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ON DESCARTES' RULE OF SIGNS FOR HYPERBOLIC POLYNOMIALS

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Abstract. We consider univariate real polynomials with all real roots and with two sign changes in the sequence of their coefficients which are all non-vanishing. Assume that one of the changes is between the linear and the constant term. By Descartes' rule of signs, such degree d polynomials have 2 positive and $d-2$ negative roots. We consider the possible sequences of the moduli of their roots on the real positive half-axis. When these moduli are distinct, we give the exhaustive answer to the question which positions can the moduli of the two positive roots occupy.

1. Introduction. In the present text we consider real univariate polynomials. A monic degree d polynomial Q is representable in the form $Q := \sum_{j=0}^{d} a_j x^j$, $a_j \in \mathbb{R}$, $a_d = 1$. The polynomial Q is *hyperbolic* if it has d

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real roots counted with multiplicity. We are interested in the *generic* case when all coefficients a_j are non-zero and the moduli of all roots are distinct.

Descartes' rule of signs (see [1, 2, 3, 4, 7, 8, 9, 16, 17]) states that the number pos of positive roots of Q is majorized by the number $\tilde{c}$ of sign changes in its sequence of coefficients. When applying this rule to $Q(-x)$, one deduces that the number neg of negative roots is majorized by the number $\tilde{p}$ of sign preservations. As $pos+neg = \tilde{c}+\tilde{p} = d$, one finds that for hyperbolic polynomials, $pos = \tilde{c}$ and $neg = \tilde{p}$. Moreover $\mathrm{sgn}(a_0) = (-1)^{pos}$.

The tropical version of Descartes' rule of signs is suggested in [6]. The problem treated in the present paper is part of a more general problem about real univariate, but not necessarily hyperbolic polynomials, considered in [5] (see also the references therein).

Definition 1. *The signs of the coefficients of the polynomial Q form the* sign pattern $\sigma(Q) := (\mathrm{sgn}(a_d), \mathrm{sgn}(a_{d-1}), \ldots, \mathrm{sgn}(a_0))$. *For monic polynomials, one has* $\mathrm{sgn}(a_d) = +$ *and knowing the sign pattern is the same as knowing the corresponding* change-preservation pattern *which is a d-vector whose jth component is c (resp. p) if $a_{d+1-j}a_{d-j} < 0$ (resp. if $a_{d+1-j}a_{d-j} > 0$). Example: for $d = 5$, the sign pattern $(+, -, -, +, -, +)$ corresponds to the change-preservation pattern cpccc.*

Given a generic monic hyperbolic polynomial Q, one can consider the moduli of its roots which are d distinct numbers on the positive half-axis and mark the positions of the moduli of the negative roots. Below we denote the positive roots by $\alpha_1 < \cdots < \alpha_{pos}$ and the moduli of negative roots by $\gamma_1 < \cdots < \gamma_{neg}$. The definition of *order* (in the sense of order of moduli of roots) and the corresponding notation should be clear from the following example:

Example 1. Suppose that $d = 7$, $pos = 4$ and $neg = 3$. Suppose that

$$\gamma_1 < \alpha_1 < \gamma_2 < \gamma_3 < \alpha_2 < \alpha_3 < \alpha_4 .$$

Then we say that the moduli of the roots of the polynomial define the order $NPNNPPP$, i. e. the letters N and P indicate the relative positions of the moduli of the negative and positive roots on the positive half-axis.

Definition 2. (1) *Given two d-vectors – a change-preservation pattern and an order – we say that they are* compatible *if the number of letters c (resp. p) of the former is equal to the number of letters P (resp. N) of the latter.*

(2) *A compatible couple (change-preservation pattern, order) (we say* couple *for short) is* realizable *if there exists a monic generic hyperbolic polynomial whose signs of the coefficients and whose order of the moduli of roots define the*

given couple. In this case we also say that the first of the components of the couple is realizable by the second one and vice versa.

The present paper studies realizability of couples, a question for which Descartes' rule of signs provides no information.

Definition 3. *For each change-preservation pattern* (*or sign pattern*) *one defines its corresponding* canonical *order as follows. One reads the change-preservation pattern from the right and to each letter c (resp. p) one puts into correspondence the letter P (resp. N). Example: for $d = 5$, to the change-preservation pattern ccpcp* (*or, equivalently, to the sign pattern* $(+, -, +, +, -, -)$) *corresponds the canonical order $NPNPP$.*

Each change-preservation or sign pattern is realizable with its canonical order, see [11, Proposition 1]. When a change-preservation or sign pattern is realizable *only* with its corresponding canonical order, it is called *canonical*.

Remarks 1. (1) It is shown in [12] that canonical are exactly these change-preservation patterns which have no isolated sign change and no isolated sign preservation, i. e. which contain no three consecutive components PNP or NPN. Hence canonical are exactly these sign patterns which have no four consecutive signs $(+, +, -, -)$, $(-, -, +, +)$, $(+, -, -, +)$ or $(-, +, +, -)$.

(2) An order realizable with a single sign pattern is called *rigid*. It turns out that rigid are exactly the *trivial* orders (when all roots are of the same sign) and the orders in which moduli of positive and negative roots alternate, see [13].

In what follows we denote by $\Sigma_{i_1,i_2,\ldots,i_s}$, $s = \tilde{c}+1$, $i_1+\cdots+i_s = d+1$, sign patterns beginning with i_1 signs $+$ followed by i_2 signs $-$ followed by i_3 signs $+$ etc. Couples in which the sign pattern has just one sign change (i. e. $\tilde{c} = 1$) have been considered in [10, Theorem 1 and Corollary 1]. The result of [10] reads:

Theorem 1. *The sign pattern $\Sigma_{m,n}$, $1 \leq n \leq m$, is realizable with and only with orders such that $\alpha_1 < \gamma_{2n-1}$. For $1 \leq m \leq n$, it is realizable with and only with orders such that $\gamma_{d-2m} < \alpha_1$.*

In the particular case $m = n$, Theorem 1 imposes no restriction on α_1, so in this case all compatible couples are realizable.

