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GENERIC PROPERTIES OF FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES *

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We prove that almost all in the Baire sense functional differential inclusions in Banach spaces have nonempty compact solution set, which depends continuously on the right-hand side and on the initial condition.

1. Introduction.

Let E be a Banach space. Denote $I = [0, 1]$ and given $\tau > 0$ we let $X = C([-\tau, 0], E)$. Consider a functional differential inclusions having the form:

$$\dot{x}(t) \in F(t, x_t), \quad x_0 = \phi \tag{1}$$

where $t \in I$ and $x_t \in X$ is given by $x_t(s) = x(t + s)$ for $s \in [-\tau, 0]$ and $F(\cdot, \cdot)$ is (almost) continuous nonempty convex and compact valued multifunction. We prove that for almost all in Baire sense (F, ϕ) the solution set $Z(F, \phi)$ of (1) is nonempty $C(I, E)$ compact and depends continuously on (F, ϕ) (of course if (1) has a solution for theese (F, ϕ)).

Such a result was first proved in [6] in case of ordinary differential equations with jointly continuous right-hand side and afterwards extended in case of almost continuous differential equations in separable Banach spaces in [1]. For the background of functional differential equations consult [4] where as in [7] some generic properties of functional differential equations are presented. Theory of functional differential inclusions is presented in [5] and in the appendix of [2].

In [8] is proven that for almost all (F, ϕ) the solution set of (1) depends continuously on (F, ϕ) when the space $E \equiv R^n$ and $\phi \in E$ (i.e. no time lag). The result is however obvious since the solution set of (1) in this case is nonempty compact and depends upper semicontinuously on (F, ϕ) hence one has only to use theorem 1 of [3] (see lemma 1).

Here we obtain similar result when, however, E is infinitely dimensional. The main difficulty in this case is to show that for almost all (in Baire sense) (F, ϕ) the solution set of (1) is nonempty $C(I, E)$ compact and depends upper semicontinuously on (F, ϕ).

Now we recall the main definitions and notations used in the paper. Note first that all the concepts not discussed in details in the sequel can be found in [2]. By $CC(E)$ denote the set of all nonempty convex and compact subsets of E, and by B the unit ball centered in the origin. We let $D_H(A, B) = \max\{\max_{a \in A} \min_{b \in B} |a - b|, \max_{b \in B} \min_{a \in A} |a - b|\}$ be the Hausdorff distance and note that $CC(E)$ equipped with this distance becomes a

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complete metric space. For $x \in E, A \in CC(E)$ denote $dist(x,A) = \min_{a\in A}|x-a|$. The Hausdorff measure of noncompactness is defined by

$$\beta(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius } \le r\},$$

where B is nonempty subset of $C(I,E)$.

Definition 1. *The multifunction $F : M \to CC(E)$ is said to be continuous at x when it is continuous with respect to the Hausdorff distance (here M is a metric space). The multifunction $F : I \times X \to CC(E)$ is said to be almost continuous when to $\varepsilon > 0$ there exists a compact $I_\varepsilon \subset I$ with Lebesgue measure greater than $1-\varepsilon$ such that F restricted on $I_\varepsilon \times X$ is continuous. Let A, B be topological spaces. The multimap $G : A \to 2^B$ is called upper semicontinuous (USC) when for every $a \in A$ and every open $U \supset G(a)$ there exists a neighbourhood $V \ni a$ such that $U \supset G(a')$ when $a' \in V$.*

We will consider the problem (1) in two cases:
1) $F(\cdot,\cdot)$ is (jointly) continuous,
2) $F(\cdot,\cdot)$ is almost continuous.
Given $K > 0$ and $\lambda(\cdot)$ - Lebesgue integrable function define the sets:

$$Y = \{(F,\phi),\ F : I\times X \to CC(E),\ \phi \in X,\ |F(t,\psi)| \le K \text{ for every } (t,\psi) \in I \times X\}$$

$$\tilde{Y} = \{(F,\phi),\ F : I\times X \to CC(E),\ \phi \in X,\ |F(t,\psi)| \le \lambda(t) \text{ for every } \psi \in X \text{ and a.e. } t \in I\}.$$

