\|, x \in K\},$$
$$C_2(t) = \inf\{g(y) - g(y_0) : t \leq \|y - y_0\|, y \in K'\},$$
$$C_3(t) = \inf\{(f\nabla g)(z) - (f\nabla g)(z_0) : t \leq \|z - z_0\|, z \in K + K'\}$$

are forcing functions associated respectively to (f, K), (g, K'), $(f\nabla g, K + K')$. Now we will show that $C_1\nabla C_2$ defined on $\mathbb{R}$ is a forcing function associated to $(f\nabla g, K + K')$. First it is easy to see that $C_1\nabla C_2 \geq 0$, $(C_1\nabla C_2)(0) = 0$; and if $(C_1\nabla C_2)(t_n) \to 0$, then $t_n \to 0$ because (f, K) and (g, K') are well-posed in the Tikhonov sense with x_0, y_0 their solutions, respectively. C_3 is a forcing function associated to $(f\nabla g, K+K')$ because the last problem is well-posed in the Tikhonov sense with solution $z_0 = x_0 + y_0$ and $(f\nabla g)(z) \geq (f\nabla g)(z_0) + C_3(\|z - z_0\|)$ for every $z \in K+K'$. To prove that $C_1\nabla C_2$ is a forcing function associated to $(f\nabla g, K+K')$ it suffices to show that $C_1\nabla C_2 \leq C_3$. If $C_3(t') = +\infty$ for some t', the last inequality is satisfied. Now assume that $C_3(t) < +\infty$ and consider $\alpha \in \mathbb{R}^+$ such that $C_3(t) < \alpha$; then there exist $x \in K$, $y \in K'$ verifying $t \leq \|x+y-x_0-y_0\|$ and $f(x)+g(y)-f(x_0)-g(y_0) < \alpha$. Let a, b be positive real numbers such that $a+b=\alpha$ and $f(x)-f(x_0) \leq a$, $g(y)-g(y_0) \leq b$. But $t \leq \|x+y-x_0-y_0\| \leq \|x-x_0\|+\|y-y_0\|$, so there exist positive real numbers t_1, t_2 such that $t_1+t_2=t$, $t_1 \leq \|x-x_0\|$ and $t_2 \leq \|y-y_0\|$; then we conclude that $C_1(t_1) \leq a$, $C_2(t_2) \leq b$ and $(C_1\nabla C_2)(t) \leq C_1(t_1)+C_2(t_2) \leq a+b=\alpha$; consequently $C_1\nabla C_2 \leq C_3$ and $(f\nabla g)(z) \geq (f\nabla g)(z_0)+C_3(\|z-z_0\|) \geq (f\nabla g)(z_0)+(C_1\nabla C_2)(\|z-z_0\|)$ for every $z \in K+K'$, which completes the proof. □

Corollary 7.2. *Let X be a normed space and $f : K \to]-\infty,+\infty]$ be a function defined on a subset K of X. Then (f,K) is well-posed in the Tikhonov sense with solution x_0 if and only if there exists a subset K' of X containing* 0 *and real numbers $p>0$, $\lambda>0$ such that $(f\nabla\lambda\|\cdot\|^p, K+K')$ is well-posed in the Tikhonov sense with solution x_0 and $f\nabla\lambda\|\cdot\|^p$ is exact at x_0.*

P r o o f. The proof is an immediate consequence of the last theorem and the fact that the problem $(\lambda\|\cdot\|^p, K')$ is well-posed in the Tikhonov sense with unique solution 0 for every subset K' of X containing 0 and every numbers $p>0$, $\lambda>0$. □

Corollary 7.3. *Let X be a reflexive Banach space and $f : X \to]-\infty,+\infty]$ be a convex proper lower semicontinuous function. Then (f,X) is well-posed in the Tikhonov sense with solution x_0 if and only if there exist $p>1$, $\lambda>0$ such that $(f\nabla\lambda\|\cdot\|^p, X)$ is well-posed in the Tikhonov sense with solution x_0.*

P r o o f. By Theorem 7.1, if (f,X) is well-posed in the Tikhonov sense with solution x_0, then $(f\nabla\lambda\|\cdot\|^p, X)$ is well-posed in the Tikhonov sense for every numbers $p>0$, $\lambda>0$ with solution x_0. Now assume that there exist $p>1$, $\lambda>0$ such that $(f\nabla\lambda\|\cdot\|^p, X)$ is well-posed in the Tikhonov sense with solution x_0. By reflexivity, convexity and coercivity, the function $x \to f(x)+\lambda\|x_0-x\|^p$ reaches its minimizer on X [22, 35], so $f\nabla\lambda\|\cdot\|^p$ is exact at x_0. Corollary 7.2 permits to conclude the proof. □

Theorem 7.4. *Let X be a normed space and f, g be two functions defined and proper lower semicontinuous, respectively, on two closed subsets K, K' of X. The following assertions are equivalent:*

(i) (f,K) and (g,K') are well-posed in the generalized sense of Tikhonov.

(ii) $(f\nabla g, K+K')$ is well-posed in the generalized sense of Tikhonov and $f\nabla g$ is exact at least at a point of $\operatorname{argmin}(f\nabla g, K+K')$.

If (i) or (ii) is satisfied, then there exist a forcing function C_1 associated to (f,K) and a forcing function C_2 associated to (g,K') such that $C_1\nabla C_2$ is forcing function associated to $(f\nabla g, K+K')$.

Proof. $(i) \Longrightarrow (ii)$ may be shown in the same way as $(i) \Longrightarrow (ii)$ of Theorem 7.1.

$(ii) \Longrightarrow (i)$. The exactness of $f\nabla g$ at a point $z_0 = x_0 + y_0 \in \operatorname{argmin}(f\nabla g, K+K')$ implies that $x_0 \in argmin(f,K)$ and $y_0 \in \operatorname{argmin}(g,K')$. Now consider a sequence $(x_n)_n$ in K such that $f(x_n) \to f(x_0)$, then $\lim_n (f\nabla g)(x_n + y_0) = \lim_n (f(x_n) + g(y_0)) = (f\nabla g)(z_0)$; consequently there exists a subsequence $(x_{n_k})_k$ converging to an element $x' \in K$ and by lower semicontinuity $f(x') \leq \underline{\lim}\, f(x_{n_k}) = f(x_0)$ so $x' \in \operatorname{argmin}(f,K)$ and (f,K) is well-posed in the generalized sense of Tikhonov. In the same way (g,K') is well-posed in the generalized sense of Tikhonov.

Now if (i) or (ii) is satified, the fact that $\operatorname{argmin}(f,K) + \operatorname{argmin}(g,K') \subset \operatorname{argmin}(f\nabla g, K+K')$ and

$$d(x+y, \operatorname{argmin}(f,K) + \operatorname{argmin}(g,K')) \leq d(x, \operatorname{argmin}(f,K)) + d(y, \operatorname{argmin}(g,K'))$$

allows us to verify as in Theorem 7.1 that $C_1\nabla C_2$ is a forcing function associated to $(f\nabla g, K+K')$, where

$$\begin{aligned} C_1(t) &= \inf\{f(x) - \min_K f : t \leq d(x, \operatorname{argmin}(f,K)), x \in K\}, \\ C_2(t) &= \inf\{g(y) - \min_{K'} g : t \leq d(y, \operatorname{argmin}(g,K')), y \in K'\} \end{aligned}$$

are forcing functions associated respectively to (f,K) and (g,K'). □

Corollary 7.5. *Let X be a normed space and $f,g : X \to]-\infty,+\infty]$ be two proper lower semicontinuous functions with f is bounded below and g is inf-compact. Then (f,X) is well-posed in the Tikhonov generalized sense if and only if $(f\nabla g, X)$ is well-posed in the same sense.*

Proof. First g reaches its minimizer over X because g is inf-compact [35]. Now consider a sequence $(x_n)_n$ such that $g(x_n) \to \min_X g$; by inf-compactness $(x_n)_n$ has a subsequence converging to an element of $\operatorname{argmin}(g,X)$, so (g,X) is well-posed in the Tikhonov generalized sense. On the other hand $f\nabla g$ is exact at any point of X [35, Proposition 6.5.5, p. 362], so the conclusion holds from Theorem 7.4. □

Corollary 7.6. *Let X be a reflexive Banach space and $f, g : X \to]-\infty, +\infty]$ be two convex proper lower semicontinuous functions. Suppose that the set $M = \bigcup_{\alpha \geq 0} \alpha(\mathrm{Dom}\, f^* - \mathrm{Dom}\, g^*)$ is a closed subspace of X^*. Then (f, X) and (g, X) are well-posed in the generalized sense of Tikhonov if and only if $(f \nabla g, X)$ is well-posed in the same sense.*

P r o o f. $f \nabla g$ is exact at any point of X because M is a closed subspace of X^* and X is reflexive [3]. Theorem 7.4 permits to conclude the proof. □

Theorem 7.7. *Let X be a normed space $f, g : X \to]-\infty, +\infty]$ be two proper functions and K, K' be two subsets of X. Assume that $(f \nabla g, K + K')$ is Levitin-Polyak well-posed with solution z_0, where $(f \nabla g)(z) = \inf_{x \in X}\{f(x) + g(z - x)\}\ \forall z \in X$ and there exists $(x_0, y_0) \in K \times K'$ such that $z_0 = x_0 + y_0$ with $(f \nabla g)(z_0) = f(x_0) + g(y_0)$. Then*

(i) (f, K) and (g, K') are well-posed in the Tikhonov sense.

(ii) Moreover if $\min_{K+K'} f \nabla g \leq \underline{\lim}_n (f \nabla g)(z_n)$ for every $(z_n)_n$ in X such that $d(z_n, K + K') \to 0$, then (f, K) and (g, K') are well-posed in the Levitin-Polyak sense.

P r o o f. (*i*) First, to avoid any confusion we denote by $(f \overline{\nabla} g)(z') = \inf\{f(x) + g(y) : z' = x + y / (x, y) \in K \times K'\}$, $z' \in K + K'$. By hypothesis $(f \nabla g)(z_0) = f(x_0) + g(y_0) \leq (f \nabla g)(z) \leq (f \overline{\nabla} g)(z)$ for every $z \in K + K'$, so $(f \overline{\nabla} g)(z_0) = f(x_0) + g(y_0) \leq (f \overline{\nabla} g)(z)\ \forall z \in K + K'$, i.e $z_0 \in \operatorname{argmin}(f \overline{\nabla} g, K + K')$ and $f \overline{\nabla} g$ is exact at z_0. On the other hand, if $z_n \in K + K'$ and $(f \overline{\nabla} g)(z_n) \to (f \overline{\nabla} g)(z_0)$, then $(f \nabla g)(z_n) \to (f \nabla g)(z_0)$ so $z_n \to z_0$. We conclude that $(f \overline{\nabla} g, K + K')$ is well-posed in the Tikhonov sense and by Theorem 7.1 (f, K) and (g, K') are well-posed in the Tikhonov sense with unique solutions x_0, y_0, respectively.

(*ii*) Consider a sequence $(x_n)_n$ such that $d(x_n, K) \to 0$ and $f(x_n) \to f(x_0)$. The sequence $w_n = x_n + y_0$ verifies $d(w_n, K + K') \to 0$ and by hypothesis one has $(f \nabla g)(z_0) \leq \underline{\lim}_n (f \nabla g)(w_n) \leq \overline{\lim}_n (f \nabla g)(w_n) \leq \overline{\lim}_n (f(x_n) + f(y_0)) = (f \nabla g)(z_0)$, so $(f \nabla g)(w_n) \to (f \nabla g)(z_0)$ and $w_n \to z_0$, i.e $x_n \to x_0$. Consequently (f, K) is well-posed in the Levitin-Polyak sense. In the same way (g, K') is well-posed in the same sense with solution y_0. □

Corollary 7.8. *Let X be a normed space, $f, g : X \to]-\infty, +\infty]$ be two proper functions and K, K' be two subsets of X. Assume that $(f \nabla g, K + K')$ is Levitin-Polyak well-posed and there exists $(x_0, y_0) \in K \times K'$ such that $\min_X f = f(x_0)$ and $\min_X g = g(y_0)$, then (f, K) and (g, K') are well-posed in the Levitin-Polyak sense.*

P r o o f. The proof is an immediate consequence of Theorem 7.7 because $f(x_0) + g(y_0) = (f\nabla g)(x_0 + y_0) \leq (f\nabla g)(z)\ \forall\, z \in X$, and then all hypotheses of Theorem 7.7 are satisfied. □

By slight modifications in the proof of Theorem 7.7 we can state the following theorem:

Theorem 7.9. *Let X be a normed space, $f, g : X \to]-\infty, +\infty]$ be two proper lower semicontinuous functions and K, K' be two closed subsets of X.*

(1) *Assume that $(f\nabla g, K + K')$ is well-posed in the generalized sense of Levitin-Polyak and the following hypothesis is verified*

$$(H) : \begin{array}{l} \exists\ \overline{z} \in \operatorname{argmin}(f\nabla g, K + K'), \exists\ (\overline{x}, \overline{y}) \in K \times K' \\ \textit{such that } \overline{z} = \overline{x} + \overline{y} \textit{ and } (f\nabla g)(\overline{z}) = f(\overline{x}) + g(\overline{y}). \end{array}$$

Then:

(*i*) *(f, K) and (g, K') are well-posed in the generalized sense of Tikhonov.*

(*ii*) *If the hypothesis considered in Theorem 7.7 (ii) holds, then (f, K) and (g, K') are well-posed in the generalized sense of Levitin-Polyak.*

(2) *If $(f\nabla g, K + K')$ is well-posed in the strong generalized sense and (H) is satisfied, then (f, K) and (g, K') are well-posed in the strong generalized sense.*

As a consequence we have the following characterizations of well-posedness in terms of a class of regularizations of parameters $\lambda > 0$, $p > 1$:

Corollary 7.10. *Let X be a normed space, $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function and K be a closed convex subset of X. Suppose that there exist $\lambda > 0$, $p > 1$ such that $(f\nabla\lambda\|\cdot\|^p, K)$ is well-posed in the generalized sense of Tikhonov and there exists $z_0 \in \operatorname{argmin}(f\nabla\lambda\|\cdot\|^p, K)$ such that $(f\nabla\lambda\|\cdot\|^p)(z_0) = f(z_0)$, then (f, K) is well-posed in the strong generalized sense.*

