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SINGULARLY PERTURBED FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES*

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Singularly perturbed delayed differential inclusions with state constraints in Banach spaces are considered. We investigate the limit behavior of the solution set when the small parameter tends to zero. To this end the limits of the "fast" components are identified with Radon probability measures.

1. Introduction. Let $E = E_1 \times E_2$ be a Banach space. Consider the singularly perturbed system

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in H(t, x_t, y_t), \ x_0 = \varphi, \ y_0 = \psi, \ t \in I = [0, 1]. \tag{1}$$

Given *locally compact convex* sets $K_i \subset E_i$, $i = 1, 2$ and $\tau > 0$ we suppose $\varphi(s) \in C([-\tau, 0], K_1)$, $\psi(s) \in C([-\tau, 0], K_2)$ where C is the corresponding space of continuous functions with the sup–norm. Denoting $C_i = C([-\tau, 0], K_i)$, we let H be a multivalued map from $I \times C_1 \times C_2$ into E while x_t (resp. y_t) is a function defined for $s \in [-\tau, 0]$ as $x_t(s) = x(t + s)$ (resp. $y_t(s) = y(t + s)$). The solution set of (1) will be denoted by $Z(\varepsilon)$. For $\varepsilon > 0$ it consists of all pairs (x, y) of AC (absolutely continuous) functions with values in K satisfying (1) a.e. in I.

The most natural question is *how* to define the solution set $Z(\varepsilon)$ of (1) at $\varepsilon = 0$. The problem is enough complex and difficult even when we are in finite dimensions, there are no delays and no state constraints. What has been done for this "simpler" case until recently is connected with either the *reduction* approach or *averaging* method.

The first one is a continuation of the idea of Tikhonov [7]: to put $\varepsilon = 0$ in (1). Then let $Z(0)$ consist of all AC x and integrable y satisfying the new "reduced" system. In [5, 8] etc. the LSC (lower semicontinuity) and/or USC (upper semicontinuity) of the mapping $\varepsilon \mapsto Z(\varepsilon)$ at $\varepsilon = 0^+$ are proved in various topologies. These results doesn't imply continuity.

The set $Z(0)$ defined in the framework of the reduction approach is not reach enough to absorb all the limits of the "slow" movements x. The contribution of the averaging method is the derivation of the limit of the "slow" part of $Z(\varepsilon)$. But it leaves open the question how to change the things concerning the "fast" variables y.

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It seems that the answer of the above question is in the embeding of $Z(0)$ in an appropriate space. In the recent publication [2], where systems of ordinary differential equations are investigated, identification of the limits of the "fast" solutions y_ε as $\varepsilon \to 0$ with invariant measures of the so-called associated system is suggested. The convergence in y is in some statistical sense, while the slow part converges in the uniform norm to a solution of specially defined "reduced" system. This idea is continued in [1], where an invariant measure for differential inclusions is introduced.

The difficulty in our case commes from the delayed structure of the inclusion which makes unclear the answer to the question "how to define $Z(0)$?". However in some special cases we are able to do this, see Theorem 2.

We give the main notations and definitions. For $A \subset E_1 \times E_2$, we denote by A_i the projection of A on E_i. Throughout the paper $\langle \cdot, \cdot \rangle$ is the duality product, $|\cdot|$ is the norm. For $x \in E$ we denote by $J(x) = \{l \in E^* : |l| = |x|, \langle l, x \rangle = |x|^2\}$ the duality mapping. For a closed, bounded (nonempty) set $A \subset E$ and $x \in E$ we denote $\hat{\sigma}(x, A) = \sup_{l \in J(x)} \sup_{a \in A} \langle l, a \rangle$. Denote $K = K_1 \times K_2$. The Bouligand cone is introduced as $T_K(z) = \{u \in E_1 \times E_2 : (1/\lambda) \liminf_{\lambda \to 0^+} d(z + \lambda u, K) = 0\}$. Let $\Omega_1 = \{\alpha \in C_1 : |\alpha(0)| = \|\alpha\|_C = \max_{-\tau \le s \le 0} |\alpha(s)|\}, \Omega_2 = \{\beta \in C_2 : |\beta(0)| = \|\beta\|_C = \max_{-\tau \le s \le 0} |\beta(s)|\}$.