Remarks 2. (1) There exist two commuting involutions acting on sign and change-preservation patterns and orders:

$$i_m \ : \ Q(x) \mapsto (-1)^d Q(-x) \ \text{ and } \ i_r \ : \ Q(x) \mapsto x^d Q(1/x)/Q(0).$$

The involution i_m exchanges the letters c and p, the letters P and N and the numbers $\tilde{c}$ and $\tilde{p}$. The involution i_r reads change-preservation patterns and orders from the right. It preserves the numbers $\tilde{c}$ and $\tilde{p}$. The factors $(-1)^d$ and $1/Q(0)$ are introduced to preserve the set of monic polynomials. Obviously, a given couple C is realizable/non-realizable together with the remaining one or three couples obtained under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action defined by i_m and i_r. One has always $i_m(C) \neq C$, but one could have $i_r(C) = C$ or $i_m i_r(C) = C$.

(2) Using the involution i_m one can reduce the study of realizability of couples to the case $\tilde{c} \leq d/2$. The sign patterns $\Sigma_{1,d}$ and $\Sigma_{d,1}$ are canonical. For $m > 1$ and $n > 1$, couples with sign patterns $\Sigma_{m,n}$ have just one sign change, and it is isolated. In this sense they are the closest to the couples with canonical sign pattern.

In this paper we give the exhaustive answer to the question of realizability of couples with sign patterns of the form $\Sigma_{m,n,1}$, i. e. with two sign changes one of which (for $n > 1$) is isolated. This is the second simplest case when the sign pattern is not canonical. (Partial results on it are obtained in [10].) One can observe that $i_r(\Sigma_{m,n,1}) = \Sigma_{1,n,m}$.

Notation 1. We remind that for $\tilde{c} = 2$, we denote by $\alpha_1 < \alpha_2$ the positive roots and by $0 < \gamma_1 < \cdots < \gamma_{d-2}$ the moduli of the negative roots. For a monic hyperbolic polynomial Q realizing a couple C with sign pattern $\Sigma_{m,n,1}$, we denote by $\nu(Q)$ or $\nu(C)$ the index j such that $\gamma_j < \alpha_2 < \gamma_{j+1}$.

Theorem 2. (A) *Suppose that $m > n$.*

(1) *If $n = 1$, then the sign pattern $\Sigma_{m,n,1}$ is canonical.*

(2) *If $n \geq 4$, then a couple C is realizable if and only if $\alpha_1 < \gamma_1$ and $\nu(C) \leq 2n - 2$.*

(3) *If $n = 2$ or 3, then a couple C is realizable if and only if either $\alpha_1 < \gamma_1$ and $\nu(C) \leq 2n - 2$ or $\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2 < \cdots < \gamma_{d-2}$.*

(B) *Suppose that $m = n > 1$. (If $m = n = 1$, then there are no negative roots and one has $\alpha_1 < \alpha_2$.)*

(4) *If $m = n \geq 4$, then a couple is realizable if and only if $\alpha_1 < \gamma_1$.*

(5) *If $m = n = 2$ or $m = n = 3$, then a couple is realizable if and only if either $\alpha_1 < \gamma_1$ or $\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2 < \cdots < \gamma_{d-2}$.*

(C) *Suppose that $m < n$.*

(6) *If $m = 1$ and $n \geq 3$, the sign pattern $\Sigma_{1,n,1}$ is canonical.*

(7) *If $m \geq 2$ and $n \geq 5$, then a couple C is realizable if and only if $\alpha_1 < \gamma_1$*

and $\nu(C) \geq d - 2m = n - m$.

(8) *The necessary and sufficient conditions for realizability of the remaining sign patterns are as follows:*

The sign pattern $\Sigma_{1,2,1}$ is realizable with all three orders NPP, PNP and PPN.

The sign pattern $\Sigma_{2,3,1}$ is realizable with and only with the orders $NPPNN$, $PPNNN$, $PNPNN$, $PNNPN$ and $PNNNP$.

The sign pattern $\Sigma_{2,4,1}$ is realizable with and only with the orders $PNNPNN$, $PNNNPN$ and $PNNNNP$.

The sign pattern $\Sigma_{3,4,1}$ is realizable with and only with the orders $PPNNNNN$, $PNPNNNN$, $PNNPNNN$, $PNNNPNN$, $PNNNNPN$ and $PNNNNNP$.

The rest of the paper contains the proof of Theorem 2.

2. Proof of part (A) of Theorem 2. Part (1) of Theorem 2 needs no proof, see part (1) of Remarks 1. For $m > n$, Theorems 3 and 4 of [10] say that either $\nu(Q) \leq 2n - 1$ or $\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2$. The latter possibility exists only for $n = 2$ and $n = 3$, and it is realizable. All cases when $\nu(Q) \leq 2n - 2$ and $\alpha_1 < \gamma_1$ are also realizable. We prove here that the case $\nu(Q) = 2n - 1$ is not realizable from which parts (2) and (3) of Theorem 2 follow.

Theorem 3. *For $m > n \geq 2$, there exists no monic hyperbolic polynomial Q defining the sign pattern $\Sigma_{m,n,1}$ and with $\nu(Q) = 2n - 1$.*

Proof. We remind that $m + n = d$ and $Q = \sum_{j=0}^{d} a_j x^j$, $a_d = 1$. The set A of hyperbolic polynomials defining the sign pattern $\Sigma_{m,n,1}$ (this is a subset of $Oa_0 \ldots a_{d-1}$) is open and connected, see [14, Theorems 2 and 3]. One knows that the subset of A of polynomials with $\nu(.) \leq 2n-2$ is non-empty ([10, Theorem 4]). If its subset of polynomials with $\nu(.) = 2n-1$ is also non-empty, then by continuity there exists a polynomial Q with $\alpha_2 = \gamma_{2n-1}$ (all other moduli of roots being distinct and $\alpha_1 < \gamma_1$). One can perform a linear change $x \mapsto gx$, $g > 0$, to obtain the condition $\alpha_2 = \gamma_{2n-1} = 1$. So now we concentrate on proving the following theorem from which Theorem 3 follows:

Theorem 4. *For $m > n \geq 2$, there exists no monic hyperbolic polynomial defining the sign pattern $\Sigma_{m,n,1}$ and with roots α_1, α_2 and $-\gamma_i$, where*

$$0 < \alpha_1 < \gamma_1 \leq \cdots \leq \gamma_{2n-2} \leq 1 = \gamma_{2n-1} = \alpha_2 < \gamma_{2n} \leq \cdots \leq \gamma_{d-2}. \qquad \square$$

P r o o f o f T h e o r e m 4. We prove Theorem 4 first in the particular case $m = n + 1$:

Proposition 1. *There exists no monic hyperbolic polynomial defining the sign pattern* $\Sigma_{n+1,n,1}$ *and with roots* α_1, α_2 *and* $-\gamma_i$, *where*

$$0 < \alpha_1 < \gamma_1 \leq \cdots \leq \gamma_{2n-2} \leq 1 = \gamma_{2n-1} = \alpha_2.$$

The proposition is proved after the proof of Theorem 4. The proof of Theorem 4 is performed by induction on m, where the induction base is the case $m = n + 1$, see Proposition 1.