P r o o f. The function $g(z) = (f\nabla\lambda\|\cdot\|^p)(z) = \inf_{x\in X}\{f(x) + \lambda\|z - x\|^p\}$, $z \in X$ is convex and continuous on X [9, Theorem 3.8]; and by [9, Corollary 4.5] and [11] (g, K) is well-posed in the strong generalized sense. Theorem 7.9 (2) permits to conclude the proof. □

Corollary 7.11. *Let X be a reflexive Banach space, $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function and K be a closed convex subset of X. Suppose that there exist $\lambda > 0$, $p > 1$ such that $(f\nabla\lambda\|\cdot\|^p, K)$ is well-posed in the generalized sense of Tikhonov and $\operatorname{argmin}(f\nabla\lambda\|\cdot\|^p, K) \cap D(f, K) = \varnothing$, where $D(f, K) = \bigcup\limits_{u\neq 0} (u + \operatorname{argmin}(f, K - u))$, then (f, K) is well-posed in the strong generalized sense.*

P r o o f. By Corollary 7.10 it is enough to show the existence of $z_0 \in \operatorname{argmin}(f\nabla\lambda\|\cdot\|^p, K)$ such that $(f\nabla\lambda\|\cdot\|^p)(z_0) = f(z_0)$. Let $z_0 \in \operatorname{argmin}(f\nabla\lambda\|\cdot\|^p, K)$. By reflexivity, convexity and coercivity there exists $u_0 \in X$ such that $(f\nabla\lambda\|\cdot\|^p)(z_0) = f(z_0 - u_0) + \lambda\|u_0\|^p$, so

$$f(z_0 - u_0) + \lambda\|u_0\|^p \leq (f\nabla\lambda\|\cdot\|^p)(z) \leq f(z - u_0) + \lambda\|u_0\|^p \quad \forall\, z \in K.$$

Consequently $z_0 \in u_0 + \operatorname{argmin}(f, K - u_0)$. If $u_0 \neq 0$, then $z_0 \in D(f, K)$ which is a contradiction. Then $u_0 = 0$ and $(f\nabla\lambda\|\cdot\|^p)(z_0) = f(z_0)$ which completes the proof. □

Remark 7.12. If (f, K), (g, K') are well-posed in the generalized sense of Levitin-Polyak it is not true that $(f\nabla g, K + K')$ is well-posed in the same sense: consider $f(x) = 0$, $g(x) = x^2$, $x \in \mathbb{R}$, $K = [0, 1]$, $K' = [0, +\infty[$. It is clear that (f, K), (g, K') are well-posed in the generalized sense of Levitin-Polyak but $(f\nabla g, K + K')$ is not well-posed in the same sense because $(f\nabla g)(z) = \inf_{x\in\mathbb{R}}\{(z - x)^2\} = 0, z \in \mathbb{R}$ and $K + K' = [0, +\infty[$.

8. Levitin-Polyak well-posedness, strong well-posedness and generalized minimizing sequences.

Theorem 8.1. *Let (X, d) be a metric space, K be a subset of X and $f : X \to \overline{\mathbb{R}}$ be a function. The following assertions are equivalent:*
(i) (f, K) is Levitin-Polyak well-posed.
(ii) There exist $x_0 \in K$ and a forcing function c such that

$$\begin{aligned} &f(x) - f(x_0) + |f(y) - f(x_0)| + d(y, K) \geq c[d(x, x_0) + d(y, x_0)], \\ &\forall\, x \in K,\ \forall\, y \in X. \end{aligned} \tag{8.1}$$

(iii) There exist $x_0 \in K$ and a forcing function c such that $f(x_0) = v(f, K)$ and

$$|f(x) - f(x_0)| + d(x, K) \geq c[d(x, x_0)],\ \forall\, x \in X. \tag{8.2}$$

(iv) There exists $x_0 \in K$ such that the minimization problem (Ω, X) is Tikhonov well-posed with solution x_0 and $\Omega(x) = |f(x) - v(f, K)| + d(x, K)$, $\forall\, x \in X$.

P r o o f. $(i) \Longrightarrow (ii)$. Let x_0 be the unique minimizer of problem (f, K) and consider the function c given by

$$\begin{aligned} c(t) = \inf\{f(x) - f(x_0) + |f(y) - f(x_0)| + d(y, K) \mid x \in K, y \in X \\ \text{and}\ \ t = d(x, x_0) + d(y, x_0)\}. \end{aligned}$$

It is clear that c is a forcing function satisfying (8.1).

$(ii) \Longrightarrow (iii)$. Taking $x = x_0$ in (8.1) we get (8.2). Afterwards, taking $y = x_0$ in (8.1) we have $f(x_0) = v(f, K)$.

$(iii) \Longrightarrow (iv)$. We have $\Omega(x_0) = \min_X \Omega = 0$. Now if $(x_n)_n$ is a sequence of X such that $\Omega(x_n) \to 0$, then $d(x_n, x_0) \to 0$ by (8.2).

$(iv) \Longrightarrow (i)$ is obvious. □

As an immediate consequence, we have the following corollary:

Corollary 8.2. *Let (X, d) be a metric space, K be a subset of X and $f : X \to \overline{\mathbb{R}}$ be a function. The following assertions are equivalent:*
(i) (f, K) is Levitin-Polyak well-posed.
(ii) There exist $x_0 \in K$, a forcing function c and two functions $\psi : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to [0, +\infty]$, $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, +\infty]$ with ψ being continuous at 0, $\psi(0) = 0$ and $\varphi(0) = 0$ satisfying $\varphi(z_n) \to 0 \Longrightarrow z_n \to 0$ such that $\min_{x \in K} f(x) = f(x_0)$ and

$$\psi(f(x)-f(x_0), f(y)-f(x_0), d(y, K)) \geq c(\varphi(d(x, x_0), d(y, x_0))),\ \forall\, x \in K,\ \forall\, y \in X.$$

(iii) There exist $x_0 \in K$, a forcing function c and two functions $\delta : \mathbb{R} \times \mathbb{R}^+ \to [0, +\infty]$, $r : \mathbb{R}^+ \to [0, +\infty]$ with δ being continuous at 0, $\delta(0) = 0$ and $r(0) = 0$ satisfying $r(z_n) \to 0 \Longrightarrow z_n \to 0$ such that $\min_{x \in K} f(x) = f(x_0)$ and

$$\delta(f(x) - f(x_0), d(x, K)) \geq c(r(d(x, x_0))),\ \forall\, x \in X.$$

(iv) There exist $x_0 \in K$ and a function $\alpha : \mathbb{R} \times \mathbb{R}^+ \to [0, +\infty]$ with α being continuous at 0, $\alpha(0) = 0$ and $\alpha(z_n) \to 0 \Longrightarrow z_n \to 0$ such that (M, X) is well-posed in the Tikhonov sense with solution $x_0 \in K$ where $M(x) = \alpha(f(x) - \inf_K f, d(x, K))$, $x \in X$.

Corollary 8.3. *Let (X, d) be a metric space, K be a subset of X and $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function on K. The following assertions are equivalent:*
(i) (f, K) is Levitin-Polyak well-posed.
(ii) $\exists x_0 \in K$ and an increasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ with $q(t) \to 0$ if $t \to 0$ such that $\forall\, x \in K$, $\forall\, y \in X$ one has:

$$\max[d(x, x_0), d(y, x_0)] \leq q\left[f(x) - \inf_K f + \left|f(y) - \inf_K f\right| + d(y, K)\right]. \tag{8.3}$$

(iii) $\exists x_0 \in K$ and an increasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ with $q(t) \to 0$ if $t \to 0$ such that $\forall\, x \in X$:

$$d(x, x_0) \leq q\left[|f(x) - \inf_K f| + d(x, K)\right].$$

(iv) $\exists x_0 \in K$ such that for every $\epsilon > 0$, $\exists\ \delta > 0$:

$$\max\left[|f(x) - \inf_K f|, d(x, K)\right] < \delta \Longrightarrow d(x, x_0) < \epsilon. \tag{8.4}$$

Proof. $(i) \Rightarrow (ii)$ It is easy to see that (f, K) is Levitin-Polyak well-posed with unique minimizer x_0 on K if and only if the problem $(h, K \times X)$ is Tikhonov well-posed with solution (x_0, x_0) on $K \times X$, where $h(x, y) = f(x) - f(x_0) + |f(y) - f(x_0)| + d(y, K)$, $(x, y) \in K \times X$; and by [21] there exists an increasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ with $q(t) \to 0$, $t \to 0$ satisfying (8.3).

$(ii) \Rightarrow (iii)$ It is enough to check that $f(x_0) = \min_K f$. Indeed, there exists a sequence $(x_n)_n$ in K such that $f(x_n) \to \inf_K f$. Now taking $x = y = x_n$ in (8.3), we get $d(x_n, x_0) \leq q[2(f(x_n) - \inf_K f)]$, so $x_n \to x_0$ and by lower semicontinuity we get $f(x_0) = \inf_K f = \min_K f$.

$(iii) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (i)$ As in the proof of $(ii) \Rightarrow (iii)$ we check that $f(x_0) = \min_K f$. Now, if $(x_n)_n$ is a sequence satisfying $d(x_n, K) \to 0$ and $f(x_n) \to f(x_0)$, then $d(x_n, x_0) \to 0$ by (8.4). □

Theorem 8.4. *Let (X, d) be a complete metric space, K be a closed subset of X and $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function at every point of K. The following assertions are equivalent:*

(i) *(f, K) is Levitin-Polyak well-posed;*

(ii) $\operatorname{diam} L'(f, K, \epsilon) \to 0$ *if* $\epsilon \to 0$.

Proof. $(i) \Rightarrow (ii)$ By Theorem 8.1 if (f, K) is Levitin-Polyak well-posed with solution x_0, then (Ω, X) is Tikhonov well-posed so $\operatorname{diam}(\epsilon{-}\operatorname{argmin}(\Omega, X)) \to 0$ if $\epsilon \to 0$ [21, Theorem 11, p. 5] which is equivalent to $\operatorname{diam} L'(f, K, \epsilon) \to 0$ if $\epsilon \to 0$ because $L'(f, K, \frac{\epsilon}{2}) \subset \epsilon - \operatorname{argmin}(\Omega, X) \subset L'(f, K, \epsilon)$.

$(ii) \Rightarrow (i)$ Let $(x_n)_n$ be a sequence such that $f(x_n) \to v(f, K)$ and $d(x_n, K) \to 0$. For every $\epsilon > 0$ there exists $\delta > 0$ such that $\operatorname{diam} L'(f, K, \delta) < \epsilon$; then for all n large enough $x_n \in L'(f, K, \delta)$ and $(x_n)_n$ is a Cauchy sequence converging to an element x^*. Now K is closed and f is lsc at every point of K, so we have $x^* \in \operatorname{argmin}(f, K)$. The uniqueness of x^* is an immediate consequence of the inclusion $\operatorname{argmin}(f, K) \subset L'(f, K, \epsilon)$ and (ii) which completes the proof. □

Corollary 8.5 ([21])**.** *Let (X, d) be a complete metric space, K be a closed subset of X and $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function at every point of K. If* $\operatorname{diam} L(f, K, \epsilon) \to 0$ *when $\epsilon \to 0$, then (f, K) is Levitin-Polyak well-posed.*

Proof. The proof is immediate by the following inclusion $L'(f, K, \epsilon) \subset L(f, K, \epsilon)$ and by Theorem 8.4. □

Remark 8.6. In general if (f, K) is Levitin-Polyak well-posed, it is not true that $\operatorname{diam} L(f, K, \epsilon) \to 0$ when $\epsilon \to 0$. An example of such case is considered in [21].

Theorem 8.7. *Under the hypotheses of Theorem* 8.4*, the following assertions are equivalent:*

(i) (f,K) *is Levitin-Polyak well-posed;*

(ii) *There exists a forcing function* c *satisfying:*

$$c(d(x,y)) \leq \max\left[\left|f(x) - \inf_K f\right| + d(x,K), \left|f(y) - \inf_K f\right| + d(y,K)\right] \quad \forall\, x,y \in X. \tag{8.5}$$

Proof. $(i) \Longrightarrow (ii)$ Condition (i) implies that (Ω, X) is Tikhonov well-posed. But $\min_X \Omega = 0$, so there exists by [21] a forcing function satisfying (8.5).

$(ii) \Longrightarrow (i)$ Pick $\epsilon > 0$ and $x, y \in \epsilon - \operatorname{argmin}(\Omega, X)$, then $\max(\Omega(x), \Omega(y)) \leq \epsilon$. By (8.5) one has $c(d(x,y)) \leq \epsilon$. Using Lemma 20 in [21], the function defined by $q(s) = \sup\{t \geq 0 : c(t) \leq s\}$, $s \geq 0$ is increasing and verifies $q(s) \to 0$ if $s \to 0$ and $t \leq q(c(t))$; accordingly $d(x,y) \leq q(c(d(x,y))) \leq q(\epsilon)$, $\operatorname{diam}(\epsilon - \operatorname{argmin}(\Omega, X)) \leq q(\epsilon)$ and then $\operatorname{diam}(\epsilon - \operatorname{argmin}(\Omega, X)) \to 0$ if $\epsilon \to 0$. But we know that $L'(f, K, \frac{\epsilon}{2}) \subset \epsilon - \operatorname{argmin}(\Omega, X) \subset L'(f, K, \epsilon)$. Theorem 8.4 permits to conclude the proof. □

Proposition 8.8. *Let* (X,d) *be a metric space,* K *be a closed subset of* X *and* $f : X \to \overline{\mathbb{R}}$ *be a function. The following assertions are equivalent:*

(i) (f,K) *is Levitin-Polyak well-posed;*

(ii) $\operatorname{argmin}(f,K)$ *contains a unique minimizer and the multifunction* $\Gamma : t \in \mathbb{R}^+ \rightrightarrows L'(f,K,t)$ *is upper semicontinuous at* 0.

Moreover if K *is compact and* f *is lower semicontinuous at every point of* K*, the previous assertions are equivalent to*

(iii) $\operatorname{argmin}(f,K)$ *is a singleton.*

Proof. $(i) \Longleftrightarrow (ii)$ This equivalence is an immediate consequence of Theorem 8.1, Proposition 22 in [21, p. 12].

$(iii) \Longrightarrow (i)$ may be shown exactly as in the proof of Theorem 23 in [21, p. 13] replacing minimizing sequences by Levitin-Polyak generalized minimizing sequences. □

Proposition 8.9. *Let* (X,d) *be a metric space,* C *be a nonempty closed subset of* X *and* $f : X \to [-\infty, +\infty]$ *be a function. Then* (f,C) *is well-posed in the generalized sense of Levitin-Polyak (resp. well-posed in the strong generalized sense) if and only if* $\operatorname{argmin}(f,C)$ *is compact and the multifunction* $\epsilon \rightrightarrows L'(f,C,\epsilon)$ *(resp.,* $\epsilon \rightrightarrows L(f,C,\epsilon)$*) is upper semicontinuous at* 0.