The first set consists of all continuous and the second of all almost continuous multimaps. It is easy to see that equipped with the metrics:

$$\rho\big((F_1,\phi_1),(F_2,\phi_2)\big) = \sup_{(t,\psi)\in I\times X} D_H(F_1(t,\psi),F_2(t,\psi)) + |\phi_1-\phi_2|_X$$

$$\tilde{\rho}\big((F_1,\phi_1),(F_2,\phi_2)\big) = \int_I \sup_{\psi\in X} D_H(F_1(t,\psi),F_2(t,\psi))\,dt + |\phi_1-\phi_2|_X$$

the sets Y and $\tilde{Y}$ become complete metric spaces. In the second case one has only to use Egorov's and Lusin's theorems.

The following lemma is theorem 1 of [3]

Lemma 1. *Let Y be a topological space and X be a metric space. If a set-valued mapping G from Y into X is USC then it is continuous in a residual (i.e. it contains a countable intersection of open and dense subsets) set in Y.*

2. The results. In this section we present and prove our main results.

Definition 2. *The multifunction $H : I \times X \to CC(E)$ is said to be locally Lipschitz iff for every $z \in I \times X$ there exists a neighbourhood $U \ni z$ and a constant $L > 0$ such that $D_H(H(t_1,\psi_1),H(t_2,\psi_2)) \le L(|t_1-t_2| + |\psi_1-\psi_2|)$ when $(t_1,\psi_1),(t_2,\psi_2) \in U$.*

The following lemma is proven in [6] in case of single valued maps.

Lemma 2. *If $G : I \times X \to CC(E)$ is continuous then to $\varepsilon > 0$ there exists a locally Lipschitz multifunction G_ε such that $D_H(G(t,\psi),G_\varepsilon(t,\psi)) < \varepsilon$ for every $(t,\psi) \in I \times X$.*

The proof is the same as in the single valued case and is omitted.

It follows from lemma 2 that the set of all locally Lipschitz functions is dense in Y and in $\tilde{Y}$ with respect to norms ρ and $\tilde{\rho}$ respectively. Further considerations are similar for Y and for $\tilde{Y}$ and will be given (mainly) in case $\tilde{Y}$.

Definition 3. *Let $\varepsilon > 0$ be given. The absolutely continuous function $x(\cdot)$ is said to be ε-solution of (1) when it is a.e. differentiable, $x_0 = \phi$ and for a.e. t the following inequality holds $dist(\dot{x}(t), F(t, x_t)) < \varepsilon$, when F is continuous respectively $dist(\dot{x}(t), F(t, x_t)) < \lambda_\varepsilon(t)$ with $\lambda_\varepsilon(t) \leq \lambda(t)$ almost all on I when it is almost continuous.*

Theorem 1. *Let $S^\varepsilon(F, \phi)$ be the set of all ε-solution of (1). If $\lim_{\varepsilon\to 0} \beta(S^\varepsilon(F, \phi)) = 0$ then (1) admits a nonempty compact solution set depending upper semicontinuously on (F, ϕ).*

Proof. Let $\{x^\varepsilon(\cdot)\}_{\varepsilon>0}$ be a net of ε-solution. Hence there exists a uniformly converging subnet $\{x^i(\cdot)\}_{i=1}^\infty$ since $\lim_{\varepsilon\to 0} \beta(S^\varepsilon(F, \phi)) = 0$ (see [2] for instance). Let $\lim_{i\to\infty} x^i(t) = x(t)$. Since $F(\cdot, \cdot)$ is (almost) continuous one can easily show that for every $t > s \in I$ we have $x(t) - x(s) \in \int_s^t F(\tau, x_\tau)\, d\tau$ and $x_0 = \phi$. Therefore $x(\cdot)$ is a solution of (1), i.e. the solution set of (1) is nonempty and $C(I, E)$ compact. Let $\phi_n \to \phi$ and $F_n \to F$ with respect to ρ (or to $\tilde{\rho}$ respectively) with F_n satisfying the conditions of the theorem. Given $\varepsilon > 0$ one has that every absolutely continuous x with

$$\dot{x}(t) \in F_n(t, x_t), \quad x_0 = \phi_n$$

is ε-solution of (1) for sufficiently large n. Thus the solution set of (1) depends USC on (F, ϕ). □