Lemma 1. *Suppose that* $d \geq 3$ *and* $\sigma(Q) = \Sigma_{m,n,1}$. *Set* $Q = (x + \gamma_i)Q_1$, $1 \leq i \leq d - 2$. *Then* $\sigma(Q_1) = \Sigma_{m_1,n_1,1}$, *where either* $m_1 = m$, $n_1 = n - 1$ *or* $m_1 = m - 1$, $n_1 = n$.

P r o o f. Indeed, it is clear that $Q_1(0) > 0$. If the coefficient of x in Q_1 is positive, then the one in Q is also positive and the polynomial Q cannot define the sign pattern $\Sigma_{m,n,1}$. Hence Q_1 defines a sign pattern of the form $\Sigma_{m_1,n_1,1}$, $m_1+n_1 = m+n-1 = d-1$. If $n_1 > n$ (resp. if $n_1 < n-1$), then the coefficient of x^{n+1} in Q is negative (resp. the coefficient of x^n is positive) and the sign pattern of Q is not $\Sigma_{m,n,1}$. □

Suppose that $m > n + 1$. Define the polynomial Q_1 as in Lemma 1. If $n_1 = n - 1$, then by [10, Theorem 4], $\nu(Q_1) \leq 2n - 3$, so $\nu(Q) \leq 2n - 3$. This means that $\alpha_2 < \gamma_{2n-2}$ – a contradiction. If $n_1 = n$, then by induction hypothesis such a polynomial Q_1 does not exist, so Q does not exist either. □

P r o o f o f P r o p o s i t i o n 1. Suppose that there exists a polynomial of the form

$$Q := \sum_{j=0}^{2n+1} t_j x^j = (x^2 - 1)R, \;\; R := x^{2n-1} + b_{2n-2}x^{2n-2} + \cdots + b_0$$

defining the sign pattern $\Sigma_{n+1,n,1}$, where the roots of the factor R are α_1, $-\gamma_1, \ldots, -\gamma_{2n-2}$, $\gamma_i \in (0, 1]$. Then

$$t_j = b_{j-2} - b_j.$$

We represent the polynomial R in the form

$$R = (x - \alpha_1)(x^{2n-2} + e_1x^{2n-3} + \cdots + e_{2n-2}),$$

where e_j is the jth elementary symmetric polynomial of the quantities γ_1, $\ldots$,

γ_{2n-2}. Hence

$$b_{n-1} = -\alpha_1 e_{n-1} + e_n \ \text{ and } \ b_{n+1} = -\alpha_1 e_{n-3} + e_{n-2}, \ \text{ so}$$
$$t_{n+1} = -\alpha_1 e_{n-1} + e_n + \alpha_1 e_{n-3} - e_{n-2}. \tag{2.1}$$

As $t_0 > 0$ and $t_1 < 0$, one obtains that $b_0 < 0$ and $b_1 > 0$, i. e. the factor R defines the sign pattern $\Sigma_{2n-1,1}$. On the other hand

$$b_0 = -\alpha_1\gamma_1 \cdots \gamma_{2n-2} \ \text{ and } \ b_1 = -b_0(1/\alpha_1 - 1/\gamma_1 - \cdots - 1/\gamma_{2n-2}), \ \text{ therefore}$$

$$1/\alpha_1 - 1/\gamma_1 - \cdots - 1/\gamma_{2n-2} > 0. \tag{2.2}$$

Lemma 2. (1) *For $\gamma_j \in (0,1]$, it is impossible to simultaneously have the inequality $t_{n+1} > 0$ (see (2.1)) and the equality*

$$1/\alpha_1 - 1/\gamma_1 - \cdots - 1/\gamma_{2n-2} = 0. \tag{2.3}$$

(2) *For $\gamma_j \in (0,1]$, it is impossible to simultaneously have the inequality $t_{n+1} > 0$ and the inequality (2.2).*

Part (2) of the lemma finishes the proof of Proposition 1. □

P r o o f o f L e m m a 2. Part (1). Indeed, the condition $t_{n+1} > 0$ is equivalent to

$$-e_{n-1} + e_{n-3} - (1/\alpha_1)(e_{n-2} - e_n) > 0. \tag{2.4}$$

Substituting $G := 1/\gamma_1 + \cdots + 1/\gamma_{2n-2}$ for $1/\alpha_1$ in the last inequality one obtains

$$\tau := -e_{n-1} + e_{n-3} - G(e_{n-2} - e_n) > 0. \tag{2.5}$$

Set $I := \{1; 2; \ldots; 2n-2\}$. Denote by Y a fixed product of $n-3$ quantities γ_i with distinct indices (hence a summand of e_{n-3}) and by J the $(n-3)$-tuple of these indices. Further we write $Y(J)$. Set $K := I \setminus J$. The coefficient of $Y(J)$ in e_{n-3} equals 1. There are exactly $\binom{n+1}{2}$ products of $n-1$ quantities γ_j with distinct indices whose $(n-1)$-tuple contains the $(n-3)$-tuple J. Indeed, the remaining two indices must be among the $n+1$ indices of the set K.