Proof. If (f,C) is well-posed in the generalized sense of Levitin-Polyak, it is clear that $\operatorname{argmin}(f,C)$ is compact. Now if $\epsilon \rightrightarrows L'(f,C,\epsilon)$ fails to be upper semicontinous (usc) at 0, there exist an open subset θ of X containing $\mathrm{argmin}(f, C)$, a sequence $(t_n)_n$ of positive numbers converging to 0 and a sequence $(x_n)_n, x_n \in L'(f, C, t_n)$ with $x_n \notin \theta$. But $d(x_n, C) \to 0$ and $f(x_n) \to \upsilon(f, C)$, so $(x_n)_n$ has a subsequence converging to an element of $\mathrm{argmin}(f, C)$; this is a contradiction because $x_n \notin \theta$. Conversely, let $(x_n)_n$ be a sequence of X such that $d(x_n, C) \to 0$, $f(x_n) \to \upsilon(f, C)$ and pick $\epsilon > 0$. The usc at 0 implies that $x_n \in (\mathrm{argmin}(f, C))_\epsilon$ for all n sufficiently large, so $d(x_n, \mathrm{argmin}(f, C)) \to 0$ and by compactness of $\mathrm{argmin}(f, C)$, $(x_n)_n$ has a subsequence converging to an element of $\mathrm{argmin}(f, C)$. In the same way we show the second equivalence replacing $L'(f, C, \epsilon)$ by $L(f, C, \epsilon)$. □

Proposition 8.10. *Let (X, d) be a metric space locally compact, C be a nonempty closed subset of X and $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous function. Suppose that for every $\epsilon > 0$, $L(f, C, \epsilon)$ is connected. The following assertions are equivalent:*

(i) $L(f, C, \epsilon_0)$ is compact for some $\epsilon_0 > 0$.

(ii) (f, C) is well-posed in the strong generalized sense.

(iii) $\mathrm{argmin}(f, C)$ *is a nonempty compact.*

Proof. First we point out that (f, C) is well-posed in the strong generalized sense if and only if (g, X) is well-posed in the Tikhonov generalized sense with $g(x) = \max(f(x) - \upsilon(f, C), d(x, C))$. Afterwards, we apply Theorem 2.1 from [8] to obtain the previous equivalences. □

Theorem 8.11. *Let (X, d) be a metric complete space, K be a nonempty closed subset of X and $f : X \to \mathbb{R}$ be a continuous function. Then (f, K) is well-posed in the generalized sense of Levitin-Polyak if and only if $\alpha(L'(f, K, \epsilon)) \to 0$ if $\epsilon \to 0$.*

Proof. If (f, K) is well-posed in the generalized sense of Levitin-Polyak, we claim that $L'(f, K, \epsilon_0)$ is bounded for some positive number ϵ_0. If this is not the case, $L'(f, K, \epsilon)$ is unbounded for every positive number ϵ, so there exists a sequence $x_n \in L'(f, K, \frac{1}{n})$ such that $d(x_n, x_0) \geq n$, where x_0 is a fixed point in X; this is a contradiction because $(x_n)_n$ has a converging subsequence; accordingly $L'(f, K, \epsilon)$ is bounded for every small positive ϵ and $\alpha(L'(f, C, \epsilon))$ exists. We remark also that (f, K) is well-posed in the generalized sense of Levitin-Polyak if and only if (Ω, X) is well-posed in the generalized sense of Tikhonov. Then the conclusion of the theorem is an immediate consequence of [21, Theorem 38, p. 25] and the trivial inclusion $L'(f, K, \frac{\epsilon}{2}) \subset \epsilon - \mathrm{argmin}(\Omega, X) \subset L'(f, K, \epsilon)$. □

Theorem 8.12. *Let (X, d) be a metric complete space, C be a nonempty* *closed subset of* X *and* $f : X \to \mathbb{R}$ *be a lower semicontinuous function. Then* (f, C) *is well-posed in the strong generalized sense if and only if* $\alpha(L(f, C, \epsilon)) \to 0$ *if* $\epsilon \to 0$.

Proof. It is clear that (f, C) is well-posed in the strong generalized sense if and only if (g, X) is well-posed in the generalized sense of Tikhonov. On the other hand, g is lower semicontinuous and X is complete, then [21, Theorem 38, p. 25] permits to conclude the proof, because for all $\epsilon > 0$ we have $\epsilon-\text{argmin}(g, X) = L(f, C, \epsilon)$. $\square$

It should be pointed out that Theorem 8.12 has been shown by Revalski and Zhivkov in [65]. The use of g has greatly simplified the proof.

Theorem 8.13. *Let* (X, d) *be a complete metric space,* C *be a nonempty closed bounded subset of* X *and* $f : X \to]-\infty, +\infty]$ *be a function.*

1) *If* f *is finite and continuous, then* (f, C) *is well-posed in the generalized sense of Levitin-Polyak if and only if there exits a forcing function* c *such that for every bounded subset* A *of* X *satisfying* $\sup_{x\in A} |f(x)| < +\infty$ *one has:*

$$c(\alpha(A)) \leq \max\left(\sup_{x\in A} |f(x) - v(f, C)|, \sup_{x\in A} d(x, C)\right) \tag{8.6}$$

2) *If* f *is lower semicontinuous, then* (f, C) *is well-posed in the strong generalized sense if and only if there exists a forcing function* c *verifying for every bounded subset* A *of* X*, such that* $\sup_{x\in A} f(x) < +\infty$*:*

$$c(\alpha(A)) \leq \max\left(\sup_{x\in A} f(x) - v(f, C), \sup_{x\in A} d(x, C)\right) \tag{8.7}$$

In particular (f, C) *is well-posed in the strong generalized sense if there exits a forcing function* c *such that:*

$$c(\alpha(A)) \leq \sup_{x\in A}(f(x) - v(f, C)) \tag{8.8}$$

for every bounded subset A *of* X*, such that* $A \cap C \neq \varnothing$ *and* $\sup_{x\in A} f(x) < +\infty$.

Proof. We will use some arguments of the proof of Theorem 39 [21, p. 26].

1) Assume that (f, C) is well-posed in the generalized sense of Levitin-Polyak, then $\alpha(L'(f, C, \epsilon)) \to 0$ if $\epsilon \to 0$ by Theorem 8.11. $Set q(\epsilon) = \alpha(L'(f, C, \epsilon))$ and $c(t) = \inf\{s \geq 0 : q(s) \geq t\}$. By [21, Lemma 20] c is an increasing forcing function satisfying $c(q(\epsilon)) \leq \epsilon$ for every $\epsilon \geq 0$. Let A be a bounded subset of X such that $\sup_{x\in A} |f(x)| < +\infty$ and consider $p = \max(\sup_{x\in A} |f(x) - v(f, C)|, \sup_{x\in A} d(x, C))$; we have $p \geq 0$ and $A \subset L'(f, C, p)$, so $c(\alpha(A)) \leq c(\alpha(L'(f, C, p)) = c(q(p)) \leq p$ and then (8.6) is satisfied. Conversely if (8.6)

holds for every bounded subset A of X such that $\sup_{x\in A}|f(x)| < +\infty$; set in particular $A = L'(f, C, \epsilon)$ which is clearly bounded so $c(\alpha(L'(f, C, \epsilon)) \leq \epsilon$, accordingly $\alpha(L'(f, C, \epsilon)) \to 0$ if $\epsilon \to 0$. Theorem 8.11 permits to conclude the proof.

2) This equivalence can be shown in the same way as point 1) using Theorem 8.12. Now suppose that (8.8) is satisfied for every bounded set A, such that $A\cap C \neq \varnothing$ and $\sup_{x\in A} f(x) < +\infty$. We will show that (8.7) is satisfied. Let B be a bounded set with $\sup_{x\in B} f(x) < +\infty$. We may always assume that c is an increasing function. If $B\cap C \neq \varnothing$ we have always

$$c(\alpha(B)) \leq \sup_{x\in B}(f(x) - v(f, C)) \leq \max\left(\sup_{x\in B}|f(x) - v(f, C)|, \sup_{x\in B} d(x, C)\right)$$

and (8.7) is satisfied. If $B \cap C = \varnothing$, we consider two cases: if there exists $(a, b) \in C \times B$ such that $f(a) \leq f(b)$, then set $B' = B \cup \{a\}$; we have

$$c(\alpha(B)) \leq c(\alpha(B')) \leq \sup_{x\in B'}(f(x) - v(f, C)) = \sup_{x\in B}(f(x) - v(f, C))$$

and (8.7) is satisfied. If for every $(a, b) \in C \times B f(b) < f(a)$, $setB(a) = B \cup \{a\}$ for every $a \in C$ such that $f(a)$ is finite; then by (8.8) we have

$$c(\alpha(B)) \leq c(\alpha(B(a))) \leq \sup_{x\in B(a)}(f(x) - v(f, C)) = f(a) - v(f, C),$$

so $c(\alpha(B)) = 0$ and again (8.7) is satisfied which completes the proof. □

Theorem 8.14. *Let (X, d) be a metric space, C be a nonempty subset of X and $f : X \to \mathbb{R}$ be a function. Assume that there exists a forcing function $\alpha : \mathbb{R} \to \mathbb{R}^+$ continuous at 0, $\alpha(0) = 0$ such that $\alpha(f - v(f, C))$ is uniformly continuous. If (f, C) is well-posed in the generalized sense of Levitin-Polyak, then we have the following implication (P) : for every sequence of functions $f_n : X \to \mathbb{R}$ converging uniformly to f on X and for every sequences $(C_n)_n$, $(X_n)_n$ of subsets of X such that $d_H(C_n, C) \to 0$, $d_H(X_n, X) \to 0$ and $x_n \in \operatorname{argmin}(f_n + d(\cdot, C_n), X_n)$ then $d(x_n, \operatorname{argmin}(f, C)) \to 0$ when $n \to +\infty$. Conversely, if the last implication is true and $\operatorname{argmin}(f, C)$ is a nonempty compact, then (f, C) is well-posed in the generalized sense of Levitin-Polyak.*

P r o o f. If (f, C) is well-posed in the generalized sense of Levitin-Polyak, it is easy to see that (R, X) is well-posed in the generalized sense of Tikhonov and $\operatorname{argmin}(f, C) = \operatorname{argmin}(R, X)$ with $R(x) = \alpha(f(x) - v(f, C)) + d(x, C)$. By hypothesis $R(x)$ is uniformly continuous, so (R, X) is well-posed in the generalized sense of Hadamard (see [65] and references therein) and (P) is satisfied. Conversely, if (P) is satisfied, let be a sequence $(x_{n)_n}$ of X verifying $d(x_n, C) \to 0$ and $f(x_n) \to \min_C f$. Set $f_n(x) = |\alpha(f(x) - \min_C f) - \alpha(f(x_n) - \min_C f)|$,

$C_n = C \cup \{x_n\}$ and $X_n = X$. It is clear that $(f_n)_n$ converges uniformly to $\alpha(f - \min_C f)$ on X, $d_H(C_n, C) \to 0$, $d_H(X_n, X) = 0$ and $x_n \in \operatorname{argmin}(f_n + d(\cdot, C_n), X)$; so $d(x_n, \operatorname{argmin}(f, C)) \to 0$ when $n \to +\infty$, which completes the proof. $\square$

9. Well-posedness, geometrical interpretation and subdifferentiability. In this section we provide several characterizations of well-posedness via epigraphical analysis and subdifferentiability.

Theorem 9.1. *Let X be a normed space with its topological dual X^*. Let K be a nonempty subset of X and $f : X \to]-\infty, +\infty]$ be a proper function on K. Consider the function $F(x) = f(x) - \langle x, x'\rangle$, $x' \in X^*$. Then*

1) (F, K) is Tikhonov well-posed with solution $x_0 \in K$ if and only if the form $(x', -1) \in X^* \times \mathbb{R}$ exposes strongly $(x_0, f(x_0))$ on $\operatorname{epi} f_K$. Consequently, (f, K) is Tikhonov well-posed if and only if the form $(0_{X^*}, -1)$ exposes strongly $(x_0, f(x_0))$ on $\operatorname{epi} f_K$.

2) If K is closed, then (f, K) is Levitin-Polyak well-posed with solution $t_0 \in K$ if and only if there exists a forcing function $\alpha : \mathbb{R} \to [0, +\infty[$ continuous at 0 with $\alpha(0) = 0$ such that the form $(0_{X^*}, -1)$ exposes strongly $(t_0, 0)$ on $\operatorname{epi} \alpha(f - v(f, K)) \cap \operatorname{epi} d(\cdot, K)$.

3) If K is closed, then (f, K) is strongly well-posed with solution $z_0 \in K$ if and only if there exists an increasing function $\beta : \mathbb{R} \to \mathbb{R}$ such that $(0_{X^*}, -1)$ exposes strongly $(z_0, 0)$ on $\operatorname{epi} \beta(f - v(f, K)) \cap \operatorname{epi} d(\cdot, K)$ where β is continuous at 0 with $\beta(0) = 0$ and the condition $\overline{\lim}\, \beta(t_n) \le 0$ implies that $\overline{\lim}\, t_n \le 0$.

Proof. 1) Assume that $(x', -1)$ exposes strongly $(x_0, f(x_0))$ on $\operatorname{epi} f_K$ and consider $(x_n)_n$ a sequence of K such that $F(x_n) \to \inf_K F$, then

$$\langle x_n, x'\rangle - f(x_n) \to \sup\{\langle x, x'\rangle - f(x) : x \in K\}$$

which is equivalent to

$$\begin{aligned}\langle (x', -1), (x_n, f(x_n))\rangle \to \sup\{\langle (x', -1), (x, \lambda)\rangle : (x, \lambda) \in \operatorname{epi} f_K\}\\ = \langle (x', -1), (x_0, f(x_0))\rangle,\end{aligned}$$

so we have $(x_n, f(x_n)) \in \operatorname{epi} f_K$ and $(x_n, f(x_n)) \to (x_0, f(x_0))$; consequently (F, K) is well-posed in the Tikhonov sense with solution x_0. Conversely assume that (F, K) is well-posed in the Tikhonov sense with solution x_0. Consider a sequence $(x_n, \lambda_n) \in \operatorname{epi} f_K$ such that

$$\langle (x', -1), (x_n, \lambda_n)\rangle \to sup\{\langle (x', -1), (x, \lambda)\rangle : (x, \lambda) \in \operatorname{epi} f_K\}.$$

Then

$$\lambda_n - \langle x', x_n\rangle \to \inf\{\lambda - \langle x', x\rangle : (x, \lambda) \in \operatorname{epi} f_K\} = \inf(F, K) = F(x_0).$$

But

$$F(x_0) \leq F(x_n) = f(x_n) - \langle x', x_n \rangle \leq \lambda_n - \langle x', x_n \rangle,$$

so $F(x_n) \to F(x_0)$, afterwards $(x_n, \lambda_n) \to (x_0, f(x_0))$; moreover $f(x_n) \to f(x_0)$.