The multifunction F from the topological space X into the topological space Y is said to be U(pper)S(emi)C(ontinuous) (L(ower)S(emi)C(ontinuous)) at $x \in X$ when to every open $V \supset F(x)$ $(V \bigcap F(x) \neq \emptyset)$ there exists a neighborghood $W \ni x$ such that $V \supset F(y)$ $(V \bigcap F(y) \neq \emptyset)$ for $y \in W$. When X and Y are metrizable (metric) spaces and F is compact valued then F is USC iff it admits a compact graph restricted to a compact subset of X. For further details on the notions used in the paper, consult with [4] or [9].

2. The Results. We can prove as in [6] the following lemma:

Lemma 1. *Suppose that*

A1. *There exist positive constants a, b, μ such that*

$$\begin{aligned} \hat{\sigma}(\alpha(0), H_1(t, \alpha, \beta)) &\le a(1 + |\alpha(0)|^2 + \|\beta\|_C^2), \quad \alpha \in \Omega_1, \beta \in C_2, \\ \hat{\sigma}(\beta(0), H_2(t, \alpha, \beta)) &\le b(1 + \|\alpha\|_C^2) - \mu|\beta(0)|^2, \quad \alpha \in C_1, \beta \in \Omega_2. \end{aligned}$$

Then there exist constants M and N such that

$$\|x^\varepsilon\|_C + \|y^\varepsilon\|_C \le M, \ |H(t, x_t^\varepsilon, y_t^\varepsilon)| \le N,$$

for every $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon), \varepsilon > 0$ and $t \in I$.

We give an example illustrating condition A1.

Example. Consider the following control system:

$$\begin{aligned} \dot{x}(t) &\in x_t w(t) + y_t, \quad x_0 \equiv 0, \\ \varepsilon \dot{y}(t) &\in x_t + f(y) w(t) - 2g(y)\|y_t\|_C, \quad y_0 \equiv 0. \end{aligned}$$

We suppose that E_i are Hilbert spaces, $w(\cdot)$ is measurable, $w(t) \in [-1, 1]$ a.e. in I. Also $f(0) = g(0) = 0$ and $f(y) = y/\sqrt{|y|^3}$, $g(y) = y/|y|$ when $y \neq 0$. Then using the simple inequality $cd \le (c^2+d^2)/2$ we get for α and β such that $\alpha(0)=x, \beta(0)=y$

$$\begin{aligned}\hat{\sigma}(x, H_1(t,\alpha,\beta)) &\le \frac{3|\alpha(0)|^2}{2} + \frac{|\beta(\cdot)|^2}{2} \le 2(1+|\alpha(0)|^2 + \|\beta\|_C^2),\\ \hat{\sigma}(y, H_2(t,\alpha,\beta)) &\le \frac{|\beta(0)|^2}{2} + \frac{|\alpha(0)|^2}{2} + |\beta(0)|^{4/3} - 2|\beta(0)|^2\\ &\le 1 + \frac{1}{2}(|\alpha(0)|^2 - |\beta(0)|^2) \le 1 + \|\alpha\|_C^2 - \frac{1}{2}|\beta(0)|^2\end{aligned}$$

(for $\beta \in \Omega_2$) since $|\beta(0)|^{4/3} \le 1 + |\beta(0)|^2$. Then $a=2, b=1, \mu=1/2$.

Due to Lemma 1 there exists a bounded set $P \subset K$ containing the values of all solutions of (1), i.e. if $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$, $\varepsilon > 0$ then $(x^\varepsilon(t), y^\varepsilon(t)) \in P$ for every $t \in I$. Denote by $\Re(P_2)$ the set of all Radon probability measures on P_2 (recall that P_2 denotes the projection of P on E_2). This set is metrizable and equipped with its weak norm is isometrically isomorphic to $C(I, P_2)^*$ (see [9]). Define the set of functions $\wp := \{\nu : I \mapsto \Re(P_2) \mid \nu(\cdot) \textit{ is measurable}\}$. Then $\nu^i \to \nu$ for $\nu^i, \nu \in \wp$ and $i = 1, 2, \dots$ if and only if

$$\int_I \left(\int_{P_2} f(t,y)\nu^i(t)\,(dy) \right) dt \to \int_I \left(\int_{P_2} f(t,y)\nu(t)\,(dy) \right) dt, \text{ for every } f \in \mathcal{F}.$$

Here $\mathcal{F}$ consists of all $f : I \times P_2 \to E_2$ such that $f(\cdot, y)$ is measurable, $f(t, \cdot)$ is continous and integrally bounded. We can represent every measurable function $y : I \to P_2$ as $\bar{\nu}(\cdot) = \delta_{y(\cdot)}$ (the Dirac measure) which is an element of $\wp$.

