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ON THE NONAUTONOMOUS GAUS'S SYSTEM

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In this paper we consider a nonautonomous periodic system which models the interaction between two species of Predator-Prey type. We give conditions under which the model has a positive periodic solution.

Introduction. Let us consider the system

$$(1)\qquad \left| \begin{array}{l} x' = \alpha x - p(x)\, y \\ y' = -\delta y + \gamma p(x)\, y \end{array} \right.$$

which models the interaction between two species of Predator-Prey type. We assume that system (1) reflects ω-periodic influence of the environment. More precisely, we will assume that the coefficients α, δ, γ are continuous ω-periodic functions of time t. When $p(x) \equiv x$, system (1) is the well-known Lotka-Volterra model.

System (1) was suggested by G. F. Gauss in 1934 and was investigated for limit cycles by Koij and Zegeling in [3]. In present paper we modify (1), assuming that the coefficients α, δ, γ are continuous ω-periodic functions of time t and we are interesting in the conditions under which (1) has at least one positive periodic solution.

The main result. For continuous ω-periodic functions g we put

$$[g] = \frac{1}{\omega}\int\limits_0^\omega g(s)\, ds,\ \ \{g\} = g - [g],\ \ g_L = \min_t g(t),\, g_M = \max_t g(t).$$

Our main result is

Theorem 1. *Let the coefficients α, δ be continuous ω-periodic functions with $[\alpha] > 0$, $[\delta] > 0$ and γ is continuous positive ω-periodic function. Let the function $p \in C^1[0,\infty)$ satisfies the conditions $p(0) = 0$, $p'(u) > 0$ for every $u \geq 0$, and let also there exist constants $B > 0$, $k_\infty > 0$ such that $p(u) \geq k_\infty u$ for $u \geq B$. Then the system (1) has at least one positive ω- periodic solution.*

Numerical example. Consider the system

$$(2)\qquad \left| \begin{array}{l} x' = \cos^2(t)\, x + x(x+1)\, y \\ y' = -\sin^2(t)\, y + \big(1 + \cos^2(t)\big)\, x(x+1)\, y \end{array} \right.$$

which satisfies all the conditions of Theorem 1. A π-periodic solution is found near the initial data

$$(3)\qquad x(0) = 0.2792278841, y(0) = 0.3487068214.$$

The calculation show that

$$|x(0) - x(\pi)| + |y(0) - y(\pi)| < 0.000005.$$

Its phase form is shown in fig.1 below. In fig.2 the phase curve that begins at the point $(0.3, 0.3)$ is traced for $t \in [0, 25\pi]$. This solution seems to be stable.

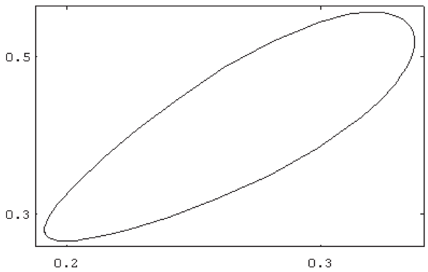

Fig. 1

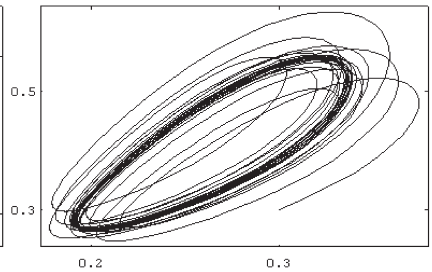

Fig. 2

Proof of Theorem 1. The proof is based on the theory of completely continuous vector fields presented by Krasnosels'kii and Zabrejko in [4]. The next theorem is extracted from [4].

Theorem 2. *Let Y be a real Banach space with a cone Q and $L : Y \to Y$ be a completely continuous and positive ($L : Q \to Q$) with respect to Q operator. Then the following assertions are valid:*

i) Let $L(0) = 0$. If for every sufficiently small $r > 0$ there is no $y \in Q$ for which $y \leq^0 L(y)$ and $\|y\|_Y = r$, then $ind(0, L; Q) = 1$.

ii) Let for every sufficiently large $R > 0$ there is no $y \in Q$ for which $\|y\|_Y = R$ and $L(y) \leq^0 y$. Then $ind(\infty, L; Q) = 0$.

iii) Let $L(0) = 0$ and $ind(\infty, L; Q) \neq ind(0, L; Q)$. Then L has nontrivial fixed points in Q.

