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NEWTON'S METHOD FOR GENERALIZED EQUATIONS UNDER WEAK CONDITIONS

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Abstract. A local convergence analysis is developed for Newton's method in order to approximate a solution of a generalized equations in a Banach space setting. The convergence conditions are based on generalized continuity conditions on the Fréchet dervative of the operator involved and the Aubin property. The specialized cases of our results extend earlier ones using similar information.

1. Introduction. A plethora of applications from diverse disciplines can be reduced using Mathematical Modelling to solving the generalized equation of the form

$$g(x) + G(x) \ni 0. \tag{1.1}$$

Here, B_1, B_2 are denoting Banach spaces, $g : B_1 \longrightarrow B_2$ is a continuously differentiable operator, and $G : B_1 \rightrightarrows B_2$ is a set-valued operator with a closed nonempty graph [12, 13]. The local convergence analysis of the Newton's method

$$g(x_n) + g'(x_n)(x_{n+1} - x_n) + G(x_{n+1}) \ni 0,\ n = 0, 1, 2, \dots \tag{1.2}$$

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for approximating a solution $x^* \in B_1$ of the generalized equation (1.1) when $G = 0$ or not has be given in [1, 2, 3, 4, 6, 7, 11, 14] under various conditions such as Lipschitz, Hölder continuity on the Fréchet derivative of the operator g. Such conditions are important in the convergence of Newton's method, since they control the derivative [5, 8, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

The local convergence analysis of Newton's method (1.2) is revisited using the Aubin property which is related to metric regularity as it is shown by Dontchev and Rockafellar in [12, 13]. It turns out that even in the specialization of Newton's method (1.2), when $G = 0$ the following advantages are obtained

(1) Larger radius of convergence.

(2) Tighter error distances on $\|x_n - x^*\|$ and

(3) An at least as precise information on the location of the solution.

Moreover, we make a special comment on the notable study by Cibulka et al. [9], where the semi-local convergence analysis of the method (1.2) is developed under Lipschitz continuouty conditions and Kantorovich-type assumptions are utilized. However, a direct comparison is not possible, since our results are local. Moreover, we use generalized Lipschitz-type conditions in our analysis in order to include a larger class of problems. However, our approach can certainly be applied to the semi-local case.

The rest of the paper includes: Preliminaries in Section 2; the local convergence in Section 3 and the special cases and numerical examples in Section 4.

2. Preliminaries. We assume familiarity with the concepts of graph (gph G), the domain (dom), the range (rge) and the inverse G^{-1} of a set-valued operator G [1, 2, 3]. Moreover, we use d, e which are the standard notations for the distance and excess, respectively between two subsets of B_1. Let $R > 0$.

Define the linearization error for some continuously differentiable function $f : [0, R) \longrightarrow (-\infty), +\infty)$

$$e_f(v_1, v_2) = f(v_2) - f(v_1) - f'(v_1)(v_2 - v_1)$$

for each $v_1, v_2 \in [0, R)$,

$$E_f(u_1, u_2) = g(u_2) - g(u_2) - f'(u_1)(u_2 - u_1)$$

for each $u_1, u_2 \in B_1$ and

$$E_{g+G,x^*}(v) = g(x) + f'(x)(v - x^*) + G(v)$$

for each $x, v \in B_1$.

The definition of the Aubin property and the following version of the contraction mapping principle are given in order to make the article as self contained as possible. More information can be found in [8, 11, 12].

The notation $U(x, \alpha)$ is used to denote an open ball centered at $x \in B_1$ and of radius $\alpha > 0$. Moreover, $U[x, \alpha]$ stands for the closure of $U(x, \alpha)$.