Now the proofs of the equivalences of 2) and 3) are an immediate consequence of 1) because with the hypotheses under consideration, it is easy to see that the Levitin-Polyak well-posedness of (f, K) with solution t_0 (resp., the strong well-posedness of (f, K) with solution z_0) is equivalent to the Tikhonov well-posedness of (p, X) with solution t_0 and $p(t_0) = 0$, where $p(x) = \max(\alpha(f - v(f, K)), d(x, K))$ (resp., is equivalent to the Tikhonov well-posedness of (q, X) with solution z_0 and $q(z_0) = 0$, where $q(x) = \max(\beta(f - v(f, K)), d(x, K)))$. □

Now we will state some results of well- posedness in relationship with the notion of subdifferentiability.

Proposition 9.2. *Let X be a normed space, $f : X \to \overline{\mathbb{R}}$ be a function and K be a nonempty subset of X. Assume that the following hypotheses hold: there exist a convex subset A of X containing K and a function $\varphi : \mathbb{R} \times \mathbb{R}^+ \to [0, +\infty]$ satisfying:*

a) φ is continuous at 0 with $\varphi(0) = 0$ and $\varphi(z_n) \to 0 \Longrightarrow z_n \to 0$;

b) $\partial H(x) \neq \varnothing$, $\forall\, x \in A_\epsilon$ for some $\epsilon > 0$, where H is the function defined by $H(x) = \varphi(f(x) - v(f, K), d(x, K))$, $x \in X$.

Then (f, K) is Levitin-Polyak well-posed if and only if there exist $x_0 \in K$ and a forcing function c such that $\langle u, x - x_0 \rangle \geq c(\|x - x_0\|)$, $\forall\, u \in \partial H(x)$, $\forall\, x \in A_\epsilon$.

Proof. By [80, Theorem 7] (see also [21, Theorem 25, p. 14]) the inequality above is equivalent to the Tikhonov well-posedness of (H, A_ϵ) with solution $x_0 \in K \subset A_\epsilon$, which is also equivalent to the Levitin-Polyak well-posedness of (f, K) with solution x_0 by Corollary 8.2 (iv). □

Corollary 9.3. *Let X be a normed space of norm assumed to be Fréchet differentiable at every $x \neq 0$, $f : X \to \overline{\mathbb{R}}$ be a function and K be a nonempty convex closed subset of X. Suppose that there exist $\epsilon > 0$ and a forcing function $\alpha : \mathbb{R} \to [0, +\infty]$ continuous at 0 with $\alpha(0) = 0$ such that:*

a) $\partial m(x) \neq \varnothing$ $\forall\, x \in K_\epsilon$, where m is the function defined by $m(x) = \alpha(f(x) - v(f, K))$, $x \in X$;

b) m is convex and lower semicontinuous;

c) $\forall\, x \in K_\epsilon$, $\operatorname{proj}_K x$ exists and is unique.

Then (f, K) is Levitin-Polyak well-posed if and only if there exist $x_0 \in K$ and a forcing function c such that

$$(9.1)\quad \langle u, x - x_0\rangle \geq c(\|x - x_0\|) - \langle y, x - x_0\rangle, \ \forall\, u \in \partial m(x), \\ \forall\, y \in T(x - \mathrm{proj}_K\, x), \ \forall\, x \in K_\epsilon.$$

Moreover if X is a Hilbert space, α is C^1 and f is continuous at every point of K_ϵ, then (9.1) *is equivalent to*

$$(9.2)\quad \alpha'(f(x) - v(f, K))\langle u, x - x_0\rangle \geq c(\|x - x_0\|) - \langle x - \mathrm{proj}_K\, x, x - x_0\rangle, \\ \forall\, u \in \partial f(x), \ \forall\, x \in K_\epsilon.$$

Proof. It is enough to apply Proposition 9.2 with $\varphi(x, y) = \alpha(x) + \frac{1}{2}y^2$ and to point out that $\partial H(x) = \partial m(x) + T(x - \mathrm{proj}_K\, x)$ (see [21, Lemma 11, p. 52] and [35, Theorem 6.6.7]). The equivalence between (9.1) and (9.2) arises from the fact that $\partial m(x) = \alpha'(f(x) - v(f, K)).\partial f(x)$ (see [17, 35] and references therein). □

Remark 9.4. Function H in Proposition 9.2 is not necessarily convex. If X is an E-space (particulary when E is a Hilbert space), then hypothesis c) of Corollary 9.3 is satisfied and $T(x - \mathrm{proj}_K\, x) = \{\theta(x - \mathrm{proj}_K\, x)\}$ [2, 21], where $\theta(x)$ is the Fréchet derivative of $n(x) = \frac{1}{2}\|x\|^2$.

Example 9.5. a) Consider $f(x) = x^3$, $K = [0, 1]$, $\alpha(x) = x^2$ and $m(x) = x^6$. All hypotheses of Corollary 9.3 are satisfied, so one may take the forcing function c as follows:

$$c(x) = \begin{cases} x^6 & \text{if } x \in [0, 1], \\ x^6 + x(x - 1) & \text{if } x \geq 1, \\ x^6 + x^2 & \text{if } x \leq 0. \end{cases}$$

b) if $K = [-1, 1]$, $f(x) = \sqrt{|x|} - \frac{1}{2}$ and $\alpha(x) = x^4$, then $m(x) = \alpha(f(x) - v(f, K)) = x^2$ is convex, so Corollary 9.3 applies.

Theorem 9.6. *Let X be a normed space of norm assumed to be Fréchet differentiable at every $x \neq 0$, $f : X \to]-\infty, +\infty]$ be a proper convex and lower semicontinuous function, K be a nonempty convex closed subset of X. The following assertions are equivalent:*

(i) (f, K) is strongly well-posed with solution x_0.

(ii) (g, X) is Tikhonov well-posed with solution x_0, where g is the function defined by $g(x) = \max(f(x) - v(f, K), d(x, K))$.

(iii) The function defined by

$$x' \in X^* \to h(x') = \inf\{\lambda \in \mathbb{R} : (x', \lambda) \in \mathrm{conv}(B)\}$$

where

$$B = [\mathrm{epi}\, f^* + (0, v(f, K))] \bigcup \left[\mathrm{epi}(\delta_K)^* \bigcap B_{X^*}(0, 1) \times \mathbb{R}\right]$$

is Fréchet differentiable at 0 *and* $h'(0) = x_0$.

Moreover if there exists $\epsilon > 0$ *such that* f *is continuous at every point of* K_ϵ *and* $\forall\, x \in K_\epsilon$, $\operatorname{proj}_K x$ *exists and is unique, then the previous assertions are equivalent to*

(iv) *There exist a forcing function* $c : \mathbb{R}^+ \to [0, +\infty]$ *and a point* $x_0 \in X$ *satisfying* $\langle u, x - x_0\rangle \geq c(\|x - x_0\|)\ \forall\, u \in \partial f(x)$ *if* $x \in K$ *or* $x \in K_\epsilon \setminus K$ *and*

$$f(x) \geq \frac{1}{2} d^2(x, K) + \upsilon(f, K); \langle w, x - x_0\rangle \geq c(\|x - x_0\|)$$

$\forall\, w \in T(x - \operatorname{proj}_K x)$ *if* $x \in K_\epsilon \setminus K$ *and* $f(x) \leq \frac{1}{2} d^2(x, K) + \upsilon(f, K)$.

Proof. Equivalence $(i) \iff (ii)$ uses classical arguments and can be omitted.

$(ii) \iff (iii)$ A simple calculation shows that $(d(\cdot, K))^* = \delta_{B_{X^*}(0,1)} + (\delta_K)^*$ and by [34] one has $g^*(x') = h(x')$ for every $x' \in X^*$. Then $(ii) \iff (iii)$ is an immediate consequence of the characterization of Tikhonov well-posedness by Asplund-Rockafellar theorem (see [21, Theorem 27, p. 15] and references therein).

$(i) \Leftrightarrow (iv)$ Set $r(x) = \max(f(x) - \upsilon(f, K), \frac{1}{2} d^2(x, K))$. It is clear that (f, K) is strongly well-posed with solution x_0 if and only if (r, X) is Tikhonov well-posed with solution x_0. Then equivalence $(i) \Leftrightarrow (iv)$ arises easily from [21, Theorem 25, p. 14; Lemma 11, p. 52], [35, Theorem 6.4.9, p. 355] and [17, Proposition 2.3.12, p. 47]. □

Theorem 9.7. *Let* X *be a normed space,* $f : X \to \mathbb{R}$ *be a real-valued function and* K *be a closed convex set of* X. *Assume that there exists a forcing function* $\alpha : \mathbb{R} \to \mathbb{R}^+$ *continuous at* 0 *with* $\alpha(0) = 0$ *such that* $\alpha(f(\cdot) - \upsilon(f, K))$ *is convex and lsc on* X. *Then* (f, K) *is Levitin-Polyak well-posed with solution* x_0 *if and only if the function* $\gamma(x') = \inf\{\lambda \in \mathbb{R}\ /\ (x', \lambda) \in \operatorname{conv} C\}$, $x' \in X^*$ *is Fréchet differentiable at* 0 *with* $\gamma'(0) = x_0$. *Here*

$$C = \operatorname{epi}\left[\alpha(f(\cdot) - \upsilon(f, K))\right]^* \bigcup \left[\operatorname{epi}(\delta_K)^* \bigcap B_{X^*}(0, 1) \times \mathbb{R}\right].$$

Proof. It is clear that (f, K) is Levitin-Polyak well-posed with solution x_0 if and only if (ϖ, X) is Tikhonov well-posed with solution x_0, where $\varpi(x) = \max(\alpha(f(x) - \upsilon(f, K)), d(x, K))$. By [21, Theorem 27, p. 15], this is equivalent to the Fréchet differentiability of ϖ^* at 0 and $(\varpi^*)'(0) = x_0$. But ϖ^* is exactly the function γ by [35, Theorem 2, p. 178], which completes the proof of theorem □

10. New generalized regularizations for saddle functions and asymptotic developments. In what follows we are concerned by generalized regularizations of saddle functions and their associated asymptotic developments.

Consider two general topological Hausdorff spaces X, Y and $f : X \times Y \to \overline{\mathbb{R}}$, $g : X \times Y \to \mathbb{R}$, $h_\epsilon : X \times Y \to \mathbb{R}$ be three functions with $\epsilon > 0$. Each function f, g is assumed to be lower semicontinuous at the first variable and upper semicontinuous at the second variable. Denote by $h_\epsilon^1 = \sup_{y\in Y} \inf_{x\in X} h_\epsilon(x, y)$ and $h_\epsilon^2 = \inf_{x\in X} \sup_{y\in Y} h_\epsilon(x, y)$ which are supposed finite for every $\epsilon > 0$ sufficiently small. Assume that the set $S = \{(a, b) \in X \times Y \mid (a, b) \text{ is a saddle point of } f\}$ is nonempty.

Definition 10.1. *A function of the kind $F_\epsilon(x, y) = f(x, y) + a_\epsilon g(x, y) + h_\epsilon(x, y)$ with $a_\epsilon > 0$, $a_\epsilon \to 0$ if $\epsilon \to 0$ is called a generalized regularization of f.*

If $h_\epsilon = 0$ and $g(x, y) = a_i\|x\|^p - b_i\|y\|^q$, a_i, b_i are positive real numbers and p, $q \in \mathbb{N}^*$, then F_ϵ reduces to the classical Tikhonov regularization.

Using a similar technique considered in the proof of Theorem 3.2 with more difficult and sophisticated arguments we can state the following result:

Theorem 10.2. *Let $(x_\epsilon, y_\epsilon)_\epsilon$ be a relatively compact sequence such that*

$$\alpha_\epsilon = \sup_y F_\epsilon(x_\epsilon, y), \beta_\epsilon = \inf_x F_\epsilon(x, y_\epsilon), \gamma_\epsilon(t) = \sup_y h_\epsilon(t, y), \delta_\epsilon(z) = \inf_x h_\epsilon(x, z)$$

are finite for every ϵ sufficiently small and every $(t, z) \in X \times Y$. Assume that the following condition holds:

$$\lim_{\epsilon\to 0} \frac{\alpha_\epsilon - \beta_\epsilon}{a_\epsilon} = lim_{\epsilon\to 0} \frac{\gamma_\epsilon(t) - \delta_\epsilon(z)}{a_\epsilon} = 0 \quad \forall\, (t, z) \in X \times Y. \tag{10.1}$$

Then:

(i) any cluster point $(\overline{x}, \overline{y})$ of $(x_\epsilon, y_\epsilon)_\epsilon$ is a saddle point of f on $X \times Y$ and is a saddle point of g on S. Furthermore for every $\alpha \in \mathbb{R}$, there exists a sequence $(\delta_\epsilon^\alpha, \theta_\epsilon^{1,\alpha}, \theta_\epsilon^{2,\alpha}) \to 0_{\mathbb{R}^3}$ if $\epsilon \to 0$ depending on the scheme under consideration such that

$$F_\epsilon(x_\epsilon, y_\epsilon) = f(\overline{x}, \overline{y}) + a_\epsilon g(\overline{x}, \overline{y}) + \alpha h_\epsilon^1 + (1 - \alpha) h_\epsilon^2 + a_\epsilon \delta_\epsilon^\alpha$$

and the sequence

$$\left(g(x_\epsilon, \overline{y}), g(\overline{x}, y_\epsilon), \frac{f(x_\epsilon, \overline{y}) - f(\overline{x}, \overline{y})}{a_\epsilon}, \frac{f(\overline{x}, y_\epsilon) - f(\overline{x}, \overline{y})}{a_\epsilon}, \frac{h_\epsilon^2 - h_\epsilon^1}{a_\epsilon} \right)_\epsilon$$

converges to $(g(\overline{x}, \overline{y}), g(\overline{x}, \overline{y}), 0, 0, 0)$ if $\epsilon \to 0$;

(ii) $F_\epsilon^i = f(\overline{x}, \overline{y}) + a_\epsilon g(\overline{x}, \overline{y}) + \alpha h_\epsilon^1 + (1-\alpha) h_\epsilon^2 + a_\epsilon \theta_\epsilon^{i,\alpha}$ and $\lim_{\epsilon\to 0} \dfrac{F_\epsilon^2 - F_\epsilon^1}{a_\epsilon} = 0$ where $F_\epsilon^1 = \sup_{y\in Y} \inf_{x\in X} F_\epsilon(x, y)$ and $F_\epsilon^2 = \inf_{x\in X} \sup_{y\in Y} F_\epsilon(x, y)$.