Given $\eta > 0$. Denote by Y_η ($\tilde{Y}_\eta$) the set of all $(F, \phi) \in Y$ ($\tilde{Y}$ such that $\lim_{\varepsilon\to 0} \beta(S^\varepsilon(F, \phi)) < \eta$).

Theorem 2. *The set $\tilde{Y}_\eta$ is open and dense in $\tilde{Y}$ for every $\eta > 0$.*

Proof. Let $(F^n, \phi^n) \in \tilde{Y} \setminus \tilde{Y}_\eta$. Suppose $\lim_{n\to\infty}(F^n, \phi^n) = (F, \phi)$. Given $\varepsilon > 0$ one has that for sufficiently large n every ε-solution of (1) with (F^n, ϕ^n) is 2ε-solurion of (1) and vice versa. Therefore $(F, \phi) \in \tilde{Y} \setminus \tilde{Y}_\eta$. I.e. $\tilde{Y} \setminus \tilde{Y}_\eta$ is closed and hence $\tilde{Y}_\eta$ is open. Furthermore $\tilde{Y}_\eta$ cotains every pair (F, ϕ) with F locally Lipschitz. Thus $\tilde{Y}_\eta$ is also dense in $\tilde{Y}$ due to lemma 2. The case of Y_η and Y can be proven in the same way. □

Denote $\tilde{Y}_\infty = \bigcap_{n=1}^\infty \tilde{Y}_{1/n}$.

Proposition 1. *If $(F, \phi) \in \tilde{Y}_\infty$, then $\lim_{\varepsilon\to 0} \beta(S^\varepsilon(F, \phi)) = 0$.*

Proof. For given n one has $\lim_{\varepsilon\to 0} \beta(S^\varepsilon(F, \phi)) < 1/n$. The proof is therefore complete since n is arbitrary. □

Denote by $S(F, \phi)$ the solution set of (1). From theorems 1 and 2 and proposition 1 we obtain the main result of the paper.

Theorem 3. *There exists a residual subset $\tilde{Y}_r$ of $\tilde{Y}$ such that $S(F, \phi)$ is nonempty and continuous on (F, ϕ).*

Proof. First $\tilde{Y}_\infty$ is a dense G_δ subset of Y (i.e. countable intersection of open and dense subsets of $\tilde{Y}$) due to theorem 2. Let $(F, \phi) \in \tilde{Y}_\infty$ the solution set $S(F, \phi)$ of (1) is nonempty compact USC depending on (F, ϕ) due to proposition 1 and theorem 1. Furthermore $\tilde{Y}_\infty$ is a Baire space. The proof is therefore complete due to lemma 1. □

Corollary 1. *Each functional differential inclusion (1) can be always approximated closedly by a functional differential inclusion whose solution set is nonempty compact and stable (i.e. depends continuously on (F, ϕ)).*

The proof follows immediately from theorem 3 and is omitted.

Remark 1. *Theorem 3 generalises theorem 3.2 of [8] and (partially) the main results of [1,6]. Corollary 1 generalises theorem 3.4 of [8].*

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ГЕНЕРИЧНИ СВОЙСТВА НА ФУНКЦИОНАЛНО-ДИФЕРЕНЦИАЛНИ ВКЮЧВАНИЯ В БАНАХОВИ ПРОСТРАНСТВА

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Доказваме, че почти всички в смисъл на Бер функционално-диференциални вкючвания в банахови пространства имат непразно и компактно множество от решения, което зависи непрекъснато от дясната част и началното условие