Definition 2.1 ([13]). *Let $\tilde{y} \in B_2$ for $\tilde{x} \in B_1$. Then, the inverse operator G^{-1} of G is said to have the Aubin property at $(\tilde{y}, \tilde{x})$ with modulus c, provided that $\tilde{x} \in G^{-1}(\tilde{y})$, if $gphG^{-1}$ is locally closed at $(\tilde{y}, \tilde{x})$, for $\tilde{x} \in G^{-1}(\tilde{y})$, and there exist constants $a, b > 0$ so that*

$$e^{-1}(G^{-1}(y) \cap U[\tilde{x}, a],\, G^{-1}(y_1)) \leq c\|y - y_1\|$$

for each $y, y_1 \in U[\tilde{y}, b]$.

Theorem 2.2 ([10]). *Let $\Psi : B_1 \rightrightarrows B_1$ be a set-valued operator and $z \in B_1$. Suppose:*
There exist scalars $\beta > 0$ and $p \in (0, 1)$ so that $gph\Psi \cap (U[z, \beta] \times U[z, \beta])$ is closed and

(*i*) $d(z, \Psi(z)) \leq \beta(1 - p)$

(*ii*) $e(\Psi(x) \cap U[z, \beta], \Psi(z_1)) \leq p\|x - z_1\|$ *for each* $x, z_1 \in U[z, \beta]$.

Then, there exists $z^* \in U[z, \beta]$ so that $z^* \in \Psi(z^*)$, i.e. Ψ admits a fixed point in $U[z, \beta]$.

3. Local Convergence. It is worth noticing that the Aubin property is related to the metric regularity [13]. Consequently, the results are provided in terms of metric regularity. But first, we need a relationship between different types of majorant conditions.

Assume $R > 0$.

Definition 3.1. *A function $h_0 : [0, R) \longrightarrow (-\infty, +\infty)$ which is continuous and non-decreasing is said to be a center-majorant function for g on $U(x^*, R)$ with modulus c_1 if for each $y \in U[x^*, R]$*

(*A*1) $c_1\|g'(y) - g'(x^*)\| \leq h_0(\|y - x^*\|)$.

(*A*2) *The function $h_0(t) - 1$ has a smallest zero denoted by ρ which satisfies $\rho \in (0, R]$.*

Definition 3.2. *A function $h = h(h_0) : [0, \rho) \longrightarrow (-\infty, +\infty)$ which is continuous and non-decreasing is said to be a restricted -majorant function for g on $U(x^*, \rho)$ with modulus c_1 if for each $\theta \in [0, 1]$, $y \in U(x^*, \rho)$*

$(A3)$ $c_1\|g'(y)-g'(x^*+\theta(y-x^*))\| \leq h((1-\theta)\|y-x^*\|)$.

Notice that the function h_0 depends on x^* and $\bar{\rho}$, where as the function h relies on x^*, ρ and h_0.

$(A4)$ Assume:

$$h_0(s) \leq h(s) \text{ for each } s \in [0,\rho). \tag{3.1}$$

Definition 3.3. *A function $h_1 : [0,R) \longrightarrow (-\infty,+\infty)$ which is continuous and non-decreasing is said to be a majorant function for g on $U(x^*,\rho)$ with modulus $c_1 > 0$ if for each $\theta \in [0,1]$, $y \in U(x^*,\bar{\rho})$.*

$(A3)'$ $c_1\|g'(y)-g'(x^*+\theta(y-x^*))\| \leq h_1((1-\theta)\|y-x^*\|)$.

It follows by these definitions that

$$h_0(s) \leq h_1(s) \text{ and } h(s) \leq h_1(s) \text{ for each } s \in [0,\rho). \tag{3.2}$$

Thus, the results in the literature using only h_1 (see e.g.[11, 14, 15] for $G=0$) can be replaced by the pair (h_0,h) resulting to finer error distances, a larger convergence radius and a more precise and larger uniqueness radius for the solution x^*. These advantages are obtained under the same computational cost, since in practice the computation of the function h_1 requires that of h_0 and h as special cases.