On the other hand in the sum $\displaystyle\sum_{J\subset I}\left(\sum_{k,\ell\in K, k<\ell}\gamma_k\gamma_\ell\right)Y(J)$ each summand of e_{n-1} is counted $\binom{n-1}{2}$ times. Therefore one can write

$$e_{n-3} = \sum_{J\subset I} Y(J) \ \text{ and } \ e_{n-1} = \sum_{J\subset I} S_2(J)Y(J)/\binom{n-1}{2}, \quad S_2(J) := \sum_{k,\ell\in K, k<\ell}\gamma_k\gamma_\ell.$$

One can similarly set

$$e_{n-2}=\sum_{J\subset I}S_1(J)Y(J)/(n-2),\quad S_1(J):=\sum_{k\in K}\gamma_k\quad\text{and}$$

$$e_n=\sum_{J\subset I}S_3(J)Y(J)/\binom{n}{3},\qquad S_3(J):=\sum_{k,\ell,m\in K,k<\ell<m}\gamma_k\gamma_\ell\gamma_m.$$

The inequality $\gamma_i\gamma_j\gamma_k\leq(\gamma_i^2+\gamma_j^2)\gamma_k/2$ implies that each term in the sum $S_3(J)$ either equals a product of terms of $S_1(J)$ and $S_2(J)$ or is majorized by the half-sum of such terms. Counting the terms in these three sums yields

$$S_3(J)/\binom{n+1}{3}\leq(S_1(J)/(n+1))\left(S_2(J)/\binom{n+1}{2}\right),\tag{2.6}$$

so for the coefficient S of $Y(J)$ in the product $-G(e_{n-2}-e_n)$ (see (2.5)) one gets

$$\begin{aligned}S&:=-S_1(J)/(n-2)+S_3(J)/\binom{n}{3}\\&\leq\ -S_1(J)\left(1/(n-2)-S_2(J)\binom{n+1}{3}/(n+1)\binom{n}{3}\binom{n+1}{2}\right).\end{aligned}\tag{2.7}$$

The following inequalities hold true:

$$GS_1(J)>S_1(J)\sum_{j\in K}1/\gamma_j\geq(n+1)^2\tag{2.8}$$

(we apply the inequality between the mean arithmetic and the mean harmonic here). One finds directly that

$$\binom{n+1}{3}/(n+1)\binom{n}{3}\binom{n+1}{2}=1/(n-2)\binom{n+1}{2}.\tag{2.9}$$

Consider the product $Y(J)$ in the different terms of the left hand-side of (2.5). Its coefficient equals

$$\begin{aligned}\kappa&:=-S_2(J)/\binom{n-1}{2}+1-GS_1(J)/(n-2)+GS_3(J)/\binom{n}{3}\\&<\ -S_2(J)/\binom{n+1}{2}+1-GS_1(J)/(n-2)+GS_1(J)S_2(J)/(n-2)\binom{n+1}{2}\\&=\ (1-S_2(J)/\binom{n+1}{2})(1-GS_1(J)/(n-2)).\end{aligned}$$

For the first term of the second line we use the inequality $\binom{n+1}{2} > \binom{n-1}{2}$; for the rightmost term of the second line we use (2.6) and (2.9). The second factor is negative, see (2.8). The first factor is non-negative, because $\gamma_i \in (0,1]$. Hence $\kappa < 0$. This is the case of κ defined for any product $Y(J)$, so $\tau < 0$. This contradicts (2.5).

Part (2). For fixed quantities $\gamma_i \in (0,1]$, the left-hand side of (2.4) decreases as $1/\alpha_1$ increases. This follows from $e_{n-2} \geq e_n$. To prove the latter inequality we denote by $Z(J^*)$ the product of $(n-2)$ quantities γ_j with distinct indices whose $(n-2)$-tuple is denoted by J^*. Then

$$e_n = \sum_{J^* \subset I} \left(\left(\sum_{k,\ell \in I \setminus J^*, k<\ell} \gamma_k \gamma_\ell \right) / \binom{n}{2} \right) Z(J^*).$$

The numerator of the coefficient of $Z(J^*)$ contains $\binom{n}{2}$ summands which are ≤ 1. Hence $e_n \leq \sum_{J^* \subset I} Z(J^*) = e_{n-2}$ and part (2) follows from part (1). $\square$

3. Proof of part (B) of Theorem 2. One knows (see the lines after Theorem 1) that for $\tilde{c} = 1$, all sign patterns $\Sigma_{n,n}$ are realizable with all possible orders. Hence there exist degree $d-1$ polynomials $Q_\flat$ with roots $-\gamma_1, \ldots, -\gamma_{d-2}$ and α_2, where $\alpha_2 < \gamma_1$ or $\gamma_k < \alpha_2 < \gamma_{k+1}$, $1 \leq k \leq d-3$, or $\gamma_{d-2} < \alpha_2$. One constructs then Q in the form $(x - \alpha_1)Q_\flat$, where $\alpha_1 > 0$ is very close to 0. Thus the respective coefficients of the polynomials $xQ_\flat$ and Q (except their constant terms) have the same signs and one has $\sigma(Q) = \Sigma_{n,n,1}$. Therefore any order in which $\alpha_1 < \gamma_1$ is realizable with the sign pattern $\Sigma_{n,n,1}$.

On the other hand for $\tilde{c} = 2$ and $n \geq 4$, one has $\alpha_1 < \gamma_1$, while for $n = 2$ and $n = 3$, if $\gamma_1 < \alpha_1$, then $\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2 < \cdots < \gamma_{d-2}$ and this order is realizable with the sign pattern $\Sigma_{n,n,1}$, see [10, Theorem 3]. This proves part (B).

4. Proof of part (C) of Theorem 2. Part (6) of the theorem follows from part (1) of Remarks 1.

To prove parts (7) and (8) of Theorem 2 we use some lemmas whose proofs are given in Section 5.

Lemma 3. *For a polynomial P defining the sign pattern $\Sigma_{n-1,n,1}$, $n \geq 5$, one has $\nu(P) \geq 1$. All couples C with sign pattern $\Sigma_{n-1,n,1}$, $n \geq 5$, and $\nu(C) \geq 1$ are realizable.*

Remark 1. Lemma 3 does not hold for $n = 4$ as shown by the following example:

$$\begin{aligned} P &:= (x+1.01)^5(x-1)(x-0.1) \\ &= x^7 + 3.95x^6 + 4.746x^5 - 0.41309x^4 - 5.11019095x^3 \\ &\quad -3.642011005x^2 - 0.63580905x + 0.105101005. \end{aligned}$$

If one perturbs the five-fold root at -1.01 to obtain five distinct real roots close to it, one gets a polynomial P with $\nu(P) = 0$.