Theorem 1. *Let A1 be fulfilled and $Z(\varepsilon) \ne \emptyset$. Then for every (generalized) sequence of solutions $(x^\varepsilon, y^\varepsilon) \in Z(\varepsilon)$, $\varepsilon > 0$ with $\varepsilon \to 0$ there exists a subsequence $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon > 0}$ (denoted in the same way) such that $x^\varepsilon \to x^0$ in C and $y^\varepsilon \to \nu^0$ in $\wp$.*

Proof. Since $(x^\varepsilon(t), y^\varepsilon(t)) \in K$ for every $\varepsilon > 0, t \in I$ and K is locally compact, by Lemma 1 and Arzela Theorem we get the needed assertion. □

Now, we give a condition which combined with A1 imply the nonemptiness of $Z(\varepsilon)$:

A2. The map H is nonempty, closed, bounded valued, bounded on the bounded sets. One of the following conditions is true:

a) H has convex values and almost closed graph, i.e. for every $\delta > 0$ there is a compact set $I_\delta \subset I$ such that $meas(I_\delta) > 1 - \delta$ and the graph of H restricted on $I_\delta \times C_1 \times C_2$ is closed. Moreover, $T_K(x,y) \cap H(t,\alpha,\beta) \ne \emptyset$ for every $x \in K_1, y \in K_2$ and $\alpha \in C_1, \beta \in C_2$ with $\alpha(0) = x, \beta(0) = y$;

b) H is almost LSC, i.e. for every $\delta > 0$ there is a compact set $I_\delta \subset I$ such that $meas(I_\delta) > 1 - \delta$ and H restricted on $I_\delta \times C_1 \times C_2$ is LSC. Moreover, $H(t,\alpha,\beta) \subset T_K(x,y)$ for every $x \in K_1, y \in K_2$ and $\alpha \in C_1, \beta \in C_2$ with $\alpha(0) = x, \beta(0) = y$.

Lemma 2. *Under conditions A1 and A2 the set $Z(\varepsilon)$ of the solutions of (1)) is not empty and is relatively compact (compact when a) of A2 is fulfilled) for every $\varepsilon > 0$.*

Proof. a) First, suppose condition a) of A2 is fulfilled. Define $H^\varepsilon(t,\alpha,\beta) = \{(u,v) \in E : (u, \varepsilon v) \in H(t,\alpha,\beta)\}$ for $\varepsilon > 0$. Since $K = K_1 \times K_2$ one has that $T_K(x,y) = T_{K_1}(x) \times T_{K_2}(y)$ hence $T_K(x,y) \cap H^\varepsilon(t,\alpha,\beta) \ne \emptyset$.

Consider the sequence of numbers $\delta_n \to 0, n = 1, 2, \dots$ monotonically and the sequence of sets $I_n \subset I, n = 1, 2, \dots$ with $meas(I_n) > 1 - \delta_n$ such that H^ε restricted on $I_n \times C$

has a closed graph. Fix n and let $P_n(t)$ be the metric projection on I_n, i.e. $P_n(t) = \{s \in I_n : |t-s| = \min_{\xi \in I_n} |t-\xi|\}$. Define $F_n(t,\alpha,\beta) = \overline{co}\ H^\varepsilon(P_n(t),\alpha,\beta)$ where $\overline{co}$ denotes the closed, convex hull. Then F_n has a closed graph and satisfies the other conditions imposed on H. Now, we can follow the proof of Lemma 2.2. of [5] up to the existence of a sequence $\{(u^n,v^n)\}_{n=1}^\infty$ of absolutely continuous functions satisfying

$$\begin{pmatrix} \dot u^n(t) \\ \dot v^n(t) \end{pmatrix} \in F_n(t,\alpha+\delta_n B_1, \beta+\delta_n B_2) + \delta_n B,\ u_n = \varphi,\ v_n = \psi,$$

where B_i and B are the closed unit balls centered at zero respectively in C_i and E.