Define the Newton iteration for solving the equation $h(s)=0$ given by

$$\begin{aligned} s_0 &= \|x_0-x^*\| \\ s_{n+1} &= \left|\frac{\int_0^1 \bar{h}((1-\theta)s_n)d\theta s_n}{1-h_0(s_n)}\right| \end{aligned} \tag{3.3}$$

and for each $n=0,1,2,\ldots$, where $\bar{h} = \begin{cases} h_0 & n=0 \\ h & n=1,2,3\ldots \end{cases}$.

Furthermore, define the set-valued operator $\Psi_x : B_1 \rightrightarrows B_1$ by

$$\Psi_x(v) = E^{-1}_{g+G,x^*}(E_g(x,v)-E_g(x^*,v)), \tag{3.4}$$

where

$$E^{-1}_{g+G,x^*}(w) = \{w_1 \in B_1 : w \in E_{g+G,x^*}(w_1)\}. \tag{3.5}$$

$(A5)$ The set valued operator $v \longrightarrow E^{-1}_{g+H,x^*}(v)$ posesses the Aubin property at zero for x^*, with modulus c_1 and related parameters $a_1, b_1 > 0$.

Define the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\varphi(s) = \left[\int_0^1 h((1-\theta)s)d\theta + h_0(s)\right] s - b_1.$$

It follows by this definition that $\varphi(0) = -b_1 < 0$ and $\lim\limits_{s\longrightarrow+\infty} \varphi(s) = +\infty$. Then, by the intermediate value theorem the equation $\varphi(s) = 0$ has solutions in $(0, +\infty)$. Denote by ρ_0 the smallest such solution.

$(A6)$ $\rho_0 \leq \rho$.

Two auxiliary results are needed.

Lemma 3.4. *Assume the conditions* $(A1)$–$(A5)$ *are valid. Then, the following items are also valid:*

(i) $c_1\|E_g(z, x^*)\| \leq e_h(\|x^* - z\|, 0)$, *for each* $z \in U(x^*, \rho_0)$ *and*

(ii) $\|E_g(z, v) - E_g(x^*, v)\| \leq b_1$, *for each* $U(x^*, s)$ *and* $s \in [0, \rho_0)$.

Proof. Notice that for each $\theta \in [0, 1]$:

$$\|x^* + (1-\theta)(z - x^*) - x^*\| = (1-\theta)\|z - x^*\| \leq \|z - x^*\| < \rho_0,$$

thus, $x^* + (1-\theta)(z - x^*) \in U(x^*, \rho_0)$. By the definition of the operator E_g, we can first write

$$c_1\|E_g(z, x^*)\| \leq c_1 \int_0^1 \|g'(z) - g'(x^* + (1-\theta)(z - x^*)\|\|z - x^*\|d\theta$$

leading to (i) by integration by parts and the definition of the function e_h. Moreover, from the definition of ρ_0, E_g and the conditions $(A1)$ and $(A5)$ we get in turn for $z \in U(x^*, s)$ and $v \in U(x^*, s)$

$$\begin{aligned}
\|E_g(z, v) - E_g(x^*, v)\| &\leq \|E_g(z, x^*)\| + \|g'(z) - g'(x^*)\|\|v - x^*\| \\
&\leq [e_h(\|z - x^*\|, 0) + h_0(\|v - x^*\|)]\|v - x^*\| \\
&\leq \left[\int_0^1 h((1-\theta)\rho_0)d\theta + h_0(\rho_0)\right] \rho_0 = b_1.
\end{aligned}$$

□

$(A7)$ The equation $\displaystyle\int_0^1 h((1-\theta)s)d\theta + h_0(s) - 1 = 0$ has a smallest solution $\rho_1 \in (0, \rho_0)$.