Proof. (*i*) Set $p_\epsilon = \dfrac{\alpha_\epsilon - \beta_\epsilon}{a_\epsilon}$ and $q_\epsilon(t, z) = \dfrac{\gamma_\epsilon(t) - \delta_\epsilon(z)}{a_\epsilon}$. Since

$\beta_\epsilon \le F_\epsilon(x_\epsilon, y_\epsilon) \le \alpha_\epsilon$ we observe first that $\dfrac{\alpha_\epsilon - F_\epsilon(x_\epsilon, y_\epsilon)}{a_\epsilon} = c_\epsilon$ and $\dfrac{F_\epsilon(x_\epsilon, y_\epsilon) - \beta_\epsilon}{a_\epsilon} = d_\epsilon$ converge to 0 if $\epsilon \to 0$. On the other hand, $\alpha_\epsilon = F_\epsilon(x_\epsilon, y_\epsilon) + a_\epsilon c_\epsilon$ and $\forall\,(x,y) \in X \times Y$,

$$\begin{aligned} f(x_\epsilon, y) + a_\epsilon g(x_\epsilon, y) + h_\epsilon(x_\epsilon, y) &\le F_\epsilon(x_\epsilon, y_\epsilon) + a_\epsilon c_\epsilon = \beta_\epsilon + a_\epsilon p_\epsilon \\ &\le f(x, y_\epsilon) + a_\epsilon g(x, y_\epsilon) + h_\epsilon(x, y_\epsilon) + a_\epsilon p_\epsilon. \end{aligned} \tag{10.2}$$

In particular we deduce that $\forall\,(a,b) \in S$,

$$0 \le f(x_\epsilon, b) - f(a, y_\epsilon) \le a_\epsilon p_\epsilon + a_\epsilon(g(a, y_\epsilon) - g(x_\epsilon, b)) + a_\epsilon q_\epsilon(a,b) \tag{10.3}$$

and

$$g(x_\epsilon, b) \le g(a, y_\epsilon) + q_\epsilon(a,b) + p_\epsilon. \tag{10.4}$$

By relative compactness, $(x_\epsilon)_\epsilon$ and $(y_\epsilon)_\epsilon$ remain in two compacts, so by semicontinuity there exist two scalars $m(b)$ and $M(a)$, such that for every $\epsilon > 0$, $m(b) \le g(x_\epsilon, b) \le g(a, y_\epsilon) + q_\epsilon(a,b) + p_\epsilon \le M(a) + q_\epsilon(a,b) + p_\epsilon$. Then (10.1) implies that $(g(x_\epsilon, b))_\epsilon$ and $(g(a, y_\epsilon))_\epsilon$ are bounded. If $(\overline{x}, \overline{y})$ is a cluster point of $((x_\epsilon, y_\epsilon))_\epsilon$ we get by semicontinuity using (10.4), that $g(\overline{x}, b) \le g(a, \overline{y})\ \forall\,(a,b) \in S$. Now returning to (10.2) and taking into account that there exist $m(y)$ and $M(x)$ in $\mathbb{R}$ satisfying $m(y) \le \inf_{\epsilon>0} g(x_\epsilon, y)$ and $\sup_{\epsilon>0} g(x, y_\epsilon) \le M(x)$, we derive that $f(x_\epsilon, y) \le f(x, y_\epsilon) + a_\epsilon(p_\epsilon + M(x) - m(y) + q_\epsilon(x,y))$; again by semicontinuity and (10.1) we deduce that $f(\overline{x}, y) \le f(x, \overline{y})\ \forall\,(x,y) \in X \times Y$, i.e $(\overline{x}, \overline{y}) \in S$, so $(\overline{x}, \overline{y})$ is a saddle point of g on S. Keeping in mind that $(g(x_\epsilon, b))_\epsilon$ and $(g(a, y_\epsilon))_\epsilon$ are bounded, (10.1) and (10.3) imply that $f(x_\epsilon, b) - f(a, y_\epsilon) \to 0$ if $\epsilon \to 0$; accordingly $f(x_\epsilon, b) \to f(a,b) = f(\overline{x}, b) = f(\overline{x}, \overline{y})$ and $f(a, y_\epsilon) \to f(a,b) = f(a, \overline{y})$. In particular $f(x_\epsilon, \overline{y}) \to f(\overline{x}, \overline{y})$ and $f(\overline{x}, y_\epsilon) \to f(\overline{x}, \overline{y})$ if $\epsilon \to 0$. Now to prove the asymptotic development, we use (10.2) with $(x,y) = (a,b) \in S$ and the following computations:

$$\begin{aligned} &f(x_\epsilon, b) - f(a,b) + a_\epsilon g(x_\epsilon, b) + h_\epsilon(x_\epsilon, b) - \inf_x h_\epsilon(x,b) \\ &\qquad \le F_\epsilon(x_\epsilon, y_\epsilon) - f(a,b) - \inf_x h_\epsilon(x,b) + a_\epsilon c_\epsilon \\ &\qquad \le f(a, y_\epsilon) - f(a,b) + a_\epsilon g(a, y_\epsilon) + h_\epsilon(a, y_\epsilon) - \inf_x h_\epsilon(x,b) + a_\epsilon p_\epsilon \\ &\qquad \le a_\epsilon p_\epsilon + a_\epsilon g(a, y_\epsilon) + \sup_y h_\epsilon(a,y) - \inf_x h_\epsilon(x,b), \end{aligned}$$

so

$$g(x_\epsilon, b) \le \frac{F_\epsilon(x_\epsilon, y_\epsilon) - f(a,b) - \inf_x h_\epsilon(x,b)}{a_\epsilon} + c_\epsilon \le g(a, y_\epsilon) + q_\epsilon(a,b) + p_\epsilon$$

and

$$g(x_\epsilon, b) - c_\epsilon \le \frac{F_\epsilon(x_\epsilon, y_\epsilon) - f(a,b) - \inf_x h_\epsilon(x,b)}{a_\epsilon} \le g(a, y_\epsilon) + q_\epsilon(a,b) + d_\epsilon;$$

so

$$\frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(a,b)-h^1_\epsilon}{a_\epsilon}\leq g(a,y_\epsilon)+q_\epsilon(a,b)+d_\epsilon.$$

Similarly, we show that

$$\begin{aligned} g(x_\epsilon,b)-q_\epsilon(a,b)-c_\epsilon &\leq \frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(a,b)-\sup_y h_\epsilon(a,y)}{a_\epsilon}\\ &\leq \frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(a,b)-h^2_\epsilon}{a_\epsilon}. \end{aligned}$$

Consequently,

$$\begin{aligned} g(x_\epsilon,b)-q_\epsilon(a,b)-c_\epsilon &\leq \frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(a,b)-h^2_\epsilon}{a_\epsilon}\\ &\leq \frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(a,b)-h^1_\epsilon}{a_\epsilon}\\ &\leq g(a,y_\epsilon)+q_\epsilon(a,b)+d_\epsilon \end{aligned} \tag{10.5}$$

since $h^1_\epsilon\leq h^2_\epsilon$. But we know that $(g(x_\epsilon,\overline{y}))_\epsilon$ and $(g(\overline{x},y_\epsilon))_\epsilon$ are bounded, and by previous arguments used in the first part of the proof of this theorem it is a routine to check that they have a unique cluster point $g(\overline{x},\overline{y})$ to which they converge; so by (10.5) applied to $(a,b)=(\overline{x},\overline{y})$ we get

$$\lim_{\epsilon\to 0}\frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(\overline{x},\overline{y})-h^2_\epsilon}{a_\epsilon}=\lim_{\epsilon\to 0}\frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(\overline{x},\overline{y})-h^1_\epsilon}{a_\epsilon}=g(\overline{x},\overline{y}) \tag{10.6}$$

from which we deduce first that for every $\alpha\in\mathbb{R}$,

$$\lim_{\epsilon\to 0}\frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(\overline{x},\overline{y})-\alpha h^1_\epsilon-(1-\alpha)h^2_\epsilon}{a_\epsilon}=g(\overline{x},\overline{y});$$

hence the asymptotic development in the last theorem is proved by setting

$$\frac{F_\epsilon(x_\epsilon,y_\epsilon)-f(\overline{x},\overline{y})-\alpha h^1_\epsilon-(1-\alpha)h^2_\epsilon}{a_\epsilon}-g(\overline{x},\overline{y})=\delta^\alpha_\epsilon.$$

By (10.1) and (10.3) applied to $(t,z)=(a,b)=(\overline{x},\overline{y})$ we get

$$0\leq\frac{f(x_\epsilon,\overline{y})-f(\overline{x},\overline{y})}{a_\epsilon}+\frac{f(\overline{x},\overline{y})-f(\overline{x},y_\epsilon)}{a_\epsilon}\leq g(\overline{x},y_\epsilon)-g(x_\epsilon,\overline{y})+q_\epsilon(\overline{x},\overline{y})+p_\epsilon,$$

so

$$\frac{f(x_\epsilon,\overline{y})-f(\overline{x},\overline{y})}{a_\epsilon}\to 0,\quad \frac{f(\overline{x},\overline{y})-f(\overline{x},y_\epsilon)}{a_\epsilon}\to 0 \text{ if } \epsilon\to 0,\quad \lim_{\epsilon\to 0}\frac{h^2_\epsilon-h^1_\epsilon}{a_\epsilon}=0$$

is an immediate consequence of (10.6) or (10.1).

(ii) From the first limit of (10.1) it is clear that

$$F^1_\epsilon\leq F^2_\epsilon\leq F_\epsilon(x_\epsilon,y_\epsilon)+a_\epsilon p_\epsilon\leq F^1_\epsilon+2a_\epsilon p_\epsilon\leq F^2_\epsilon+2a_\epsilon p_\epsilon,$$

hence by setting $\mu_\epsilon^i = \dfrac{F_\epsilon^i - F_\epsilon(x_\epsilon, y_\epsilon)}{a_\epsilon}$ we get $|\mu_\epsilon^i| \le p_\epsilon$ for $i = 1, 2$. Consequently, $F_\epsilon^i = F_\epsilon(x_\epsilon, y_\epsilon) + a_\epsilon \mu_\epsilon^i$ and $\theta_\epsilon^{i,\alpha} = \delta_\epsilon^\alpha + \ \mu_\epsilon^i \to 0$ if $\epsilon \to 0$, $\lim_{\epsilon\to 0} \dfrac{F_\epsilon^2 - F_\epsilon^1}{a_\epsilon} = 0$ is an immediate consequence of the first limit of (10.1) or from $\lim_{\epsilon\to 0}(\mu_\epsilon^2 - \mu_\epsilon^1) = 0$ which completes the proof. □

Remark 10.3. The first limit in (10.1) is straightforward satisfied if (x_ϵ, y_ϵ) is a saddle point of F_ϵ. Also we observe that there exists a wide class of functions $h_\epsilon : X \times Y \to \mathbb{R}$ satisfying the second limit in (10.1). Take for instance the functions of the kind $h_\epsilon(x, y) = \sum_{i=1}^n b_\epsilon^i g_i(x, y)$ where $m_i(y) \le g_i(x, y) \le M_i(x)\ \forall\,(x, y) \in X \times Y$. Here $m_i(y)$, $M_i(x)$ are real numbers and $b_{\epsilon.}^i \ge 0$ satisfying $\lim_{\epsilon\to 0} \dfrac{b_{\epsilon.}^i}{a_\epsilon} = 0\ \forall\, i$. Then

$$0 \le \frac{\sup_{y\in Y} h_\epsilon(a, y) - \inf_{x\in X} h_\epsilon(x, b)}{a_\epsilon} \le \frac{\sum_{i=1}^n b_{\epsilon.}^i (M_i(a) - m_i(b))}{a\epsilon}$$

and

$$\lim_{\epsilon\to 0} \frac{\sum_{i=1}^n b_{\epsilon.}^i (M_i(a) - m_i(b))}{a\epsilon} = 0 \qquad \forall\,(a, b) \in X \times Y.$$

In particular one can consider the classical functions used in many schemes of saddle point approximation methods $g_i(x, y) = \alpha_i \|x - x_i\|^{p_i} - \beta_i \|y - y_i\|^{q_i}$ where x_i, y_i are given points in the normed spaces X, Y, $p_i, q_i \in \mathbb{N}^*$ and $\alpha_i, \beta_i > 0$ (for instance, see [62, 66, 67, 68] and references therein). More generally one may take $h_\epsilon(x, y) = \sum_{i=1}^n b_\epsilon^i g_{\epsilon i}(x, y)$ where there exist two real functions $m_{\epsilon i}(y)$ and $M_{\epsilon i}(x)$ such that for every (x, y) we have $m_{\epsilon i}(y) \le g_{\epsilon i}(x, y) \le M_{\epsilon i}(x)$ and

$$\lim_{\epsilon\to 0} \frac{\sum_{i=1}^n b_{\epsilon.}^i (M_{\epsilon i}(a) - m_{\epsilon i}(b))}{a\epsilon} = 0 \qquad \forall\,(a, b) \in X \times Y,$$

then

$$\lim_{\epsilon\to 0} \frac{\sup_{y\in Y} h_\epsilon(a, y) - \inf_{x\in X} h_\epsilon(x, b)}{a_\epsilon} = 0.$$

For example, see the regularization function considered in Theorem 11.5.

