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CONFORMAL MANIFOLDS WITH PARALLEL SPINORS

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We prove that on a conformal spin manifold any Weyl structure admitting a parallel spinor of weight $k \neq 0$ must be closed. The same is true when $k = 0$ and the dimension is at least 3, with the additional assumption that in dimension 4 the manifold is compact.

1. Introduction. The Riemannian spin manifolds admitting parallel spinors have been studied by Hitchin [5] and Wang [7]. Using the Berger-Simons theorem (see for example [1]) they have proved that a complete simply connected irreducible non-symmetric spin manifold admits a parallel spinor iff its holonomy is $SU(n)$, $Sp(n)$, G_2 or $Spin(7)$.

Now, on conformal spin manifolds there exists a natural 1-parameter family of spinor bundles – the bundles of spinors of weight k, $k \in \mathbb{R}$ (cf. [4]). Recall that a Weyl connection (or Weyl structure) on a conformal manifold is a torsion-free connection which preserves the conformal structure. A particular example is the Levi-Civita connection of any metric in the conformal class. Thus on conformal manifolds the Weyl connections naturally generalize the Levi-Civita connection.

The purpose of this paper is to study the Weyl structures admitting parallel spinors of weight k. In Theorem 4.1 we prove that when $k \neq 0$ any such structure must be closed. In particular, in the simply connected case we essentially obtain a Riemannian manifold admitting a parallel spinor (with respect to the Levi-Civita connection). For $k = 0$ and dimension greater than 2 the same result has been proved by Moroianu [6] under the additional assumption that in the 4-dimensional case the manifold is compact. In fact, in dimension 4 he proved more precise results. Namely, a compact 4-dimensional conformal spin manifold with a parallel spinor of weight 0 is hyper-Hermitian and hence (using the classification of Boyer [2]) it is conformally equivalent to a flat torus or a K3-surface or a coordinate quaternionic Hopf-surface with its standard locally conformally flat metric. As shown in [6], there are examples of non-closed Weyl structures on non-compact 4-dimensional conformal spin manifolds which admit parallel spinors of weight 0. We show that when the dimension is 2 even the assumption that the manifold is compact does not ensure the closeness of a Weyl structure admitting a parallel spinor of weight 0.

2. Algebraic preliminaries.

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Let $CO_+(n) = \mathbb{R}_+ \times SO(n)$ be the conformal orthogonal group. This group is embedded in $GL(n,\mathbb{R})$ by mapping $(a_1, a_2) \in CO_+(n)$ to $a_1 a_2 \in GL(n,\mathbb{R})$. Thus we obtain a representation ρ of $CO_+(n)$ in $\mathbb{R}^n$, $\rho(a_1, a_2)x = a_1 a_2 x$, $(a_1, a_2) \in CO_+(n)$, $x \in \mathbb{R}^n$. The Lie algebra of $CO_+(n)$ is $co(n) = \mathbb{R} \oplus so(n)$ and the corresponding Lie algebra representation is given by

$$\rho(A_1, A_2)x = A_1 x + A_2 x, \qquad (A_1, A_2) \in co(n), \quad x \in \mathbb{R}^n. \tag{2.1}$$

The representation ρ gives rise to a representation of $CO_+(n)$ in $(\mathbb{R}^n)^*$, which we shall denote by ρ^*. Identifying $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ by the standard inner product, we obtain that $\rho^*(a_1, a_2)\alpha = a_1^{-1} a_2 \alpha$, $(a_1, a_2) \in CO_+(n)$, $\alpha \in \mathbb{R}^n$.

Let $CSpin(n) = \mathbb{R}_+ \times Spin(n)$ be the conformal spin group. Then $CSpin(n)$ double covers $CO_+(n)$ (for $n \geq 2$) and the projection is $id \times \pi$, where π is the projection of $Spin(n)$ on $SO(n)$. Denote by ρ_k the representation of $CSpin(n)$ in the space of spinors Σ given by $\rho_k(a_1, \widetilde{a}_2)\psi = a_1^k \widetilde{a}_2 \psi$, $(a_1, \widetilde{a}_2) \in CSpin(n)$, $\psi \in \Sigma$. When considered as a $CSpin(n)$-module with respect to this representation, we shall denote the space of spinors by $\Sigma^{(k)}$ and call it *the space of spinors of weight* k (cf. [4]). The corresponding representation of the Lie algebra $cspin(n) = \mathbb{R} \oplus spin(n)$ of $CSpin(n)$ on $\Sigma^{(k)}$ is

$$\rho_k(A_1, \widetilde{A}_2)\psi = kA_1\psi + \widetilde{A}_2\psi, \qquad (A_1, \widetilde{A}_2) \in cspin(n), \quad \psi \in \Sigma. \tag{2.2}$$