Lemma 4. *If for the hyperbolic polynomial Q, one has $\sigma(Q) = \Sigma_{2,n,1}$, $n \geq 4$, then $\nu(Q) \geq n-2$.*

Lemma 5. *If for the hyperbolic polynomial Q, one has $\sigma(Q) = \Sigma_{3,5,1}$, then $\nu(Q) \geq 2$.*

We prove part (7) of Theorem 2 by induction on n and m. The proof of part (7) for $\Sigma_{2,5,1}$, $\Sigma_{3,5,1}$ and $\Sigma_{4,5,1}$ follows from Lemmas 4, 5 and 3 respectively. We observe that part (7) is true for couples with sign patterns $\Sigma_{1,n,1}$ (which are canonical).

Suppose that part (7) is proved

$$\text{for } n \leq n_0 \ (n_0 \geq 5), \ 1 \leq m \leq n_0 - 1 \qquad \text{and}$$

$$\text{for } n = n_0 + 1, \ 1 \leq m \leq m_0 \ (1 \leq m_0 \leq n_0 - 1).$$

Consider a couple C with sign pattern $\Sigma_{m_0+1,n_0+1,1}$. If $m_0 = n_0 - 1$, then we apply Lemma 3.

Suppose that $1 \leq m_0 \leq n_0 - 2$. We apply Lemma 1. We represent a polynomial Q (with $\sigma(Q) = \Sigma_{m_0+1,n_0+1,1}$) realizing the couple C in the form $(x+\gamma_1)Q_1$. If $\sigma(Q_1) = \Sigma_{m_0,n_0+1,1}$, then by inductive assumption we get

$$\nu(Q_1) \geq n_0 + 1 - m_0 > n_0 + 1 - (m_0+1), \ \text{ so } \ \nu(Q) \geq n_0 + 1 - (m_0+1).$$

If $\sigma(Q_1) = \Sigma_{m_0+1,n_0,1}$, then $\nu(Q_1) \geq n_0 - (m_0+1)$ which for $m_0 \leq n_0 - 2$ is positive. In this case, as γ_1 is the smallest of moduli of negative roots, the inequality $\nu(Q_1) \geq 1$ implies $\gamma_2 < \alpha_2$ and $\nu(Q) = \nu(Q_1) + 1 \geq n_0 + 1 - (m_0+1)$. This proves part (7).

Part (8). The claim about $\Sigma_{1,2,1}$ follows from [10, Example 2].

The statement about $\Sigma_{2,3,1}$ results from [10, Theorems 3 and 4] (for the orders $NPPNN$, $PNNNP$, $PNNPN$, $PNPNN$ and the ones with which it is not realizable). It results for the order $PPNNN$ from the following example:

$$\begin{aligned} & (x+1.3)(x+1.2)(x+1.1)(x-1)(x-0.5) \\ = \quad & x^5+2.1x^4-0.59x^3-2.949x^2-0.419x+0.858. \end{aligned}$$

The result concerning the sign pattern $\Sigma_{2,4,1}$ is proved in [15, Subsection 3.2].

Realizability of the sign pattern $\Sigma_{3,4,1}$ with the order $PPNNNNN$ follows from Remark 1. Its realizability with the orders $PNPNNNN$, $PNNPNNN$, $PNNNPNN$, $PNNNNPN$ and $PNNNNNP$, as well as its non-realizability with the remaining orders, can be deduced from [10, Theorems 3 and 4].

5. Proofs of Lemmas 3, 4 and 5.

Proof of Lemma 3. The second claim of the lemma follows from [10, Part (2) of Theorem 4], so we prove only its first claim. One has to show that there exists no polynomial P with $\sigma(P)=\Sigma_{n-1,n,1}$ and $\nu(P)=0$. Suppose that such a polynomial exists. Then there exists also a polynomial P with $\sigma(P)=\Sigma_{n-1,n,1}$ and such that $\alpha_2=\gamma_1=1$ (this is proved exactly as in the beginning of the proof of Theorem 3). Set

$$U:=\prod_{j=1}^{2n-3}(x+\gamma_j)=\sum_{j=0}^{2n-3}u_{2n-3-j}x^j,\ \ u_j>0, \qquad \text{so}$$

$$P=(x^2-(1+\alpha_1)x+\alpha_1)U=\sum_{j=0}^{2n-1}p_{2n-1-j}x^j.$$

Thus

$$p_{n-1}=u_{n-1}-(1+\alpha_1)u_{n-2}+\alpha_1u_{n-3}.$$

We set $u_j:=e_j+e_{j-1}$, where e_j is the jth elementary symmetric polynomial of the quantities γ_2, …, γ_{2n-3}, with $e_0=1$ and $e_{-1}=0$. Hence

$$\begin{aligned} & e_{n-1}+e_{n-2}-(1+\alpha_1)(e_{n-2}+e_{n-3})+\alpha_1(e_{n-3}+e_{n-4})<0,\ \text{ i. e.} \\ & e_{n-1}-\alpha_1e_{n-2}-e_{n-3}+\alpha_1e_{n-4}<0. \end{aligned} \tag{5.1}$$

We apply a reasoning similar to the one used in the proof of Lemma 2. Namely, we prove that it is impossible to have simultaneously (5.1) and

$$1/\alpha_1>\sum_{j=2}^{2n-3}1/\gamma_j. \tag{5.2}$$

To this end we first show that one cannot have at the same time (5.1) and

$$1/\alpha_1 = \sum_{j=2}^{2n-3} 1/\gamma_j. \tag{5.3}$$

So suppose that the couple of conditions (5.1) and (5.3) is possible. Set $1/\alpha_1 = S_{-1} := \sum_{j=2}^{2n-3} 1/\gamma_j$. This means that

$$S_{-1}(e_{n-1} - e_{n-3}) - e_{n-2} + e_{n-4} < 0. \tag{5.4}$$

Set $I := \{2; 3; \ldots; 2n-3\}$. Denote by Y a fixed product of $n-4$ quantities γ_i with distinct indices (hence a summand of e_{n-4}) and by J the $(n-4)$-tuple of these indices. Further we write $Y(J)$. Set $K := I \setminus J$. The coefficient of $Y(J)$ in e_{n-4} equals 1. There are exactly $\binom{n}{2}$ products of $n-2$ quantities γ_j with distinct indices whose $(n-2)$-tuple contains the $(n-4)$-tuple J. Indeed, the remaining two indices must be among the n indices of the set K.