Corollary 10.4. *Let X, Y be two convex compacts of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume that $F_\epsilon : X \times Y \to \mathbb{R}$ is finite, convex-concave and continuous with $h_\epsilon(x, y) = \sum_{i=1}^p b_\epsilon^i g_i(x, y)$ where $g_i : X \times Y \to \mathbb{R}$, $i = 1, \ldots, p$ are such that $m_i(y) \le g_i(x, y) \le M_i(x)\ \forall\,(x, y) \in X \times Y$. $m_i(y)$, $M_i(x)$ are real numbers and $a_\epsilon > 0$, $b_{\epsilon.}^i \ge 0$ satisfying $\lim_{\epsilon\to 0} \frac{b_{\epsilon.}^i}{a_\epsilon} = 0\ \forall\, i$, $\lim_{\epsilon\to 0} a_\epsilon = 0$. Then*

$$\lim_{\epsilon\to 0}\frac{\min\limits_{x\in X}\max\limits_{y\in Y}(f(x,y)+a_\epsilon g(x,y)+\sum\limits_{i=1}^{p} b_\epsilon^i g_i(x,y))-\min\limits_{x\in X}\max\limits_{y\in Y}(f(x,y))}{a_\epsilon}$$
$$=\min_{x\in X_1}\max_{y\in X_2}(g(x,y))=\max_{y\in X_2}\min_{x\in X_1}(g(x,y))=g(a,b)$$

for some $(a,b)\in S$*, where* $S=X_1\times X_2$*,* $X_1=\operatorname{proj}_X S$ *and* $X_2=\operatorname{proj}_Y S$*.*

Proof. By [69] F_ϵ has a saddle point (x_ϵ,y_ϵ) and the sequence $((x_\epsilon,y_\epsilon))_\epsilon$ is relatively compact. The limits in (10.1) are obviously satisfied, so the conclusions of Theorem 10.2 hold. But for every $(x,y)\in X\times Y$ we have

$$\sum_{i=1}^{p} b_{\epsilon.}^i m_i(y))\le h_\epsilon^1\le h_\epsilon^2\le\sum_{i=1}^{p} b_{\epsilon.}^i M_i(x),$$

then $\lim_{\epsilon\to 0}\dfrac{h_\epsilon^1}{a_\epsilon}=\lim_{\epsilon\to 0}\dfrac{h_\epsilon^2}{a_\epsilon}=0$. On the other hand the optimal saddle value of $g(x,y)$ on S can be written as

$$\min_{x\in X_1}\max_{y\in X_2}(g(x,y))=\max_{y\in X_2}\min_{x\in X_1}(g(x,y))=g(a,b)$$

for some $(a,b)\in S$ that is (a,b) is a saddle point of g on S, where $S=X_1\times X_2$, $X_1=\operatorname{proj}_X S$ and $X_2=\operatorname{proj}_Y S$ (see [58, p. 49]) which completes the proof. □

Theorem 10.2 can provide for instance in dimensional setting an interesting tool for application to the conjugacy of bivariate functions as follows: Fix (x^*,y^*) in $\mathbb{R}^n\times\mathbb{R}^m$ and set

$$K(x^*,y^*)=\sup_{y\in D}\inf_{x\in C}(\langle x^*,x\rangle+\langle y^*,y\rangle+g(x,y))$$

(for instance see [69] for the importance of this function in saddle functions theory and conjugacy) where C, D are two convex compact sets of $\mathbb{R}^n$, $\mathbb{R}^m$, respectively. Set

$$K_\epsilon(x^*,y^*)=\sup_{y\in D}\inf_{x\in C}(f(x,y)+\epsilon g_1(x,y)+\epsilon^2 g_2(x,y)+\cdots+\epsilon^n g_n(x,y)),\ \epsilon>0,$$

where $f,\ g,\ g_i:C\times D\to\mathbb{R}$ are given convex-concave continuous functions, $i=1,2\ldots,n$ and $f(x,y)=\langle x^*,x\rangle+\langle y^*,y\rangle+g(x,y)$. Denote by S_f and S_{g_k}, respectively, the sets of saddle points of f and g_k on $C\times D$, which are nonempty by [69, Corollary 37.6.2, p. 397]. Furthermore they are compact and $S_f=\operatorname{proj}_C S_f\times\operatorname{proj}_D S_f$, $S_{g_k}=\operatorname{proj}_C S_{g_k}\times\operatorname{proj}_D S_{g_k}$ [58].

Proposition 10.5. *We have the following formulas:*

If $n = 2p + 1$

$$\begin{aligned} K_\epsilon(x^*, y^*) &= f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2 g_2(x_2, y_2) + \epsilon^3 g_3(x_2, y_2) + \cdots \\ &+\epsilon^{2k} g_{2k}(x_{2k}, y_{2k}) + \epsilon^{2k+1} g_{2k+1}(x_{2k}, y_{2k}) + \cdots \\ &+\epsilon^{2p} g_{2p}(x_{2p}, y_{2p}) + \epsilon^{2p+1} g_{2p+1}(x_{2p}, y_{2p}) + \epsilon\gamma_{2p+1,\epsilon}(x^*, y^*) \end{aligned} \tag{10.7}$$

for some $\gamma_{2p+1,\epsilon}(x^*, y^*)$ *converging to* 0 *if* $\epsilon \to 0$.

If $n = 2p$

$$\begin{aligned} K_\epsilon(x^*, y^*) &= f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2 g_2(x_2, y_2) + \epsilon^3 g_3(x_2, y_2) + \cdots \\ &+\epsilon^{2k} g_{2k}(x_{2k}, y_{2k}) + \epsilon^{2k+1} g_{2k+1}(x_{2k}, y_{2k}) + \cdots \\ &+\epsilon^{2p-2} g_{2p-2}(x_{2p-2}, y_{2p-2}) + \epsilon^{2p-1} g_{2p-1}(x_{2p-2}, y_{2p-2}) \\ &+\epsilon^{2p}(\alpha g^1_{2p} + (1-\alpha) g^2_{2p}) + \epsilon r^\alpha_{2p,\epsilon}(x^*, y^*) \end{aligned} \tag{10.8}$$

with $r^\alpha_{2p,\epsilon}(x^*, y^*) \to 0$ *if* $\epsilon \to 0$

and $g^1_{2p} = \sup_{y\in D} \inf_{x\in C} g_{2p}(x, y)$, $g^2_{2p} = \inf_{x\in C} \sup_{y\in D} g_{2p}(x, y)$, $\alpha \in \mathbb{R}$. *Here* (x_0, y_0) *is a saddle point of* f *on* $C \times D$ *and is also a saddle point of* g_1 *on* S_f. $(x_{2k}, y_{2k}) \in S_{g_{2k}}$ *and is a saddle point of* g_{2k+1} *on* $S_{g_{2k}}$.

Proof. We prove the proposition by reccurence. For $n = 1$, $K_\epsilon(x^*, y^*) = \sup_{y\in D} \inf_{x\in C}(f(x, y) + \epsilon g_1(x, y))$. The function $F_\epsilon(x, y) = f(x, y) + \epsilon g_1(x, y)$ is convex-concave and continuous has a saddle point (x_ϵ, y_ϵ) on $C \times D$ [69]. All hypotheses in Theorem 10.2 are satisfied with $\alpha_\epsilon = \beta_\epsilon$ and $h_\epsilon = 0$, so

$$K_\epsilon(x^*, y^*) = \max_{y\in D} \min_{x\in C}(f(x, y) + \epsilon g_1(x, y)) = f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon\delta_{1,\epsilon}(x^*, y^*),$$

$\lim_{\epsilon\to 0} \delta_{1,\epsilon}(x^*, y^*) = 0$, where (x_0, y_0) is a saddle point of f on $C \times D$ and is also a saddle point of g_1 on S_f.

For $n = 2$,

$$K_\epsilon(x^*, y^*) = \sup_{y\in D} \inf_{x\in C}(f(x, y) + \epsilon g_1(x, y) + \epsilon^2 g_2(x, y))$$

and the function $G_\epsilon(x, y) = f(x, y) + \epsilon g_1(x, y) + \epsilon^2 g_2(x, y)$ has a saddle point on $C\times D$ by the previous argument; and by Remark 10.3 all assumptions of Theorem 10.2 are fulfilled, then

$$\begin{aligned} K_\epsilon(x^*, y^*) &= \max_{y\in D} \min_{x\in C}(f(x, y) + \epsilon g_1(x, y) + \epsilon^2 g_2(x, y)) \\ &= f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2(\alpha g^1_2 + (1-\alpha) g^2_2) + \epsilon r^\alpha_{2,\epsilon}(x^*, y^*), \end{aligned}$$

$\lim_{\epsilon\to 0} r^\alpha_{2,\epsilon}(x^*, y^*) = 0$.

Now assume that (10.7) is verified for $n = 1, 3, \ldots, 2p-1$ and show that is satisfied for $2p+1$.

$$K_\epsilon(x^*, y^*) = \sup_{y \in D} \inf_{x \in C} (f(x,y) + \epsilon g_1(x,y) + \epsilon^2 H_\epsilon(x,y)),$$

where

$$H_\epsilon(x,y) = g_2(x,y) + \epsilon g_3(x,y) + \cdots + \epsilon^{2p-1} g_{2p+1}(x,y).$$

H_ϵ has a saddle point on $C \times D$ and it is easy to see by Remark 10.3, that $\epsilon^2 H_\epsilon$ satisfies the second limit in (10.1), so by Theorem 10.2 one has

$$K_\epsilon(x^*, y^*) = f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2 \max_{y \in D} \min_{x \in C} H_\epsilon(x,y) + \epsilon \delta_\epsilon(x^*, y^*),$$

$\lim_{\epsilon \to 0} \delta_\epsilon(x^*, y^*) = 0$; and by the recurrence hypothesis

$$\begin{aligned} \max_{y \in D} \min_{x \in C} H_\epsilon(x,y) &= g_2(x_2, y_2) + \epsilon g_3(x_2, y_2) + \cdots + \epsilon^{2k-2} g_{2k}(x_{2k}, y_{2k}) \\ &\quad + \epsilon^{2k-1} g_{2k+1}(x_{2k}, y_{2k}) + \cdots + \epsilon^{2p-2} g_{2p}(x_{2p}, y_{2p}) \\ &\quad + \epsilon^{2p-1} g_{2p+1}(x_{2p}, y_{2p}) + \epsilon \gamma_{2p,\epsilon}(x^*, y^*), \end{aligned}$$

$\lim_{\epsilon \to 0} \gamma_{2p,\epsilon}(x^*, y^*) = 0$; $(x_{2k}, y_{2k}) \in S_{g_{2k}}$ and is a saddle point of g_{2k+1} on $S_{g_{2k}}$, $k = 1, \ldots, p$. Then

$$\begin{aligned} K_\epsilon(x^*, y^*) &= f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2 g_2(x_2, y_2) + \epsilon^3 g_3(x_2, y_2) + \cdots \\ &\quad + \epsilon^{2k} g_{2k}(x_{2k}, y_{2k}) + \epsilon^{2k+1} g_{2k+1}(x_{2k}, y_{2k}) + \cdots + \epsilon^{2p} g_{2p}(x_{2p}, y_{2p}) \\ &\quad + \epsilon^{2p+1} g_{2p+1}(x_{2p}, y_{2p}) + \epsilon \gamma_{2p+1,\epsilon}(x^*, y^*) \end{aligned}$$

with

$$\gamma_{2p+1,\epsilon}(x^*, y^*) = \delta_\epsilon(x^*, y^*) + \epsilon^2 \gamma_{2p,\epsilon}(x^*, y^*) \to 0$$

when $\epsilon \to 0$. In the same way we prove (10.8) by recurrence which completes the proof. □

11. Well-posedness of generalized regularizations for bivariate functions. In the sequel we investigate well-posedness of generalized regularizations of saddle functions. Let X, Y be two reflexive Banach spaces renormed by strictly convex norms $\|\cdot\|_X$, $\|\cdot\|_Y$ making them E-spaces and $f : X \times Y \to \overline{\mathbb{R}}$, $g : X \times Y \to \mathbb{R}$, $h_\epsilon : X \times Y \to \mathbb{R}$ be three functions weakly lsc at the first variable for each fixed y and weakly usc at the second variable for each fixed x. Consider $F_\epsilon(x,y) = f(x,y) + \epsilon g(x,y) + h_\epsilon(x,y)$. In what follows we state sufficient conditions ensuring that $(F_\epsilon, X \times Y)$ is well-posed.

Theorem 11.1. *Assume that $f(x,y)$ satisfies (H_1) and (H_2) of Section 2 and the following hypotheses are verified:*

(i) $F_\epsilon(\cdot, y)$ is strictly convex and lsc $\forall\, y \in Y$;

(ii) $F_\epsilon(x, \cdot)$ is strictly concave and usc $\forall\, x \in X$;

(iii) $\exists y_0 \in Y$ such that for every $\lambda \in \mathbb{R}$, $A_\lambda = \{x \in X \mid F_\epsilon(x, y_0) \leq \lambda\}$ is bounded;

(iv) $\exists x_0 \in X$ such that for every $\lambda \in \mathbb{R}$, $B_\lambda = \{y \in Y \mid F_\epsilon(x_0, y) \geq \lambda\}$ is bounded;

(v) $\exists (a, b) \in X \times Y$ such that $f(a, b)$ is finite.

Then:

(a) $\inf_x \sup_y F_\epsilon(x, y) = \sup_y \inf_x F_\epsilon(x, y)$;

(b) F_ϵ has a unique saddle point $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ and $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ is finite;

(c) every minimaximizing sequence $(x_n, y_n)_n$ of F_ϵ converges weakly to $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ if $n \to +\infty$ and $F_\epsilon(x_n, \overline{y}_\epsilon), F_\epsilon(\overline{x}_\epsilon, y_n), F_\epsilon(x_n, y_n)$ converge to $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$, when $n \to +\infty$.

Moreover if there exist two functions $p, q \in \{f, g, h_\epsilon\}$ (eventually identical) such that $p(x_n, \overline{y}_\epsilon) \to p(\overline{x}_\epsilon, \overline{y}_\epsilon)$ and $x_n \rightharpoonup \overline{x}_\epsilon$ ($\rightharpoonup$ denotes the weak convergence) imply that $\lim_n \|x_n - \overline{x}_\epsilon\| = 0$; and $q(\overline{x}_\epsilon, y_n) \to q(\overline{x}_\epsilon, \overline{y}_\epsilon)$ and $y_n \rightharpoonup \overline{y}_\epsilon$ imply that $\lim_n \|y_n - \overline{y}_\epsilon\| = 0$; then $(F_\epsilon, X \times Y)$ is well-posed.