Here $ind(., L; Q)$ denotes the index of a point with respect to L and Q. The sign $\leq^0$ denotes the semiordering generated by Q.

We introduce the following notations

$$D_x^- = \min_{0 \leq t,s \leq \omega} e^{\int_{t+s}^{t} \{\alpha\}(\tau) d\tau}, \qquad D_x^+ = \max_{0 \leq t,s \leq \omega} e^{\int_{t+s}^{t} \{\alpha\}(\tau) d\tau},$$

$$D_y^- = \min_{0 \leq t,s \leq \omega} e^{-\int_{t-s}^{t} \{\delta\}(\tau) d\tau}, \quad D_y^+ = \max_{0 \leq t,s \leq \omega} e^{-\int_{t-s}^{t} \{\delta\}(\tau) d\tau},$$

$$C_x = \frac{D_x^-}{D_x^+} e^{-[\alpha]\omega}, \qquad C_y = \frac{D_y^-}{D_y^+} e^{-[\delta]\omega}.$$

One can easy verify the validity of

Lemma 1. *Let δ and g be continuous ω-periodic functions and $[\delta] > 0$. Then the equation*

$$x' = -\delta(t)x + g(t)$$

has a unique ω-periodic solution for which it holds the representation

$$x(t)=\int_0^\omega \frac{e^{-[\delta]s}}{1-e^{-[\delta]\omega}}e^{-\int_{t-s}^{t}\{\delta\}(\tau)d\tau}g(t-s)\,ds.$$

Furthermore, there exists a unique ω-periodic solution to the equation

$$x'=\delta(t)\,x-g(t)\,,$$

for which it holds the representation

$$x(t)=\int_0^\omega \frac{e^{-[\delta]s}}{1-e^{-[\delta]\omega}}e^{\int_{t+s}^{t}\{\delta\}(\tau)d\tau}g(t+s)\,ds.$$

Put

$$G_x(t,s)=\frac{e^{-[\alpha]s}}{1-e^{-[\alpha]\omega}}e^{\int_{t+s}^{t}\{\alpha\}(\tau)d\tau},\quad G_y(t,s)=\frac{e^{-[\delta]s}}{1-e^{-[\delta]\omega}}e^{-\int_{t-s}^{t}\{\delta\}(\tau)d\tau}.$$

Using Lemma 1, the problem for ω-periodic solutions of (1) is reduced to the problem for ω-periodic solutions of the following operator system

$$\text{(4)}\qquad \left|\begin{array}{l} x(t)=\int_0^\omega G_x(t,s)\,p(x(t+s))\,y(t+s)\,ds\overset{def}{=}X(x,y)\\ y(t)=\int_0^\omega G_y(t,s)\,p(x(t-s))\,\gamma(t-s)\,y(t-s)\,ds\overset{def}{=}Y(x,y)\end{array}\right.$$

Put $P(x,y)=(X(x,y),Y(x,y))$ and let $C(\omega)$ be the space of the continuous ω-periodic functions and let H be the Banach space $H=C(\omega)\otimes C(\omega)$, provided with the usual norm

$$\|(x,y)\|=\max_t|x(t)|+\max_t|y(t)|\,.$$

Let $C_+(\omega)\subseteq H$ be the cone

$$C_+(\omega)=\{(x,y)\in H\;:\;x_L\geq C_x x_M,\;y_L\geq C_y y_M\}\,.$$

As in [2], it is easy to verify that the completely continuous operator P is positive with respect to $C_+(\omega)$, i.e. $P\;:\;C_+(\omega)\to C_+(\omega)$. Furthermore, the derivate of the operator P in zero is zero and from Theorem 2i) follows $ind\,(0,P;C_+(\omega))=1$.