Recall that if $\widetilde{A}_2 \in spin(n)$ and $\pi\widetilde{A}_2 = A_2 \in so(n)$ then

$$\widetilde{A}_2\psi = \frac{1}{4}\sum_{i,j}(A_2 e_i, e_j)e^i e^j \psi, \qquad \psi \in \Sigma, \tag{2.3}$$

where $\{e_i\}$ is an orthonormal frame of $\mathbb{R}^n$, $\{e^i\}$ is its dual and $\alpha\psi$ denotes the Clifford product of $\alpha \in (\mathbb{R}^n)^*$ and $\psi \in \Sigma$. The Clifford action of $\mathbb{R}^n$ on Σ is obtained by the identification of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ mentioned above.

The Clifford action of $(\mathbb{R}^n)^*$ is still defined in the conformal framework, but it sends $\Sigma^{(k)}$ to $\Sigma^{(k-1)}$, i.e.

$$(\rho^*(\pi\widetilde{a})\alpha).(\rho_k(\widetilde{a})\psi) = \rho_{k-1}(\widetilde{a})(\alpha\psi), \qquad \alpha \in (\mathbb{R}^n)^*, \quad \psi \in \Sigma, \quad \widetilde{a} \in CSpin(n).$$

3. Weyl connections.

Let (M, c) be an oriented conformal manifold of dimension n and let $CO_+(M)$ be the bundle of oriented conformally orthogonal frames with respect to the conformal class c. A connection ∇^W is called Weyl connection (or Weyl structure) with respect to c if it satisfies the following conditions:

(i) ∇^W preserves c;

(ii) ∇^W is torsion-free.

The condition (i) means that ∇^W is a connection on the principal bundle $CO_+(M)$, or equivalently, for any metric $g \in c$ $\nabla^W g = \theta^g \otimes g$ for some 1-form θ^g.

Denote by ∇^g the Levi-Civita connection of g (which is an example of Weyl connection with respect to c). Then we have

$$\nabla^W_X Y = \nabla^g_X Y - \frac{1}{2}\theta^g(X)Y - \frac{1}{2}\theta^g(Y)X + \frac{1}{2}g(X,Y)(\theta^g)^\#,$$

where $(\theta^g)^\#$ is the dual vector field of θ^g with respect to g. Equivalently,

$$\nabla^W_X = \nabla^g_X - \frac{1}{2}\theta^g(X)Id - \frac{1}{2}\theta^g \wedge X, \tag{3.4}$$

where $\theta^g \wedge X$ is the skew-symmetric (with respect to g) endomorphism of TM defined by $(\theta^g \wedge X)(Y) = \theta^g(Y)X - g(X,Y)(\theta^g)^\#$.

If $\widetilde{g} = e^f g$, then $\theta^{\widetilde{g}} = \theta^g + df$. Hence, a Weyl connection coincides with the Levi-Civita connection of some metric from c iff θ^g is exact. Such Weyl structures are called exact. If θ^g is closed, then the Weyl structure is called closed. Any exact Weyl structure is closed and the converse is true if M is simply connected.

A $CSpin$-structure on (M,c) is a $CSpin(n)$-principal bundle $CSpin(M)$ which double covers $CO_+(M)$. Given a $CSpin$-structure on (M,c) we can define the spinor bundles of weight k: these are the associated fibre bundles $\Sigma^{(k)}M = CSpin(M) \times_{\rho_k} \Sigma^{(k)}$.

Since ∇^W and ∇^g are connections on $CO_+(M)$, they can be lifted to connections on $CSpin(M)$. By (2.1), (2.2), (2.3) and (3.4) we obtain

$$\nabla^W_X \psi = \nabla^g_X \psi - \frac{2k-1}{2}\theta^g(X)\psi + \frac{1}{4} X \theta^g \psi, \qquad \psi \in \Gamma(\Sigma^{(k)}M). \tag{3.5}$$

This equality has to be understood as follows: the bundle $\Sigma^{(k)}M$ is identified with the bundle of spinors ΣM of g and the Clifford product in the right-hand side is with respect to this identification.