Proof. (a) By (i), (iii), (v) and [4] one has

$$\inf_x \sup_y F_\epsilon(x, y) = \sup_y \inf_x F_\epsilon(x, y).$$

(b) Set $\varphi_\epsilon(x) = \sup_y F_\epsilon(x, y)$ which is convex lsc and $\varphi_\epsilon(x) \geq F_\epsilon(x, y_0)$, so $\{x \in X \mid \varphi_\epsilon(x) \leq \lambda\}$ is bounded for every λ by (iii) and $\min_x \varphi_\epsilon(x) = \varphi_\epsilon(\overline{x}_\epsilon)$ [35] for some $\overline{x}_\epsilon$. Using (ii), (iv) and [35] a symmetric argument shows that $\max_y \psi_\epsilon(y) = \psi_\epsilon(\overline{y}_\epsilon)$ for some $\overline{y}_\epsilon$, where $\psi_\epsilon(y) = \inf_x F_\epsilon(x, y)$; then by (a) $\varphi_\epsilon(\overline{x}_\epsilon) = \psi_\epsilon(\overline{y}_\epsilon)$, i.e $\sup_y F_\epsilon(\overline{x}_\epsilon, y) = \inf_x F_\epsilon(x, \overline{y}_\epsilon) = F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ which are finite because $f(x, y)$ satisfies (H_1) and (H_2); consequently $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ is a saddle point of F_ϵ on $X \times Y$. The uniqueness is immediate from the strict convexity of $F_\epsilon(\cdot, y)$ $\forall\, y$ and the strict concavity of $F_\epsilon(x, \cdot)$ $\forall\, x$.

(c) First we observe by Theorem 2.6 and (a) that the set of minimaximizing sequences of F_ϵ is nonempty. Now let $(x_n, y_n)_n$ be a minimaximizing sequence of F_ϵ. We have $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) \leq F_\epsilon(x_n, \overline{y}_\epsilon) \leq F_\epsilon(x_0, y_n) + \epsilon_n$, $\epsilon_n > 0$, $\epsilon_n \to 0$ and $(y_n)_n$ is bounded by (iv). In the same way $F_\epsilon(x_n, y_0) \leq F_\epsilon(\overline{x}_\epsilon, y_n) + \epsilon_n \leq F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) + \epsilon_n$ and $(x_n)_n$ is bounded by (iii). By weak relative compactness of $(x_n, y_n)_n$, semicontinuity and uniqueness of the saddle point $(\overline{x}_\epsilon, \overline{y}_\epsilon)$, it is a routine to check that $x_n \rightharpoonup \overline{x}_\epsilon$ and $y_n \rightharpoonup \overline{y}_\epsilon$. On the other hand, there exist three scalars m, M, α such that

$$m \leq F_\epsilon(x_n, \overline{y}_\epsilon) \leq F_\epsilon(x_0, y_n) + \epsilon_n \leq M$$

and

$$m \leq F_\epsilon(x_n, \overline{y}_\epsilon) \leq F_\epsilon(\overline{x}_\epsilon, y_n) + \epsilon_n \leq \alpha$$

for every n, so $(F_\epsilon(x_n, \overline{y}_\epsilon))_n, (F_\epsilon(\overline{x}_\epsilon, y_n))_n$ are bounded; and by a classical argument they have a unique cluster point $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ to which they converge. But $F_\epsilon(x_n, \overline{y}_\epsilon) - \epsilon_n \leq F_\epsilon(x_n, y_n) \leq F_\epsilon(\overline{x}_\epsilon, y_n) + \epsilon_n$, then $F_\epsilon(x_n, y_n) \to F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ when $n \to +\infty$. Since

$$F_\epsilon(x_n, \overline{y}_\epsilon) \to F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) = \min_x F_\epsilon(x, \overline{y}_\epsilon), F_\epsilon(\overline{x}_\epsilon, y_n) \to F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) = \max_y F_\epsilon(\overline{x}_\epsilon, y),$$

$(x_n)_n$and $(y_n)_n$ are, respectively, minimizing and maximizing sequences for the two last extremum problems; and as in the proof of Theorem 5.2 we show that the sequence $(f(x_n, \overline{y}_\epsilon), g(x_n, \overline{y}_\epsilon), h_\epsilon(x_n, \overline{y}_\epsilon))_n$ converges to $(f(\overline{x}_\epsilon, \overline{y}_\epsilon), g(\overline{x}_\epsilon, \overline{y}_\epsilon), h_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon))$. A symmetric argument shows that

$$(f(\overline{x}_\epsilon, y_n), g(\overline{x}_\epsilon, y_n), h_\epsilon(\overline{x}_\epsilon, y_n)) \to (f(\overline{x}_\epsilon, \overline{y}_\epsilon), g(\overline{x}_\epsilon, \overline{y}_\epsilon), h_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)),$$

when $n \to +\infty$; so $(x_n, y_n)_n$ converges in the norm topology to $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ by hypothesis which completes the proof of the theorem. □

As an immediate consequence of the previous theorem and the fact that $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are E-spaces, we have the following corollaries:

Corollary 11.2. *Assume that $f : X \times Y \to \mathbb{R}$ is convex-concave lsc at the first variable for each fixed y and usc at the second variable for each fixed x and there exist (x_0, y_0) in $X \times Y$ and scalars m, M such that $f(x, y_0) \geq m$, $f(x_0, y) \leq M$ for every $(x, y) \in X \times Y$. Set*

$$F_\epsilon(x, y) = f(x, y) + \sum_{i=1}^{p} \varphi_\epsilon^i(\|x - x_i\|^{p_i}) - \sum_{j=1}^{q} \psi_\epsilon^j(\|y - y_j\|^{q_j}),$$

where ϵ is a positive parameter, p, q, p_i, $q_j \in \mathbb{N}^*$ and x_i, y_j, $i = 1, \ldots, p$, $j = 1, \ldots, q$ are given points in X and Y, respectively, φ_ϵ^i, $\psi_\epsilon^j : [0, +\infty[\to \mathbb{R}$ are continuous functions at 0, convex and strictly increasing such that for every $\lambda \in \mathbb{R}$ the sets

$$\left\{ x \in X / \sum_{i=1}^{p} \varphi_\epsilon^i(\|x - x_i\|^{p_i}) \leq \lambda \right\}, \quad \left\{ y \in Y / \sum_{j=1}^{q} \psi_\epsilon^j(\|y - y_j\|^{q_j}) \leq \lambda \right\}$$

are bounded, then $(F_\epsilon, X \times Y)$ is well-posed.

Corollary 11.3. *Assume that $f : X \times Y \to \mathbb{R}$ is convex-concave lsc at the first variable for each fixed y and usc at the second variable for each fixed x.*

Set

$$F_\epsilon(x,y) = f(x,y) + \sum_{i=1}^{p} a_\epsilon^i \|x - x_i\|^{p_i} - \sum_{j=1}^{q} b_\epsilon^j \|y - y_j\|^{q_j},$$

where ϵ, a_ϵ^i, $b_\epsilon^j \in \mathbb{R}_+^*$, p, q, p_i, $q_j \in \mathbb{N}^*$ *and* x_i, y_j, $i = 1,\ldots,p$, $j = 1,\ldots,q$ *are given points in* X *and* Y*, respectively. If there exist* (x_0, y_0) *in* $X \times Y$ *and scalars* m, M *such that* $f(x, y_0) \geq m, f(x_0, y) \leq M$ *for every* $(x,y) \in X \times Y$*, then* $(F_\epsilon, X \times Y)$ *is well-posed.*

Corollary 11.4. *Assume that* $f : X \times Y \to \mathbb{R}$ *is convex-concave lsc at the first variable for each fixed* y *and usc at the second variable for each fixed* x*. Set*

$$F_\epsilon(x,y) = f(x,y) + \sum_{i=1}^{p} \alpha_i \epsilon e^{\frac{1}{\epsilon}(\|x-x_i\|^{p_i}+c_i)+\varpi_i} - \sum_{j=1}^{q} \theta_j \epsilon e^{\frac{1}{\epsilon}(\|y-y_j\|^{q_j}+d_j)+\delta_j},$$

where ϵ, α_i, $\theta_j \in \mathbb{R}_+^*$, c_i, d_j, ϖ_i, $\delta_j \in \mathbb{R}$, p, q, p_i, $q_j \in \mathbb{N}^*$ *and* x_i, y_j, $i = 1,\ldots,p$, $j = 1,\ldots,q$ *are given points in* X *and* Y*, respectively. If there exist* (x_0, y_0) *in* $X \times Y$ *and scalars* m, M *such that* $f(x, y_0) \geq m$, $f(x_0, y) \leq M$ *for every* $(x,y) \in X \times Y$*, then* $(F_\epsilon, X \times Y)$ *is well-posed.*

Finally we end our investigation by the following theorem in finite dimensional setting which combines the results of Theorem 10.2 and the ones of Theorem 11.1:

Theorem 11.5. *Let* $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $f_i : \mathbb{R}^m \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1\ldots,p$, $j = 1\ldots,q$ *be real-valued functions such that* f *is continuous convex-concave and* f_i, g_j *are convex. Assume that the sets*

$$X = \{x \in \mathbb{R}^m \mid f_i(x) \leq 0, i = 1\ldots,p\}, \quad Y = \{y \in \mathbb{R}^n \mid g_j(y) \leq 0, j = 1,\ldots,q\}$$

are nonempty and bounded. Set

$$F_\epsilon(x,y) = f(x,y) + a_\epsilon \left(\sum_{k=1}^{r} \alpha_k \|x - x_k\|^{p_k} - \sum_{k=1}^{s} \beta_k \|y - y_k\|^{q_k} \right) + \sum_{i=1}^{p} a_{i\epsilon} r_{i\epsilon} e^{\frac{1}{r_{i\epsilon}}(f_i(x))+h_i(x)} - \sum_{j=1}^{q} b_{j\epsilon} t_{j\epsilon} e^{\frac{1}{t_{j\epsilon}}(g_j(y))+k_j(y)},$$

where a_ϵ, α_k, β_k, $a_{i\epsilon}$, $b_{j\epsilon}$, $r_{i\epsilon}$, $t_{j\epsilon}$ *are real positive numbers, such that* $a_\epsilon \to 0$ *if* $\epsilon \to 0$, h_i, k_j *are convex continuous functions defined on* X, Y*, respectively, such that for every* $(x,y) \in X \times Y$, $h_i(x) \leq \varpi_i$, $k_j(y) \leq \delta_j$ *and*

$$\lim_{\epsilon \to 0} \frac{\sum_{i=1}^{p} a_{i\epsilon} r_{i\epsilon} e^{\varpi_i} + \sum_{j=1}^{q} b_{j\epsilon} t_{j\epsilon} e^{\delta_j}}{a_\epsilon} = 0,$$

r, s, p, q, p_k, $q_k \in \mathbb{N}^*$ and x_k, y_k are given points in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If $z = (z_i)_i \in \mathbb{R}^w$, $w \in \{m, n\}$ we denote $\|z\| = \left(\sum_{i=1}^{w} z_i^2\right)^{\frac{1}{2}}$ the strictly convex norm of $\mathbb{R}^w$. Then we have the following results:

1) *$(F_\epsilon, X \times Y)$ is well-posed;*
2) *The conclusions of Theorem 10.2 hold.*

For its proof we need the following lemma:

Lemma 11.6 ([69, Corollary 37.6.2, p. 397])**.** *Let C and D be nonempty closed bounded convex sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and K be a continuous finite convex-concave function on $C \times D$. Then K has a saddle point with respect to $C \times D$.*

Proof of Theorem 11.5. 1) The proof of this point is an immediate consequence of Lemma 11.6 (applied to $K = F_\epsilon$ and $C \times D = X \times Y$) and various arguments used in the proof of Theorem 11.1 and the fact that the norm $\|z\| = \left(\sum_{i=1}^{w} z_i^2\right)^{\frac{1}{2}}$ is strictly convex, X, Y are convex compact sets and the weak convergence reduces to the norm convergence in finite dimensional setting.

2) If (x_ϵ, y_ϵ) is the unique saddle point of F_ϵ on $X \times Y$, the sequence $((x_\epsilon, y_\epsilon))_\epsilon$ is relatively compact and the first limit in (10.1) is straightforward satisfied. On the other hand it is easy to see that

$$-\sum_{j=1}^{q} b_{j\epsilon} t_{j\epsilon} e^{\delta_j} \leq h_\epsilon(x, y) \leq \sum\nolimits_{i=1}^{p} a_{i\epsilon} r_{i\epsilon} e^{\varpi_i}$$

for every $(x, y) \in X \times Y$, so

$$0 \leq \frac{\sup_{y \in Y} h_\epsilon(x, y) - \inf_{x \in X} h_\epsilon(x, y)}{a_\epsilon} \leq \frac{\sum_{i=1}^{p} a_{i\epsilon} r_{i\epsilon} e^{\varpi_i} + \sum_{j=1}^{q} b_{j\epsilon} t_{j\epsilon} e^{\delta_j}}{a_\epsilon}$$

which goes to 0 if $\epsilon \to 0$ and the second limit in (10.1) is also satisfied, which completes the proof. □

12. Characterization of well-posedness of saddle point problems in metric spaces In this section we characterize well-posedness of saddle point problems considered in Section 2. Many examples will illustrate the difference between the notions of well-posedness under consideration. Consider two complete metric spaces $(X, d), (Y, d')$ and $X \times Y$ is the complete metric space endowed with the product topology associated to the metric $d((x, y), (x', y')) = \max(d(x, x'), d'(y, y'))$. $F : X \times Y \to \overline{\mathbb{R}}$ is a lower semicontinuous function at the first variable and upper semicontinuous at the second variable. Suppose also that $\sup_{y\in Y}\inf_{x\in X} F(x,y)$, $\inf_{x\in X}\sup_{y\in Y} F(x,y)$ are finite and the function $W(x,y) = G(x) - H(y)$ is well defined as in Section 2, with $G(x) = \sup_{y\in Y} F(x,y)$ and $H(y) = \inf_{x\in X} F(x,y)$. The next theorem provides a characterization of well-posedness of the saddle point problem $(F, X\times Y)$ in the sense of Definition 2.7:

Theorem 12.1. *$(F, X\times Y)$ is well-posed if and only if for every $\epsilon > 0$, $\epsilon - \operatorname{argminmax} F \neq \emptyset$ and $\operatorname{diam}(\epsilon - \operatorname{argminmax} F) \to 0$ if $\epsilon \to 0$.*