On the other hand, in the sum $\sum_{J\subset I}(\sum_{k,\ell\in K,k<\ell} \gamma_k\gamma_\ell)Y(J)$ each summand of e_{n-2} is counted $\binom{n-2}{2}$ times. Therefore one can write

$$e_{n-4} = \sum_{J\subset I} Y(J) \ \text{ and } \ e_{n-2} = \sum_{J\subset I} S_2(J)Y(J)/\binom{n-2}{2}, \quad S_2(J) := \sum_{k,\ell\in K,k<\ell} \gamma_k\gamma_\ell.$$

One can similarly set

$$e_{n-3} = \sum_{J\subset I} S_1(J)Y(J)/(n-3), \qquad S_1(J) := \sum_{k\in K} \gamma_k \quad \text{and}$$

$$e_{n-1} = \sum_{J\subset I} S_3(J)Y(J)/\binom{n-1}{3}, \quad S_3(J) := \sum_{k,\ell,m\in K,k<\ell<m} \gamma_k\gamma_\ell\gamma_m.$$

For a given $(n-4)$-tuple J, the coefficient of $Y(J)$ in the left hand-side of (5.4) equals

$$\phi := S_{-1}(S_3(J) - S_1(J)) - S_2(J) + 1.$$

The quantity $S_1(J)$ (resp. $S_3(J)$) contains n (resp. $\binom{n}{3}$) terms. As all quantities γ_j, $j \geq 2$, are > 1, one has

$$S_3(J) > S_1(J)\binom{n}{3}/n.$$

Hence

$$\phi > \frac{\binom{n}{3} - n}{\binom{n}{3}} S_3(J)S_{-1} - S_2(J) + 1.$$

We set $S_{-1}(J) = \sum_{j \in K} 1/\gamma_j$. One has

$$S_3(J)S_{-1} > S_3(J)S_{-1}(J) \geq \binom{n}{3} nS_2(J) / \binom{n}{2} \text{ and}$$

$$\frac{\binom{n}{3} - n}{\binom{n}{3}} \cdot \frac{\binom{n}{3}n}{\binom{n}{2}} = \frac{n}{n-1} \cdot \frac{(n-1)(n-2)-6}{3} =: \psi.$$

Thus

$$\phi > (\psi - 1)S_2(J) + 1.$$

For $n \geq 5$, one has $\psi > 1$ and $\phi > 0$. Summing up over all sets J one obtains a contradiction with (5.4).

If one supposes that (5.1) and (5.2) are valid simultaneously, then one again arrives at a contradiction with (5.4). Indeed, if in the product $S_{-1}(e_{n-1} - e_{n-3})$ (see (5.4)) one replaces S_{-1} by a larger positive quantity, then the left-hand side of (5.4) increases. This is due to the fact that the symmetric elementary polynomials e_{n-1} and e_{n-3} contain one and the same number of summands, but the quantities γ_j are > 1, so $e_{n-1} > e_{n-3}$. □

Proof of Lemma 4. We use the involution i_r (see part (1) of Remarks 2) and consider polynomials defining the sign pattern $\Sigma_{1,n,2}$ instead of $\Sigma_{2,n,1}$. We denote again the positive roots by $\alpha_1 < \alpha_2$ and the moduli of the negative roots by $\gamma_1 < \cdots < \gamma_n$. One has $\gamma_n < \alpha_2$ (see [10, Theorem 3] and part (1) of Remarks 2). In the new setting we have to prove that no hyperbolic polynomial Q with $\sigma(Q) = \Sigma_{1,n,2}$ has the property $\gamma_3 < \alpha_1$.

Suppose that such a polynomial exists, so γ_1, γ_2, $\gamma_3 \in (0, \alpha_1)$. We set

$$A_1 := \alpha_1 + \alpha_2, \qquad A_{-1} := 1/\alpha_1 + 1/\alpha_2, \qquad A_{-2} := 1/\alpha_1\alpha_2$$

$$G_1 := \gamma_1 + \gamma_2 + \gamma_3 \qquad G_{-1} := 1/\gamma_1 + 1/\gamma_2 + 1/\gamma_3, \quad H_1 := \gamma_4 + \cdots + \gamma_n,$$

$$H_{-1} := 1/\gamma_4 + \cdots + 1/\gamma_n \quad L_{-2} := \sum_{1 \leq i < j \leq n} 1/\gamma_i\gamma_j, \qquad \delta := \alpha_1\alpha_2\gamma_1 \cdots \gamma_n.$$

The coefficients of x^{n+1} and x^2 of Q are given by

$$c_{n+1} := -A_1 + G_1 + H_1 \quad \text{and}$$

$$c_2 := (-A_{-1}(G_{-1} + H_{-1}) + A_{-2} + L_{-2})\delta.$$

We show that it is impossible to have simultaneously the inequalities

$$\begin{array}{lll} & \gamma_i \leq \alpha_1, \ \ i = 1, \ 2, \ 3, & \gamma_j \geq \alpha_1, \ \ j \geq 4 \\ (5.5) & & \\ & c_{n+1} \leq 0 \qquad \text{and} & c_2 < 0. \end{array}$$

Suppose that these inequalities except the last one hold true. We show that then the minimal possible value of c_2/δ is positive. Fix the sum $g_{12} := \gamma_1 + \gamma_2$. The terms in the expression for c_2/δ containing γ_1 or γ_2 are:

$$((-A_{-1} + 1/\gamma_3 + H_{-1})g_{12} + 1)/\gamma_1\gamma_2 =: \eta/\gamma_1\gamma_2.$$

As $\gamma_3 \leq \alpha_1$ and $H_{-1} \geq 1/\gamma_4$, the quantity η is not smaller than

$$(-1/\alpha_2 + 1/\gamma_4)g_{12} + 1 = (\gamma_1\alpha_2 + \gamma_2\alpha_2 + (\alpha_2 - \gamma_1 - \gamma_2)\gamma_4)/\alpha_2\gamma_4.$$

Conditions (5.5) except the last of them imply $\alpha_2 > \gamma_1 + \gamma_2$. Hence $\eta > 0$. This means that for fixed g_{12}, the quantity c_2/δ is minimal when $1/\gamma_1\gamma_2$ is minimal, i. e. when $\gamma_1 = \gamma_2$. In the same way one obtains that minimality of c_2/δ is possible only for $\gamma_1 = \gamma_2 = \gamma_3$ which we suppose to hold true from now on.