Suppose that $(u^n,v^n),\ n=1,2,\dots$ exist on the whole interval I and are bounded. Then by Lemma 1 and A2 it follows that all (u^n,v^n) are Lipschitz with a common constant. Furthermore $(u^n,v^n) : I \mapsto K,\ n=1,2,\dots$ thus by Arzela Theorem one can conclude (passing to subsequences if necessary) that $(u^n,v^n) \to (u^0,v^0),\ n\to\infty$ in C–topology. It is standard to prove that

$$\begin{pmatrix} \dot u^0(t) \\ \dot v^0(t) \end{pmatrix} \in H^\varepsilon(t,u_t^0,v_t^0)$$

and (u^0,v^0) will be the solution demanded.

Now, we will show the existence of (u^0,v^0) on the whole I. Since H is bounded on the bounded sets, one can prove the existence of (u^n,v^n) at least locally on say $[0,T)$ with $T>0$. On this interval $(u^n,v^n),\ n=1,2,\dots$ are bounded and Lipschitz uniformly. Therefore the solution (u^0,v^0) exists also on $[0,T)$. Let T be the least upper bound of the right ends of intervals of the existence of solutions (u^0,v^0) of (1). By Lemma 1 and A2 one can conclude that $|H^\varepsilon(t,u_t^0,v_t^0)| \le N/\varepsilon$ on $[0,T)$ for all such solutions ($\varepsilon>0$ is fixed!). Hence we can define $u^0(T) = \lim_{t\to T^-} u^0(t)$ and $v^0(T) = \lim_{t\to T^-} v^0(t)$. Therefore one can prove the existence of solutions of (1) on $[T,T+\lambda),\ \lambda>0$ if $T<1$. Thus $T=1$.

b) Let H^ε be the mapping defined in the proof of a). Again by $K = K_1\times K_2$ it follows that $H(t,\alpha,\beta) \subset T_K(x,y)$. Obviously H^ε is almost LSC too and if $(x^\varepsilon,y^\varepsilon)$ is a solution of (1) then $|H^\varepsilon(t,x_t^\varepsilon,y_t^\varepsilon)| \le N/\varepsilon$ on I. Let $I\setminus S = \bigcup_{n=1}^\infty I_n$ where $\{I_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint compacts, $S\subset I$ is a null set and H^ε is LSC on $I_n\times C_1\times C_2, n=1,2,\dots$. From Theorem 2 of [3] we know that there exist $\Gamma^{(N/\varepsilon)+1}$–continuos selections $f_n(t,\alpha,\beta) \in H^\varepsilon(t,\alpha,\beta)$ on $I_n\times C_1\times C_2,\ n=1,2,\dots$. Define the multifunction

$$F(t,\alpha,\beta) = \begin{cases} \bigcap_{\delta>0} \overline{co}\ f_n(t,\alpha+\delta B_1,\beta+\delta B_2), & (t,\alpha,\beta)\in I_n\times C_1\times C_2, \\ \bigcap_{\delta>0} \overline{co}\ H^\varepsilon(t,\alpha+\delta B_1,\beta+\delta B_2), & \text{elsewhere.} \end{cases}$$

It is easy to show that F is (jointly) measurable, see the proof of Theorem 6.2 of [4]. Moreover F has almost closed graph and $T_K(x,y)\cap F(t,\alpha,\beta) \ne \emptyset$. Then we are in case a) for the function F. Thus there is a solution (u^0,v^0) on I of

$$\begin{pmatrix} \dot u(t) \\ \dot v(t) \end{pmatrix} \in F(t,u_t,v_t),\ u_0=\varphi,\ v_0=\psi.$$

As in the proof of Lemma 6.1 of [4] one can show that

$$\begin{pmatrix} \dot u^0(t) \\ \dot v^0(t) \end{pmatrix} = f_n(t,u_t^0,v_t^0),\ t\in I_n,\ n=1,2,\dots.$$

Hence (u^0, v^0) is a solution of (1) and the nonemptiness of $Z(\varepsilon)$ is proved.