Define the radius

$$\rho^* = \min\{a_1, \rho_1\}.$$

Notice that if $z \in \Psi_x(z)$, then

$$g(x) + f'(x)(z - x) + G(z) \ni 0.$$

Lemma 3.5. *Assume that the conditions* $(A1)$–$(A7)$ *are valid. Then, the conditions of the Theorem* 2.2 *are also valid if*

$$\beta = \frac{\int_0^1 h((1-\theta)s)d\theta}{s(1-h_0(s))}\|x^* - x\|$$

and

$$p := h_0(\|x^* - x\|) \in [0, 1)$$

for $x \in U(x^*, s)$ *and* $s \in (0, \rho^*)$. *Moreover, there exists* $z \in \Psi_x(z)$ *so that*

$$\|x^* - z\| \leq \frac{\int_0^1 h((1-\theta)s)d\theta}{s(1-h_0(s))}\|x^* - x\|.$$

Proof. Notice that by the choice of x and ρ^*, $p \in [0, 1)$. By the definitions (3.4) and the excess e and since

$$x^* \in L^{-1}_{g+G,x^*}(\theta) \cap U[x^*, a_1]$$

and $E_g(x^*, x^*) = 0$ it follows

$$d(x^*, \Psi_x(x^*) \leq c_1\|E_g(x, x^*)\| \leq e_g(\|x^* - x\|, 0).$$

The definition of e_g and β give

$$\frac{e_g(\|x^* - x\|, 0)}{1 - h_0(\|x^* - x\|)} \leq \frac{\int_0^1 h((1-\theta)s)d\theta}{s(1-h_0(s))}\|x^* - x\|$$

showing item (i) of the Theorem 2.2. Moreover, we have that for $x \in U(x^*, s)$, $\rho < \rho^* < a_1$. Then, by (3.4)

$$\begin{aligned} & e(\psi_x(z) \cap U[x^*, \beta], \Psi_x(u)) \\ = \; & e(L^{-1}_{g+G,x^*}(E_g(x, z) - E_g(x^*, z)) \cap U[x^*, a_1] \\ = \; & L^{-1}_{g+G,x^*}(E_g(x, z) - E_g(x^*, z)). \end{aligned}$$

But $\beta < \rho^*$ and $z, u \in U[x^*, \beta]$, so we have $z, u \in U[x^*, \rho^*]$. Consequently, by the Lemma 3.4 and $(A6)$ we obtain in turn

$$\|E_g(x, z) - E_g(x^*, z)\| \leq b_1,$$

$$\|E_g(x,u) - E_g(x^*,u)\| \leq b_1,$$

and

$$\begin{array}{rl} & e(\psi_x(z) \cap U[x^*,\beta], \Psi_x(u)) \\ \leq & c_1\|E_g(x,z) - E_g(x^*,z) - E_g(x,u) + E_g(x^*,u)\| \\ \leq & c_1\|g'(x) - g'(x^*)\|\|u - z\| \\ = \ \leq & h_0(\|x - x^*\|)\|u - z\|, \end{array}$$

showing the item (ii) in the Theorem 2.2. Moreover, $x_{n+1} \in \Psi(x_{n+1})$ exists and satisfies (1.2).

Define the sequence $\{s_n\}$ given by the formula (3.6). Notice that

$$s_1 - s_0 = \left(\frac{\int_0^1 h_0((1-\theta)s_0)d\theta}{1 - h_0(s_0)} \right) s_0 \leq 0,$$

so $0 \leq s_1 \leq s_0$. It follows by this definition and a simple inductive argument that

$$s_{n+1} - s_n = \left(\frac{\int_0^1 \bar{h}((1-\theta)s_n)d\theta}{1 - h_0(s_n)} \right) s_n \leq 0.$$

Thus, the sequence $\{s_n\}$ is non-decreasing and bounded from below by 0 and as such it converges to some $\bar{s} \in [0, s_0]$. By letting $n \longrightarrow +\infty$ in the definition of the sequence $\{s_n\}$, we get

$$\bar{s}(1 - h_0(\bar{s})) = \int_0^1 h((1-\theta)\bar{s})d\theta\bar{s}.$$

If $\bar{s} \neq 0$, then

$$\int_0^1 h((1-\theta)\bar{s})d\theta + h_0(\bar{s}) = 1$$

for $\bar{s} \in (0, s_0)$ so, $\bar{s} < \rho_1$. Hence, $\bar{s} = \lim_{n \longrightarrow +\infty} s_n = 0$. Therefore, the sequence $\{s_n\}$ is convergent to zero. □

Notice that by its definition this sequence is non-increasing if the function $\mu(s) = \dfrac{\int_0^1 h((1-\theta)s)d\theta}{s(1 - h_0(s))}$ is non-increasing in $(0, \rho_1)$.