Fix the sum A_1. The condition $c_{n+1} \leq 0$ implies

$$A_1 \geq 3\gamma_1 + H_1 \geq 3\gamma_1 + \gamma_4.$$

The terms containing α_1 or α_2 in c_2/δ are:

$$\begin{aligned} & (-A_1(3/\gamma_1 + H_{-1}) + 1)A_{-2} \leq (-A_1(3/\gamma_1 + 1/\gamma_4) + 1)A_{-2} \\ = \ & (-A_1(3\gamma_4 + \gamma_1) + \gamma_1\gamma_4)A_{-2}/\gamma_1\gamma_4 < (-(3\gamma_4 + \gamma_1)(3\gamma_1 + \gamma_4) + \gamma_1\gamma_4)A_{-2}/\gamma_1\gamma_4 \\ = \ & (-3\gamma_4^2 - 9\gamma_1\gamma_4 - 3\gamma_1^2)A_{-2}/\gamma_1\gamma_4 < 0. \end{aligned}$$

This expression is minimal when A_{-2} is maximal, i. e. when $\alpha_1 = \gamma_1$. In this case the coefficient which multiplies $1/\alpha_2$ in the quantity c_2/δ equals

$$-3/\gamma_1 - H_{-1} + 1/\gamma_1 = -2/\gamma_1 - H_{-1} < 0,$$

so c_2/δ is minimal when $1/\alpha_2$ is maximal, i. e. when α_2 is minimal, so $c_{n+1} = 0$ and $\alpha_2 = H_1 + 2\gamma_1$. Set $H_{-2} := \sum_{4 \leq i < j \leq n} 1/\gamma_i\gamma_j$. Thus we have

$$L_{-2} = 3/\gamma_1^2 + (3/\gamma_1)H_{-1} + H_{-2} \ \text{ and}$$

$$\begin{aligned} c_2/\delta &= -(1/\gamma_1 + 1/(H_1 + 2\gamma_1))(3/\gamma_1 + H_{-1}) \\ &\quad +1/\gamma_1(H_1 + 2\gamma_1) + 3/\gamma_1^2 + (3/\gamma_1)H_{-1} + H_{-2} \\ &= -2/\gamma_1(H_1 + 2\gamma_1) + (2/\gamma_1)H_{-1} - H_{-1}/(H_1 + 2\gamma_1) + H_{-2}. \end{aligned}$$

For $n = 4$, the term H_{-2} is absent, one has $H_1 = \gamma_4$, $H_{-1} = 1/\gamma_4$, so

$$c_2/\delta = -2/\gamma_1(\gamma_4 + 2\gamma_1) + 2/\gamma_1\gamma_4 - 1/\gamma_4(\gamma_4 + 2\gamma_1) = 3/\gamma_4(\gamma_4 + 2\gamma_1) > 0.$$

For $n \geq 5$, one gets $H_{-1} > 1/(H_1 + 2\gamma_1)$ and $H_{-2}(H_1 + 2\gamma_1) > H_{-1}$, so again $c_2/\delta > 0$. Thus one cannot have all conditions (5.5) fulfilled at the same time which proves the lemma. □

Proof of Lemma 5. As in the proof of Lemma 4 we use the involution i_r (see part (1) of Remarks 2). We consider polynomials defining the sign pattern $\Sigma_{1,5,3}$ instead of $\Sigma_{3,5,1}$. Thus for the positive roots $\alpha_1 < \alpha_2$ and for the moduli of the negative roots $\gamma_1 < \cdots < \gamma_6$ one has $\gamma_6 < \alpha_2$ (see [10, Theorem 3] and part (1) of Remarks 2). We have to show that no hyperbolic polynomial Q with $\sigma(Q) = \Sigma_{1,5,3}$ satisfies $\gamma_5 < \alpha_1$.

Suppose that such a polynomial exists, so $\gamma_1, \ldots, \gamma_5 \in (0, \alpha_1)$. Making a linear change $x \mapsto bx$, $b > 0$, one obtains the condition $\alpha_1 = 1$. We set

$$\begin{array}{lll} A_1 := \alpha_1 + \alpha_2, & A_{-1} := 1/\alpha_1 + 1/\alpha_2, & A_{-2} := 1/\alpha_1\alpha_2 \\ H_1 := \displaystyle\sum_{j=1}^{6} \gamma_j & H_{-1} := \displaystyle\sum_{j=1}^{6} 1/\gamma_j, & H_{-2} := \displaystyle\sum_{1\leq i<j\leq 6} 1/\gamma_i\gamma_j, \\ H_{-3} := \displaystyle\sum_{1\leq i<j<k\leq 6} 1/\gamma_i\gamma_j\gamma_k, \quad \text{and} & & \delta := \alpha_1\alpha_2\gamma_1\cdots\gamma_6. \end{array}$$

The coefficients of x^7 and x^3 of Q are equal to

$$\begin{aligned} c_7 &:= -A_1 + H_1 \qquad \text{and} \\ c_3 &:= (-A_{-1}H_{-2} + A_{-2}H_{-1} + H_{-3})\delta. \end{aligned}$$

We show that it is impossible to simultaneously obtain the conditions

$$\gamma_1 \leq \cdots \leq \gamma_5 \leq 1 \leq \gamma_6, \;\; c_7 < 0 \;\text{ and }\; c_3 < 0.$$

We prove that for fixed sum $g_1 := \gamma_1 + \gamma_2$, the quantity c_3/δ is minimal for $\gamma_1 = \gamma_2$. To this end we set

$$g_{-1} := 1/\gamma_1 + 1/\gamma_2 = g_1/\gamma_1\gamma_2, \;\; G_{-1} := 1/\gamma_3 + \cdots + 1/\gamma_6, \;\; G_{-2} := \sum_{3\leq i<j\leq 6} 1/\gamma_i\gamma_j$$

and observe that the terms in c_3/δ containing γ_1 or γ_2 are:

$$\begin{aligned} & -A_{-1}(1/\gamma_1\gamma_2+g_{-1}G_{-1})+A_{-2}g_{-1}+(1/\gamma_1\gamma_2)G_{-1}+g_{-1}G_{-2} \\ = \quad & (-A_{-1}(1+g_1G_{-1})+A_{-2}g_1+G_{-1}+g_1G_{-2})/\gamma_1\gamma_2 \\ = \quad & ((-A_{-1}+G_{-1})+g_1(-A_{-1}G_{-1}+A_{-2}+G_{-2}))/\gamma_1\gamma_2. \end{aligned}$$