Now, fix $\varepsilon > 0$ and let $\{(x^n, y^n)\}_{n=1}^{\infty}$ be a sequence of solutions of (1). Since the pair (x^n, y^n) is N/ε for every n and K is locally compact the Arzela Theorem is applicable. Hence $Z(\varepsilon)$ is relatively compact. In case a) of A2 it is also closed. □

Corollary. *Suppose A1 and A2 are satisfied. Then for every sequence of solutions* $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ *of (1) with* $\varepsilon \to 0$ *there exist a subsequence* $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ *(denoted in the same way) such that* $x^\varepsilon \to x^0$ *in* C *and* $y^\varepsilon \to \nu^0$ *in* $\wp$.

Consider a functional–differential inclusion having the form:

$$(2) \qquad \begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in H(t, x_t, y, y(t-h(t))),\ x_0 = \varphi,\ y(s) = \psi(s),\ s \in [-\tau, 0].$$

Define $\hat{H} \equiv H$ when condition a) from A2 is met and

$$\hat{H}(t, \alpha, y, y_1) = \bigcap_{\delta>0} \overline{co}\ H(t, \alpha + \delta B_1, (y + \delta\tilde{B}) \cap K_2, (y_1 + \delta\tilde{B}) \cap K_2)$$

for every $\alpha \in C_1,\ y, y_1 \in E_2$, when condition b) is true. Here B_1 and $\tilde{B}$ are the closed unit ball centered at zero in C_1, respectively in E_2.

Theorem 2. *Suppose* E_i *are reflexive, A2 and the following is true:*

A1′. *There exist constants* $a, b, \mu > 0$ *such that for every* $(x(t), y(t)) \in E$

$$\begin{aligned} \hat{\sigma}(x(t), H_1(t, x_t, y, y(t-h(t)))) &\le a(1 + |x(t)|^2 + |y(t)|^2 + |y(t-h(t))|^2), \\ \hat{\sigma}(y(t), H_2(t, x_t, y, y(t-h(t)))) &\le b(1 + |x(t)|^2 + |y(t-h(t))|^2) - \mu|y(t)|^2. \end{aligned}$$

A3. *If* $\inf_{t\in I} h(t) = 0$ *then* $\mu > 2b$.

Then to every generalized sequence $\{(x^\varepsilon, y^\varepsilon)\}_{\varepsilon>0}$ *of solutions of (2) there exists a subsequence (denoted in the same way) such that* $x^\varepsilon \to x^0$ *and* $y^\varepsilon \to \nu^0$ *in the same topologies as in Theorem 1 and*

$$(3) \qquad \begin{pmatrix} \dot{x}^0(t) \\ 0 \end{pmatrix} \in \int_{P_2} \hat{H}(t, x_t^0, z)\mu^0(t)\,(dz),\ x_0 = \varphi,$$

where $\mu^0(t) = \nu^0(t) \otimes \nu^0(t-h(t))$ *and* $\nu^0(s) = \delta_{\psi(s)},\ s \in [-\tau, 0]$.

Proof. Using A1′ and A3 we can prove boundedness result anologous to Lemma 1, see e.g. [6]. Substitute $z(t) = (y(t), y(t-h(t)))$. Then if $\varepsilon_i \to 0$ and $(x^i, y^i) \in Z(\varepsilon_i),\ i = 1, 2, \dots$ by Theorem 1 (passing to subsequences if necessary) $(x^i, z^i) \to (x^0, \mu^0)$ in considered topologies and $(\dot{x}^i(\cdot), \varepsilon_i\dot{y}^i(\cdot)) \to (\dot{x}_0(\cdot), 0)$ in $L^1(I, E)$–weak. The second convergence is a standard observation, see e.g. [5].

The rest of the proof is the same as the proof of Theorem 4 of [6]. □

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ФУНКЦИОНАЛНО–ДИФЕРЕНЦИАЛНИ ВКЛЮЧВАНИЯ СЪС СУНГУЛЯРНО СМУЩЕНИЕ В БАНАХОВИ ПРОСТРАНСТВА

Цанко Дончев Дончев, Йордан Иванов Славов

Разглеждат се функционално–диференциални включвания в банахови пространства с малък параметър пред част от производните и фазови ограничения. Изследва се поведението на множеството от решения, когато малкият параметър клони към нула. За тази цел границите на „бързите“ компоненти се отъждествяват с вероятностни мерки на Радон.