P r o o f. If $(F, X\times Y)$ is well-posed with a unique saddle point $(\overline{x}, \overline{y})$, obviously $(\overline{x}, \overline{y}) \in \epsilon-\operatorname{argminmax} F$ for every $\epsilon > 0$. If $\lim_{\epsilon\to 0}\operatorname{diam}(\epsilon-\operatorname{argminmax} F) = 0$ fails, there exist $a > 0$ and a sequence $(\epsilon_k)_k \downarrow 0$, such that

$$\operatorname{diam}(\epsilon_k - \operatorname{argminmax} F) > a$$

for every k, so there exist sequences $(u_k, v_k), (w_k, z_k) \in \epsilon_k - \operatorname{argminmax} F$ such that $\max(d(u_k, w_k), d'(v_k, z_k)) > a$. Accordingly, $(u_k, v_k), (w_k, z_k)$ are minimaximizing sequences for $(F, X\times Y)$; and by hypothesis $(u_k, v_k)_k, (w_k, z_k)_k$ converge to $(\overline{x}, \overline{y})$ which is a contradiction because $\max(d(u_k, w_k), d'(v_k, z_k)) \to 0$, $k \to +\infty$ and $\max(d(u_k, w_k), d'(v_k, z_k)) > a > 0$ for every k. Conversely, it is easy to check that the nonemptness of $\epsilon - \operatorname{argminmax} F$ for every $\epsilon > 0$ is equivalent to the existence of at least of a minimaximizing sequence. Now consider a minimaximizing sequence $((x_n, y_n))_n$ of problem $(F, X\times Y)$. For every $\alpha > 0$ there exists $\beta > 0$ such that for every $0 < \epsilon < \beta$ one has $\operatorname{diam}(\epsilon - \operatorname{argminmax} F) < \alpha$. For a fixed $\epsilon \in]0, \beta[$ there exists $N_\epsilon \in \mathbb{N}$ such that $\forall\, n \geq N_\epsilon$ we have $(x_n, y_n) \in \epsilon - \operatorname{argminmax} F$; consequently $((x_n, y_n))_n$ is a Cauchy sequence which converges to a point $(x', y') \in X\times Y$. By lower semicontinuity of F at the first variable, and its upper semicontinuity at the second variable it is easy to see that (x', y') is a saddle point of F on $X\times Y$. The uniqueness of this point is an immediate consequence of $\lim_{\epsilon\to 0}\operatorname{diam}(\epsilon - \operatorname{argminmax} F) = 0$, which completes the proof. □

Remark 12.2. It is possible to give a short proof of the previous result as follows: By [16], the saddle point problem $(F, X\times Y)$ is well-posed if and only if $(W, X\times Y)$ is Tikhonov well-posed and $\rho = 0$, which is equivalent by [21] to $\lim_{\epsilon\to 0}(\operatorname{diam}\epsilon - \operatorname{argmin} W) = 0$; that is $\lim_{\epsilon\to 0}(\operatorname{diam}\epsilon - \operatorname{argminmax} F) = 0$ because $\epsilon - \operatorname{argmin} W \subseteq \epsilon - \operatorname{argminmax} F \subseteq 2\epsilon - \operatorname{argmin} W$ if $\rho = 0$.

In the proof of the next theorem we will show that the large variational set $\Delta(F, \epsilon)$ is nonempty for every $\epsilon > 0$ and contains many interesting variational subsets in relationship with classical optimization and variational analysis.

Theorem 12.3. *Assume that F is finite and continuous and X, Y are complete, then the following assertions are equivalent:*

(*i*) $(F, X \times Y)$ *is strongly well-posed;*
(*ii*) $\operatorname{diam} \Delta(F, \epsilon) \to 0$ *if* $\epsilon \to 0$.

Proof. The proof of $(i) \Longrightarrow (ii)$ is similar to the proof of the first implication of Theorem 12.1.

$(ii) \Longrightarrow (i)$ *Step 1.* We claim that there exists a unique point $(\overline{x}, \overline{y}) \in X \times Y$ satisfing $F(\overline{x}, \overline{y}) = \inf_X \sup_Y F = \sup_Y \inf_X F$. To this end consider for every $\epsilon > 0$ the sets defined in Section 2 by

$$\Delta_1(F, \epsilon) = \{(x, y) \in X \times Y \mid \inf_X G - \epsilon \leq F(x, y) \leq \inf_X G + \epsilon\}$$

and

$$\Delta_2(F, \epsilon) = \{(x, y) \in X \times Y \mid \sup_Y H - \epsilon \leq F(x, y) \leq \sup_Y H + \epsilon\},$$

which are of course contained in $\Delta(F, \epsilon)$. For $\epsilon > 0$ there exists $x_\epsilon \in X$ such that $G(x_\epsilon) \leq \inf_X G + \epsilon$ with $G(x_\epsilon) = \sup_{y \in Y} F(x_\epsilon, y)$, so there exists $y_\epsilon \in Y$ with $\inf_X G - \epsilon \leq G(x_\epsilon) - \epsilon \leq F(x_\epsilon,\, y_\epsilon) \leq G(x_\epsilon) \leq \inf_X G + \epsilon$, i.e $(x_\epsilon, y_\epsilon) \in \Delta_1(F, \epsilon)$, which is then nonempty. Similarly, $\Delta_2(F, \epsilon) \neq \emptyset$. So $\Delta(F, \epsilon)$ is nonempty. Now consider two sequences $((x_n, y_n))_n$, $((t_n, z_n))_n \in X \times Y$ satisfying $F(x_n, y_n) \to \inf_X G$ and $F(t_n, z_n) \to \sup_Y H$. From (ii), for every $\alpha > 0$, $\exists r > 0$ such that $\forall \epsilon \in]0, r[$ we have $\operatorname{diam} \Delta(F, \epsilon) < \alpha$. For a fixed $\epsilon \in]0, r[$, $\exists N_\epsilon$ such that $\forall n \geq N_\epsilon$, $(x_n, y_n) \in \Delta_1(F, \epsilon)$ and $(t_n, z_n) \in \Delta_2(F, \epsilon)$; consequently $((x_n, y_n))_n$, $((t_n, z_n))_n$ are Cauchy sequences converging to (x', y'), (t', z'), respectively, and by continuity of F, $F(x', y') = \inf_X G$ and $F(t', z') = \sup_Y H$ from which we deduce that $(x', y'), (t', z') \in \Delta(F, \epsilon)$ and then $(x', y') = (t', z') = (\overline{x}, \overline{y})$ is the unique point such that $F(\overline{x}, \overline{y}) = \inf_X G = \sup_Y H$.

Step 2. $(\overline{x}, \overline{y})$ is a saddle point of problem $(F, X \times Y)$. First we observe that $(\epsilon-\operatorname{argmin} G) \times (\epsilon-\operatorname{argmax} H) \subset \Delta(F, \epsilon)$ (which implies again that $\Delta(F, \epsilon) \neq \emptyset$), so $\lim_{\epsilon \to 0} \operatorname{diam}(\epsilon - \operatorname{argmin} G) = \lim_{\epsilon \to 0} \operatorname{diam}(\epsilon - \operatorname{argmax} H) = 0$ and by [21] the minimization problems (G, X), $(-H, Y)$ are well-posed in the Tikhonov sense. In particular they have solutions $\overline{a} \in X$, $\overline{b} \in Y$ with

$$G(\overline{a}) = \min_X G = \max_Y H = H(\overline{b})$$

so $(\overline{a}, \overline{b})$ is a saddle point of $(F, X \times Y)$; but $(\overline{a}, \overline{b})$, $(\overline{x}, \overline{y}) \in \Delta(F, \epsilon)$ and (ii) is satisfied, then $(\overline{a}, \overline{b}) = (\overline{x}, \overline{y})$.

Step 3. Every sequence $((x_n, y_n))_n$ such that $F(x_n, y_n) \to F(\overline{x}, \overline{y})$ converges to $(\overline{x}, \overline{y})$. First we observe the existence of such sequence, because every minimaximizing sequence $((u_n, v_n))_n$ satisfies $F(u_n, v_n) \to F(\overline{x}, \overline{y})$ [16]. Now if $F(x_n, y_n) \to F(\overline{x}, \overline{y})$, then by a classical argument $((x_n, y_n))_n$ is a Cauchy sequence converging to a point $(c, d) \in \Delta(F, \epsilon)$, $so (c, d) = (\overline{x}, \overline{y})$ because $\operatorname{diam} \Delta(F, \epsilon) \to 0$ if $\epsilon \to 0$ which completes the proof of Theorem 12.3. $\square$

Remarks 12.4. a) If $(F, X \times Y)$ is strongly well-posed, then it is well-posed because every minimaximizing sequence $((x_n, y_n))_n$ verifies $F(x_n, y_n) \to \sup_{y \in Y} \inf_{x \in X} F(x, y) = \inf_{x \in X} \sup_{y \in Y} F(x, y)$ (see [16]).

b) If $(F, X \times Y)$ is well-posed with its unique saddle point $(\overline{x}, \overline{y})$, every sequence $((u_n, v_n))_n$ such that $F(u_n, v_n) \to F(\overline{x}, \overline{y})$ is not necessarily a minimaximizing sequence. For example take $X = Y = [0, 1]$ and

$$F(x,y) = \begin{cases} x - y & \text{if} \quad (x,y) \in [0,1[\times[0,1]; \\ y + 1 & \text{if} \quad x = 1, y \in [0,1]. \end{cases}$$

It is clear that $(0, 0)$ is the unique saddle point of F on $X \times Y$ and $G(x) = x$ if $x \in [0, 1[$, $G(1) = 2$; $H(y) = -y$, $y \in [0, 1]$ and $W(x, y) = x + y$ if $(x, y) \in [0, 1[\times[0, 1]$, $W(x, y) = 2 + y$ if $x = 1$, $y \in [0, 1]$. Then $((x_n, y_n))_n$ is a minimaximizing sequence if and only if $(x_n, y_n) \in [0, 1[\times[0, 1]$ and $(x_n, y_n) \to (0, 0)$, so $(F, X \times Y)$ is well-posed but not strongly well-posed because if $F(u_n, v_n) \to F(0, 0)$, then $u_n - v_n \to 0$ and $(u_n, v_n) \in [0, 1[\times[0, 1]$, but in general $(u_n, v_n) \nrightarrow (0, 0)$, for example take $u_n = v_n = 1 - |\cos n|$.

c) If $(F, X \times Y)$ is strongly well-posed with unique saddle point $(\overline{x}, \overline{y})$, a sequence $((u_n, v_n))_n$ such that $F(u_n, v_n) \to F(\overline{x}, \overline{y})$ is not necessarily a minimaximizing sequence. Take $X = Y = [0, 1]$ and

$$F(x,y) = \begin{cases} -x + y & \text{if} \quad (x,y) \in [0,1[\times [0,1] \\ -y - 1 & \text{if} \quad x = 1, y \in [0,1]\,. \end{cases}$$

It is clear that $(1, 0)$ is the unique saddle point of F on $X \times Y$ and $G(x) = 1 - x$ if $x \in [0, 1[$, $G(1) = -1$; $H(y) = -y - 1$, $y \in [0, 1]$ and $W(x, y) = 2 - x + y$ if $(x, y) \in [0, 1[\times [0, 1]$, $W(x, y) = y$ if $x = 1$, $y \in [0, 1]$. $((x_n, y_n))_n$ is a minimaximizing sequence if and only if $x_n = 1$, $y_n \in [0, 1]$ and $y_n \to 0$. On the other hand, if $F(u_n, v_n) \to F(1, 0) = -1$ and $(u_n, v_n) \in [0, 1[\times [0, 1]$, then $(u_n, v_n) \to (1, 0)$ but $((u_n, v_n))_n$ is not a minimaximizing sequence because $W(u_n, v_n) \to 1$.

d) If $(F, X \times Y)$ is well-posed, even under strong regularity conditions as differentiability, compactness of $X \times Y$, convexity-concavity of F, it is not true that $(F, X \times Y)$ is strongly well-posed. Consider for instance $F(x, y) = x^2 - y^2$ and $X = Y = [0, 1]$, then $(0, 0)$ is the unique saddle point of F on $X \times Y$. $((x_n, y_n))_n$ is a minimaximizing sequence if and only if $(x_n, y_n) \in [0, 1] \times [0, 1]$ and $(x_n, y_n) \to (0, 0)$. Now if $(u_n, v_n) \in [0, 1] \times [0, 1]$ such that $F(u_n, v_n) \to F(0, 0) = 0$, in general $(u_n, v_n) \nrightarrow (0, 0)$, for instance take $u_n = v_n = |\cos n|$.

In what follows we investigate the Levitin-Polyak well-posedness of saddle point problems. Consider two complete metric spaces X_1, Y_1 of X and Y, respectively, which are not necessarily complete, $F : X \times Y \to \overline{\mathbb{R}}$ be a function lower semicontinuous at the first variable and upper semicontinuous at the second variable. Assume that $\sup_{y\in Y_1}\inf_{x\in X_1} F(x,y)$, $\inf_{x\in X_1}\sup_{y\in Y_1} F(x,y)$ are finite and the function $Z(x,y) = J(x) - K(y)$ is well defined by $J(x) = \sup_{y\in Y_1} F(x,y)$, $x \in X$ and $K(y) = \inf_{x\in X_1} F(x,y)$, $y \in Y$ as in Section 2. With slight modifications in the proofs of the previous theorems of this section we state the following theorems:

Theorem 12.5. *If $A(F,\epsilon) \neq \varnothing$ for every $\epsilon > 0$ and $\operatorname{diam} A(F,\epsilon) \to 0$ when $\epsilon \to 0$, then the saddle point problem $(F, X_1 \times Y_1)$ is Levitin-Polyak well-posed. $(F, X_1 \times Y_1)$ is Levitin-Polyak well-posed if and only if $B(F,\epsilon) \neq \varnothing$ for every $\epsilon > 0$ and $\operatorname{diam} B(F,\epsilon) \to 0$, $\epsilon \to 0$.*

Theorem 12.6. *Assume that $F : X \times Y \to \mathbb{R}$ is continuous, then the saddle point problem $(F, X_1 \times Y_1)$ is strongly Levitin-Polyak well-posed if and only if $\operatorname{diam} \Delta'(F,\epsilon) \to 0$ when $\epsilon \to 0$.*

Acknowledgements. The author would like to express his thanks to the referee of this paper for the invaluable help and advice. He is grateful to him for his helpful comments and suggestions which played a crucial role in this revised form.

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Laboratory of Operations Research, Optimization and Statistics
Department of Mathematics
Faculty of Sciences
Ibn Tofail University
Kénitra, PC 14000, Morocco
e-mails: dri_mentagui@yahoo.fr, d_mentagui@hotmail.com

Received November 5, 2017
Revised March 4, 2019