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A SIMPLE WAY OF COMPUTING THE MATRIX EXPONENTIAL

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This paper deals with a simple way of computing the matrix exponential. This elementary approach can be applied in the University courses on ordinary differential equations. The advantage of this approach is the avoidance of computation of the Jordan normal form of a real constant coefficient matrix, as well as the method of indefinite coefficients.

As it is well known, the solution of the system of ordinary differential equations

$$\begin{aligned} x' &= Ax, x = (x_1, \ldots, x_n), \\ x(0) &= x_0, x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

A- $n \times n$ real-valued matrix with constant elements is expressed by the formula $x = e^{At}x_0$.

In computing the matrix exponential we must find at first a real Jordan normal form of the matrix A. The corresponding computations are rather heavy. Another way to solve the Cauchy problem (1) is to use the method of the indefinite coefficients. In this case we have to solve a linear system of algebraic equations containing n^2 unknown coefficients. Certainly, a tiresome work has to be done too.

The aim of this report is to comment and to develop another way of computing e^{At} proposed at first in [1] (see also [2]).

Our starting point will be the Euler approach for solving a linear ordinary differential equation with constant coefficients. Euler looked for a solution of the type $e^{\lambda t}$ and concluded that λ satisfied an algebraic equation (characteristic equation).

Imitating his approach we consider the characteristic equation of the matrix A:

$$p(\lambda) = det(A - \lambda E) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0, \tag{2}$$

$$a_0 = (-1)^n.$$

According to the famous Hamilton-Cayley theorem, the matrix A satisfies (2), i.e.

$$a_0A^n + a_1A^{n-1} + \ldots + a_nE = 0 \tag{3}$$

and E is the unit matrix in $\mathbb{R}^n$.

Having in mind (3) we conclude that the matrix exponential e^{At} is a solution of the following matrix differential equation:

$$L\Phi \equiv a_0\Phi^{(n)} + a_1\Phi^{(n-1)} + \ldots + a_nE = 0. \tag{4}$$

Moreover, $\Phi = e^{At}$ satisfies the next Cauchy data:

$$\Phi(0) = E, \Phi'(0) = A, \ldots, \Phi^{(n-1)}(0) = A^{n-1}. \tag{5}$$

So our main problem is to find another solution in a simpler form of (4), (5). This "new" solution will coincide with e^{At} due to the next proposition.

Proposition 1. *The homogeneous Cauchy problem* $L\Phi = 0$, $\Phi(0) = 0, \dots, \Phi^{(n-1)}(0) = 0$ *possesses the unique matrix solution* $\Phi \equiv 0$.

The proof is obvious as if $\Phi = (a_{ij})_{i,j=1}^n$ then $La_{ij} = 0$, $a_{ij}(0) = \dots = a_{ij}^{(n-1)}(0) = 0$ and the Cauchy problem for the previous scalar equation possesses the trivial solution 0 only.

Suppose now that the functions $\{\varphi_1, \dots, \varphi_n\}$ form a real basis of solutions of the scalar differential equation

(6) $$L(\varphi_j) = 0, 1 \le j \le n.$$

Then the matrix $\varphi_j A^{j-1}$ satisfies the matrix ordinary differential equation $L(\varphi_j A^{j-1}) = 0$, i.e.

(7) $$0 = \sum_{j=1}^n L(\varphi_j A^{j-1}) = L(\sum_{j=1}^n \varphi_j A^{j-1}).$$

Having in mind that the matrix function $\Phi(t) = \sum_{j=1}^n \varphi_j A^{j-1}$ satisfies the Cauchy data

$$\Phi(0) = \sum_{j=1}^n \varphi(0) A^{j-1}, \dots, \Phi^{(n-1)}(0) = \sum_{j=1}^n \varphi_j^{(n-1)}(0) A^{j-1}$$

we conclude that if φ_j, $1 \le j \le n$, are such that

(8) $$L(\varphi_j) = 0, \varphi_j^{(i)}(0) = \delta_{j,i+1}, 0 \le i \le n-1$$

($\delta_{j,i+1}$ stands for the Kronecker symbol) then

$$e^{At} = \varphi_1 E + \dots + \varphi_{n-1} A^{n-1}.$$

This our main result.

Theorem 1. *Let* φ_j, $1 \le j \le n$ *satisfy the Cauchy problems (8). Then* $e^{At} = \sum_{j=1}^{n-1} \varphi_j A^{j-1}$.

Of course, $\{\varphi_1, \dots, \varphi_n\}$ is a very special basis for the linear ordinary differential equation (8) and to find it we must solve n-Cauchy problems. So assume that $\{\psi_1, \dots, \psi_n\}$ is another real basis of the solutions of the same scalar equation (8):$L(\psi_j) = 0$. Denote by $W_\varphi(t)$, $W_\psi(t)$ the Wronsky matrices corresponding to the two basises.