Finally, recall that the conformal scalar curvature k^g with respect to $g \in c$ of a Weyl structure ∇^W is defined to be the scalar curvature of the curvature tensor of ∇^W (the trace is taken with respect to g). It is well-known (cf. [3]) that

$$k^g = s^g - (n-1)\delta\theta^g - \frac{(n-1)(n-2)}{4}|\theta^g|^2, \tag{3.6}$$

where s^g is the scalar curvature of g and the co-differential operator and the norm are taken with respect to g.

4. Weyl structures with parallel spinors.

Theorem 4.1 *Let ∇^W be a Weyl connection on a conformal spin manifold (M,c) of dimension n. Suppose there exists a parallel spinor of weight $k \neq 0$ with respect to ∇^W. Then ∇^W is closed. In particular, if M is simply connected then there exists a metric $g \in c$ such that (M,g) admits a parallel spinor (with respect to the Levi-Civita connection). The same is true when $k = 0$, $n \geq 3$, $n \neq 4$ and when $k = 0$, $n = 4$ and M is compact.*

Proof: If (M,c) admits a parallel section of $\Sigma^{(k)}M$ with respect to ∇^W then there exists $\psi \in \Sigma^{(k)}$ on which the holonomy group of ∇^W acts trivially. Let $(A_1, \widetilde{A}_2) \in cspin(n)$ be such that $\rho_k(A_1, \widetilde{A}_2)\psi = 0$. Then

$$kA_1\psi + \widetilde{A}_2\psi = 0 \tag{4.7}$$

and hence $kA_1 < \psi, \psi > + < \widetilde{A}_2\psi, \psi >= 0$. But $kA_1 < \psi, \psi >$ is real and $< \widetilde{A}_2\psi, \psi >$ is purely imaginary. Thus $kA_1 = 0$ and if $k \neq 0$ it follows that $A_1 = 0$, i.e. $(A_1, \widetilde{A}_2) \in spin(n)$. This shows that the restricted holonomy group of ∇^W as a connection on $CSpin(M)$ is contained in $Spin(n)$ and therefore the restricted holonomy group of ∇^W as a connection on $CO_+(M)$ is contained in $SO(n)$. Hence, if M is simply connected the holonomy group is subgroup of $SO(n)$ and thus ∇^W is a torsion-free connection on a sub-bundle of $CO_+(M)$ with structure group $SO(n)$, i.e. ∇^W is the Levi-Civita connection of a metric $g \in c$. In particular, θ is exact and (M,g) admits a parallel spinor (with respect to the Levi-Civita connection). When M is not simply connected the above is true for its universal covering space and hence θ is closed.

The case $k = 0$ follows from [6], since (3.5) shows that the connection on the spinor bundle of a Riemannian spin manifold, which is called there "the Weyl connection", actually coincides with the Weyl connection ∇^W acting on sections of $\Sigma^{(0)}M$. $\Box$

Remark: When the dimension is 2 even the assumption that M is compact does not ensure the closeness of a Weyl structure admitting a parallel spinor of weight 0. To see this, notice that when $k = 0$ and $n = 2$ (4.7) yields that $A_2 = 0$, i.e. the restricted holonomy group of ∇^W is contained in $\mathbb{R}_+$ and thus the conformal scalar curvature is zero. So, from (3.6) we obtain that the scalar curvature of a metric $g \in c$ is $s^g = \delta\theta^g$. Hence, M is homeomorphic to the torus $T^2 = S^1 \times S^1$. Now, take on T^2 the conformal class of the standard flat metric g. Then it follows easily from (3.5) that there exists a parallel spinor with respect to ∇^W iff $\dfrac{\partial\theta_1^g}{\partial x^1} + \dfrac{\partial\theta_2^g}{\partial x^2} = 0$, where $\theta^g = \theta_1^g dx^1 + \theta_2^g dx^2$ and x^1 and x^2 are the coordinates on the first and the second factor of T^2 respectively. The above equation obviously has solutions θ^g, which are not closed.

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КОНФОРМНИ МНОГООБРАЗИЯ С ПАРАЛЕЛНИ СПИНОРИ

Богдан Александров

Доказваме, че върху конформно спин многообразие всяка Вайлова структура допускаща паралелен спинор с тегло $k \neq 0$ е затворена. Същото е вярно, когато $k = 0$ и размерността е поне 3, при допълнителното предположение, че в размерност 4 многообразието е компактно.