$(A8)$ The equation $\displaystyle\int_0^1 h_0(\theta s)ds - 1 = 0$ has a smallest solution $\delta \in (0, \bar{\rho})$.

Define the parameter $\gamma > 0$ by

$$\gamma = \min\{c_1 b_1, \delta, \rho_0\}.$$

The isolation of x^* as a solution of the equation (1.1) is determined in the next result.

Proposition 3.6. *Assume that the conditions* $(A1)$ *and* $(A5)$ *are valid and the set valued operator* $v \longrightarrow L^{-1}_{g+G,x^*}(v)$ *is single valued in* $U(0, b_1)$. *Then, the equation (1.1) is uniquely solvable by* x^* *in the ball* $U(x^*, \delta)$.

Proof. Suppose that $\bar{x} \in U(x^*, \gamma)$ for $0 < \|\bar{x} - x^*\| < \gamma$ is a solution of the equation (1.1). Let $z = x^* + \theta(\bar{x} - x^*)$. Then, by the condition (A1) and the definition of the parameters γ and ρ_0, we get in turn

$$\begin{aligned} c_1\|E_g(x^*, \bar{x})\| &\leq \int_0^1 c_1\|g'(x^* + \theta(\bar{x} - x^*)) - g'(x^*)\|\|\bar{x} - x^*\|d\theta \\ &\leq \int_0^1 h_0(\theta\|\bar{x} - x^*\|\|\bar{x} - x^*\|d\theta \\ &\leq h_0(\|\bar{x} - x^*\|)\|\bar{x} - x^*\| \\ &< \|\bar{x} - x^*\| \end{aligned}$$

and

$$\|E_g(x^*, \bar{x})\| = \frac{\|\bar{x} - x^*\|}{c_1} \leq b_1.$$

But $\bar{x}$ solves equation (1.1) and

$$0 \in g(\bar{x}) + G(\bar{x}) = E_g(x^*, \bar{x}) + L_{g+G,x^*}(\bar{x}).$$

Thus, $-E_g(x^*, \bar{x}) \in L_{g+G,x^*}(\bar{x})$ and consequently $\bar{x} \in L^{-1}_{g+G,x^*}(-E_g(x^*, \bar{x}))$.

Moreover, the condition $(A6)$ gives

$$e(L^{-1}_{g+G,x^*}(0) \cap U[x^*, a_1], L^{-1}_{g+G,x^*}(-E_g(x^*, \bar{x})) \leq c_1\|E_g(x^*, \bar{x}).$$

Then, since $\|E_g(x^*, \bar{x})\| \leq b_1$,

$$x^* \in L^{-1}_{g+G,x^*}(0) \cap U[x^*, a_1],$$

$$\bar{x} \in L_{g+G,x^*}(-E_g(x^*, \bar{x}))$$

and the hypothesis that the operator $u \longrightarrow L^{-1}_{g+G,x^*}(u)$ is single value in $U(0, b_1)$. It follows

$$L^{-1}_{g+G,x^*}(0) \cap U[x^*, a_1] = \{x^*\}$$

and

$$L_{g+G,x^*}^{-1}(-E_g(x^*,\bar{x})) = \{\bar{x}\},$$

leading to

$$\|\bar{x} - x^*\| \le c_1\|E_g(x^*,\bar{x})\| < \|\bar{x} - x^*\|.$$

Hence, we conclude $\bar{x} = x^*$. □

The local convergence for the Newton's method (1.2) follows in the next result.