Then we have that

(9) $$\begin{cases} \varphi_1 = c_{11}\psi_1(t) + \dots + c_{1n}\psi_n(t), & c_{1j} = \text{const.} \in \mathbb{R} \\ \dots \\ \varphi_n = c_{n1}\psi_1(t) + \dots + c_{nn}\psi_n(t), & c_{nj} = \text{const.} \in \mathbb{R} \end{cases}$$

Certainly, for each j, $1 \le j \le n-1$:

$$\varphi_1^{(j)} = c_{11}\psi_1^{(j)} + \dots + c_{1n}\psi_n^{(j)}$$
$$\dots$$
$$\varphi_n^{(j)} = c_{n1}\psi_1^{(j)} + \dots + c_{nn}\psi_n^{(j)}.$$

As $W_\varphi(t) = \begin{pmatrix} \varphi_1 & \dots & \varphi_n \\ \vdots & \vdots & \vdots \\ \varphi_1^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{pmatrix}$ and

$$C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \dots & & \\ c_{n1} & \dots & c_{nn} \end{pmatrix},$$

det $C \neq 0$, we can use the next assertion.

Proposition 2. *Let $\{\varphi_1, \dots, \varphi_n\}$, $\{\psi_1, \dots, \psi_n\}$ be two real basises of solutions of the scalar differential equation $Lx = 0$ for which (9) holds.*

Then $W_\varphi(t) = W_\psi(t)C^*$ and C^* is the adjoint matrix of C.

So we conclude that $C^* = (W_\psi(0))^{-1}$, i.e.

$$C = (W_\psi^*(0))^{-1} = (W_\psi^{-1}(0))^*$$

as $W_\varphi(0) = E$.

Thus C maps the basis $\{\psi_1, \dots, \psi_n\}$ onto the basis $\{\varphi_1, \dots, \varphi_n\}$.

We shall summarize the results just obtained in the next form, useful for applications.

1. We solve the algebraic equation (2).

2. We write down a real basis of solutions of the scalar differential equation $Lx = 0$, namely $\{\psi_1, \dots, \psi_n\}$.

3. We compute the constant coefficient matrix

$$(W_\psi^{-1}(0))^* = \begin{pmatrix} \psi_1(0) & {\psi_1}'(0) & \dots & \psi_1^{(n-1)}(0) \\ \dots & & & \\ \psi_n(0) & {\psi_n}'(0) & \dots & \psi_n^{(n-1)}(0) \end{pmatrix}^{-1}$$

and write the basis $(\varphi_1, \dots, \varphi_n)$ given by

$$\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} = (W_\psi^{-1}(0))^* \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}.$$

Then $e^{At} = \varphi_1 E + \varphi_2 A + \dots + \varphi_n A^{n-1}$.

As the students p. 1,2 are familiar with minor technical difficulty can only arrise in p. 3.

Example 1. Consider the real valued matrix $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $b \neq 0$. Its eigenvalues are $\lambda_{1,2} = a \pm ib$. Then $x(t) = A_1 e^{at} \cos\ bt + B_1 e^{at} \sin\ bt$ is the general solution of the scalar equation $Lx = 0$ with A_1, B_1-arbitrary constants.

Let $x(0) = 1$, $x'(0) = 0$. Then $x_1(t) = e^{at}(\cos\ bt - \frac{a}{b} \sin\ bt)$. In a similar way $x(0) = 0$, $x'(0) = 1 \Rightarrow x_2(t) = \frac{e^{at}}{b} \sin\ bt$.

Thus, $e^{At} = x_1(t)E + x_2 A = e^{at} \begin{pmatrix} \cos\ bt & \sin\ bt \\ -\sin\ bt & \cos\ bt \end{pmatrix}$.

Example 2. Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = 0$, $\lambda_{2,3} = -3$. Then $\psi_1(t) = 1$, $\psi_2(t) = e^{-3t}$, $\psi_3(t) = te^{-3t}$ form a basis of the corresponding scalar differential equation. Obviously,

$$W^*_\psi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \end{pmatrix}, (W^*_\psi(0))^{-1} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 6 & -6 & -9 \\ 1 & -1 & -3 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 6 & -6 & -9 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ e^{-3t} \\ te^{-3t} \end{pmatrix} =$$

$$= \frac{1}{9}\begin{pmatrix} 9 \\ 6 - 6e^{-3t} - 9te^{-3t} \\ 1 - e^{-3t} - 3te^{-3t} \end{pmatrix}.$$

Therefore

$$e^{At} = E + \frac{1}{3}(2 - 2e^{-3t} - 3te^{-3t})\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{pmatrix} +$$

$$+\frac{1}{9}(1 - e^{-3t} - 3te^{-3t})\begin{pmatrix} 1 & -2 & 4 \\ 4 & 1 & -20 \\ -5 & 1 & 16 \end{pmatrix}.$$

We shall not complete the obvious computation.

REFERENCES

[1] I. E. Leonard. The matrix exponential. *SIAM Review*, **38** (1996), 507-512.
[2] E. Liz. A note on the matrix exponential. *SIAM Review*, **40** (1998), 700-702.

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ЕДИН ЕЛЕМЕНТАРЕН НАЧИН ЗА ПРЕСМЯТАНЕ НА ЕКСПОНЕНТА НА МАТРИЦА

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В тази статия се разглежда един елементарен начин за пресмятане на експонента на матрица. Той може да се използва в университетските курсове по обикновени диференциални уравнения. Предимствата на този подход при решаване на системи от обикновени диференциални уравнения с постоянни коефициенти са следните: избягва се както привеждането на матрица в жорданова нормална форма, така и метода на неопределените коефициенти.