Obviously $-A_{-1}+G_{-1}>1/\gamma_3+1/\gamma_4\geq 2/\gamma_4$. Notice that $g_1\leq 2$. We show that

$$1/\gamma_4-A_{-1}G_{-1}+A_{-2}+G_{-2}\geq 0 \tag{5.6}$$

which means that c_3/δ is minimal when $\gamma_1\gamma_2$ is maximal, i. e. when $\gamma_1=\gamma_2$. Inequality (5.6) results from the two inequalities

$$\begin{aligned} (1/\alpha_2)G_{-1} & \leq 1/\gamma_3\gamma_6+1/\gamma_4\gamma_6+1/\gamma_5\gamma_6+A_{-2} \quad \text{and} \\ G_{-1} & \leq 1/\gamma_4+1/\gamma_3\gamma_4+1/\gamma_3\gamma_5+1/\gamma_4\gamma_5. \end{aligned}$$

The first of them follows from $1/\gamma_i\gamma_6\geq 1/\gamma_i\alpha_2$, $i=3$, 4, 5 and $1/\alpha_2\gamma_6\leq 1/\alpha_2$. The second results from

$$G_{-1}\leq 1/\gamma_3+1/\gamma_4+2/\gamma_5\leq 1/\gamma_4+1/\gamma_3\gamma_4+1/\gamma_3\gamma_5+1/\gamma_4\gamma_5.$$

It should be noticed that if instead of γ_1 and γ_2 one chooses other two quantities γ_i and γ_j, $1\leq i<j\leq 5$, the proof that c_3/δ is minimal for $\gamma_i=\gamma_j$ can be performed in the same way.

Thus one needs to consider only the situation $\gamma_1=\cdots=\gamma_5$ in which case

$$c_3/\delta=-(1+1/\alpha_2)(10/\gamma_1^2+5/\gamma_1\gamma_6)+(1/\alpha_2)(5/\gamma_1+1/\gamma_6)+10/\gamma_1^3+10/\gamma_1^2\gamma_6.$$

The coefficient of $1/\alpha_2$ is

$$(-10/\gamma_1^2+5/\gamma_1)+(-5/\gamma_1\gamma_6+1/\gamma_6)<0.$$

Hence c_3/δ is minimal when α_2 is minimal, i. e. when $\alpha_2=5\gamma_1+\gamma_6-1$. Further we shorten the notation as follows: we set $a:=\alpha_2$, $r:=\gamma_1$ and $w:=\gamma_6$. Hence $0<r\leq 1\leq w\leq a$. For $a=5r+w-1$, we compute the product

$$ar^3wc_3/\delta=(5r+w-1)r^3wc_3/\delta=:-2K.$$

One obtains

$$\begin{aligned} K & = 12r^3+25r^2w+5rw^2-25r^2-30rw-5w^2+5r+5w \\ & = -(1-r)(12r^2+25rw+5w^2-5r-5w)-8r^2<0. \end{aligned}$$

The minimal possible value of c_3/δ being positive, one cannot have $c_7<0$ and $c_3<0$ at the same time. This proves the lemma. $\square$

REFERENCES

[1] F. Cajori. A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity. *Colorado Coll. Publ.* **12** (1910), 171–215.

[2] D. R. Curtiss. Recent extensions of Descartes' rule of signs, *Ann. of Math. (2)* **19**, *4* (1918), 251–278.

[3] J.-P. de Gua de Malves. Démonstrations de la Règle de Descartes, Pour connoître le nombre des Racines positives & négatives dans les Équations qui n'ont point de Racines imaginaires. Memoires de Mathématique et de Physique tirés des registres de l'Académie Royale des Sciences (1741), 72–96.

[4] R. Decartes. The Geometry of René Descartes with a facsimile of the first edition, translated by D. E. Smith and M. L. Latham. New York, Dover Publications, Inc., 1954.

[5] J. Forsgård, V. P. Kostov, B. Z. Shapiro. Could René Descartes have known this? *Exp. Math.* **24**, *4* (2015), 438–448.

[6] J. Forsgård, D. Novikov, B. Shapiro. A tropical analog of Descartes' rule of signs. *Int. Math. Res. Not. IMRN 12* (2017), 3726–3750, arXiv:1510.03257 [math.CA].

[7] J. Fourier. Sur l'usage du théorème de Descartes dans la recherche des limites des racines. Bulletin des sciences par la Société philomatique de Paris (1820) 156–165, 181–187; œuvres 2, 291–309, Gauthier-Villars, 1890.

[8] C. F. Gauss. Beweis eines algebraischen Lehrsatzes. *J. Reine Angew. Math.* **3** (1828), 1–4.

[9] J. L. W. V. Jensen. Recherches sur la théorie des équations. *Acta Math.* **36**, *1* (1913), 181–195.

[10] V. P. Kostov. Descartes' rule of signs and moduli of roots. *Publ. Math. Debrecen* **96**, *1–2* (2020), 161–184.

[11] V. P. Kostov. Hyperbolic polynomials and canonical sign patterns. *Serdica Math. J.* **46**, *2* (2020) 135–150.

[12] V. P. Kostov. Which sign patterns are canonical?. *Results Math.* **77**, *6* (2022), paper No. 235, 12 pp.

[13] V. P. Kostov. Hyperbolic polynomials and rigid orders of moduli. *Publ. Math. Debrecen* **100**, *1–2* (2022), 119–128.

[14] V. P. Kostov. Univariate polynomials and the contractibility of certain sets. God. Sofii. Univ. "Sv. Kliment Okhridski". Fac. Mat. Inform. (Annual of Sofia University "St. Kliment Ohridski", Faculty of Mathematics and Informatics) **107** (2020), 81–105.

[15] V. P. Kostov. Moduli of roots of hyperbolic polynomials and Descartes' rule of signs, Constructive Theory of Functions, Sozopol 2019 (Eds B. Draganov, K. Ivanov, G. Nikolov and R. Uluchev). Sofia, Prof. M. Drinov Acad. Publ. House, 2020, 131–146.

[16] E. Laguerre. Sur la théorie des équations numériques. Journal de Mathématiques pures et appliquées, s. 3, t. 9, 99-146 (1883); œuvres 1, Paris, 1898, Chelsea, New-York, 1972, 3–47.

[17] B. E. Meserve. Fundamental Concepts of Algebra. New York, Dover Publications, 1982.

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