Theorem 3.7. *Assume that the conditions* (*A*1)–(*A*7) *are valid and choose* $x_0 \in U(x^*, s_0) - \{x^*\}$ *for* $\|x_0 - x^*\| \le s_0 < \rho^*$. *Then, there exists a sequence* $\{x_n\} \in U(x^*, \rho^*)$ *generated by the Newton's method* (1.2) *convergent to* x^* *and so that* $\|x^* - x_{n+1}\| \le s_{n+1}$ *for* $n = 0, 1, 2, \ldots$. *Additionally, if there exists* $r \in [0, \bar{\rho})$ *satisfying the equation*

$$\int_0^1 h((1-\theta)r)d\theta + h_0(r) - 1 = 0,$$

then, $\rho^* = r$ *is the largest convergence radius for the Newton's method* (1.2). *Moreover, under the conditions of the Proposition* 3.6, *the sequence* $\{x_n\}$ *is unique and* x^* *is also the unique solution of the equation* (1.1) *in the open ball* $U(x^*, \gamma)$.

Proof. Mathematical induction shall establish that there exists $x_{n+1} \in \Psi_{x_n}(x_{n+1})$ so that

$$\|x^* - x_n\| \le s_n \tag{3.6}$$

for $n = 0, 1, 2, \ldots$. By hypothesis $x_0 \in U(x_0, s_0)$. Then, since $\|x_0 - x^*\| \le s_0 < \rho^*$ by the Lemma 3.5 and the definition of the sequence $\{s_n\}$ there exists $x_1 \in \Psi_{x_0}(x_1)$ and the estimates (3.6) are valid if $n = 0$. Assume that there exist $x_j \in U(x^*, \rho^*)$, $j = 0, 1, 2, \ldots, n$ satisfying (3.6). Then, again by the Lemma 3.5 there exists $x_{n+1} \in \Psi_{x_n}(x_{n+1})$ which satisfies the first estimate in (3.6). But, then we have, since $\|x^* - x_n\| \le s_n$ that

$$\begin{aligned} \|x^* - x_{n+1}\| &\le \frac{\int_0^1 \bar{h}((1-\theta)s_n)d\theta\|x_n - x^*\|}{1 - h_0(s_n)} \\ &\le \frac{\int_0^1 \bar{h}((1-\theta)s_n)d\theta s_n}{1 - h_0(s_n)} \\ &= s_{n+1}. \end{aligned}$$

Thus, the induction for the assertion (3.6) is completed. The proof of the part about the largest radius is standard and can be found e.g. in [14].

In order to show the uniqueness of the sequence $\{x_n\}$, assume there exist $y_{n+1}, x_{n+1} \in U(x^*, s_n) \subset U[x^*, \rho^*]$ so that $y_{n+1} \in \Psi_{x_n}(y_{n+1})$ and $x_{n+1} \in \Psi_{x_n}(x_{n+1})$. But the operator $u \longrightarrow L_{g+G,x^*}(u)$ is single valued in the open ball $U(0, b_1)$ so by part (ii) of Lemma 3.4, we deduce $y_{n+1} \in \Psi_{x_n}(y_{n+1})$ and $x_{n+1} \in \Psi_{x_n}(x_{n+1})$. Assume $y_{n+1} \neq x_{n+1}$. Then, it follows as in Lemma 3.4 that

$$\begin{aligned} \|y_{n+1} - x_{n+1}\| &= e(\Psi_{x_n}(y_{n+1}) \cap U[x^*, s_n], \Psi_{x_n}(x_{n+1})) \\ &\leq (1 - h_0(\|x_n - x^*\|))\|y_{n+1} - x_{n+1}\| \\ &< \|y_{n+1} - x_{n+1}\|, \end{aligned}$$

which is a contradiction. Therefore, we conclude that $y_{n+1} = x_{n+1}$, for $n = 0, 1, 2, \ldots$. □

Remark 3.8. We used the same constant c_1 in Definition 3.1, Definition 3.2 and Definition 3.3 for simplicity although they differ in general. If we were to use d_1, d_2 and d_3 instead of c_1, respectively in these definitions, then $d = \max\{d_1, d_2\}$ can be used instead of c_1 in the aforementioned results.

4. Special cases.

Special case 1 (Lipschitz). Let $G = 0$ and $c_1 = 1$. Define functions $h_0(s) = \frac{\ell_0}{2}s^2 - s$, $h(s) = \frac{\ell}{2}s^2 - s$ and $h_1(s) = \frac{\ell_1}{2}s^2 - s$ for some Lipschitz constants ℓ_0, ℓ and ℓ_1. Then, we have

$$\ell_0 \leq \ell_1 \text{ and } \ell \leq \ell_1. \tag{4.1}$$

Thus, $h_0(s) \leq h_1(s)$, $h(s) \leq h_1(s)$, $h_0'(s) \leq h_1'(s)$, and $h'(s) \leq h_1'(s)$ for each $s \in [0, \rho)$. According to Theorem 3.7, the radius ρ^* can be found if we solve the equation

$$\frac{\ell s}{2(1 - \ell_0 s)} = 1,$$

so

$$\rho^* = \frac{2}{2\ell_0 + \ell}$$

or

$$\frac{\ell s}{2(1 - \ell s)} = 1 \text{ (by } (A3))$$

so

$$\rho^* = \frac{2}{3\ell} \tag{4.2}$$

and

$$\rho^* \leq \rho_0^*.$$

The Newton iteration (3.3) becomes

$$s_{n+1} = \frac{\ell s_n^2}{2(1-\ell s_n)}. \tag{4.3}$$

If only the function h_1 is used we must solve the equation

$$\frac{\ell_1 s}{2(1-\ell_1 s)} = 1$$

resulting to

$$\rho_1^* = \frac{2}{3\ell_1}.$$

The value of ρ_1^* is attributed to Traub [27] and Rheinboldt [21]. The corresponding iteration is

$$\bar{s}_{n+1} = \frac{\ell_1 \bar{s}_n^2}{2(1-\ell_1 \bar{s}_n)}.$$

Notice that

$$\rho_1^* \leq \rho^*$$

and

$$s_n \leq \bar{s}_n,$$

for $n = 1, 2, \ldots$. Moreover, the radius of the uniqueness ball using (h_0, h) is $\frac{2}{\ell}$ which is at least as large than the one using $h-1$ which is $\frac{2}{\ell_1}, \frac{2}{\ell_1} < \frac{2}{\ell}$.

Special case 2 ($G = 0$ and $c_1 = 1$). Define the functions

$$h_0(s) = H_0'(s) - H_0'(0)$$

$$h((1-\theta)s) = H'(s) - H'(\theta s)$$

and

$$h_1((1-\theta)s) = H_1'(s) - H_1'(\theta s),$$

where H_0, H, H_1 are twice continuously differentiable functions satisfying:

$$H_0(0) = H(0) = H_1(0) = 0,\ H_0'(0) = H'(0) = H_1'(0) = -1$$

and all three H functions being convex and strictly increasing. Then, we have

$$h_0(s) \leq h_1(s) \text{ and } h(s) \leq h_1(s).$$

Therefore, the local results using only the function H_1 (see [4, 15]) are improved if instead the functions (H_0, H) are utilized with advantages (1)–(3) as stated in the introduction (see also Special case 1). Here, we also assume

$$H_0(s) \leq H(s).$$

Otherwise the preceding results hold with H_0, H replaced by the function h_2 which is defined to be the largest of h_0 and h on the interval $[0, R)$.

Notice also that if $h(s) \leq h_0(s)$ or $H(s) \leq H_0(s)$. Then, clearly the results hold with h_0 or H_0 replacing h or H.

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