Let us find $ind\,(\infty,P;C_+(\omega))$. Let $B_*=B/C_x$ and $N=\inf_{0\leq u\leq B_*}p'(u)$. We have $N>0$. It is easy to see that $x_L\geq B$ whenever $x_M\geq B_*$. Let R be sufficiently large and $R>\max\left(\frac{B}{C_x},\frac{[\delta]}{\gamma_L k_\infty D_y^- C_x},\frac{[\alpha]}{ND_x^- C_y},\frac{[\alpha]}{k_\infty D_x^- C_y}\right)$.

We define

$$P_*(x,y)=\left(\frac{D_x^-}{\omega[\alpha]}\int_0^\omega p(x(t))\,y(t)\,dt+1,\;\frac{1}{\omega}\int_0^\omega y(t)\,dt+1\right).$$

At first we will show that the completely continuous and positive vector fields $I-P$ and $I-P_*$ are linear homotopic at $x_M+y_M=2R$. By a contradiction argument we assume that there exists $(\tilde{x}, \tilde{y}) \in C_+(\omega)$ and $\theta \in [0,1]$ for which

$$\theta X(\tilde{x}, \tilde{y}) + (1-\theta) \frac{D_x^-}{[\alpha]} \int_0^\omega p(\tilde{x}(s)) \tilde{y}(s) \, ds + (1-\theta) = \tilde{x}(t), \tag{5}$$

$$\theta Y(\tilde{x}, \tilde{y}) + (1-\theta) \frac{1}{\omega} \int_0^\omega \tilde{y}(s) \, ds + (1-\theta) = \tilde{y}(t). \tag{6}$$

Consider two cases.

1) Let $\tilde{x}_M \geq R$. Then $\tilde{x}_L \geq B$ and $\tilde{x}_L \geq RC_x$ and from (6) we obtain the following inequality

$$\theta \gamma_L k_\infty RC_x \int_0^\omega G_y(t,s) \tilde{y}(t-s) \, ds + (1-\theta) \frac{1}{\omega} \int_0^\omega \tilde{y}(s) \, ds + (1-\theta) \leq \tilde{y}(t),$$

which after integrating at $[0,\omega]$ yields

$$\theta \gamma_L k_\infty RC_x \frac{D_y^-}{[\delta]} [\tilde{y}] + (1-\theta) [\tilde{y}] + (1-\theta) \leq [\tilde{y}].$$

In view of the choice of R, the last inequality is valid iff $\tilde{y} \equiv 0$ and $\tilde{\theta} \equiv 1$. Then substituting the values found for $\tilde{y} \equiv 0$ and $\tilde{\theta} \equiv 1$ in (5), we get $\tilde{x} \equiv 0$ which is a contradiction.

2) Let $\tilde{y}_M \geq R$. Then $\tilde{y}_L \geq C_y R$. We will prove that $\tilde{x}_M \leq B_*$ is not valid. Let $\tilde{x}_M \leq B_*$. Then from the mean value theorem, it follows $p(\tilde{x}(t)) \geq N\tilde{x}(t)$ and from (5) we have

$$\theta C_y RN \int_0^\omega G_x(t,s) \tilde{x}(t+s) \, ds + (1-\theta) C_y RN \frac{D_x^-}{[\alpha]} \int_0^\omega \tilde{x}(s) \, ds \leq \tilde{x}(t).$$

Integrating the last inequality at $[0,\omega]$, we get

$$\theta C_y RN \frac{D_x^-}{[\alpha]} [\tilde{x}] + (1-\theta) C_y RN \frac{D_x^-}{[\alpha]} [\tilde{x}] \leq [\tilde{x}],$$

which is a contradiction. Consequently $\tilde{x}_M \geq B_*$ and $\tilde{x}_L \geq B$. Now from (5) follows

$$\theta C_y Rk_\infty \int_0^\omega G_x(t,s) \tilde{x}(t+s) \, ds + (1-\theta) C_y Rk_\infty \frac{D_x^-}{[\alpha]} \int_0^\omega \tilde{x}(s) \, ds \leq \tilde{x}(t).$$

Hence, after integrating at $[0,\omega]$ we get the impossible inequality

$$RC_y k_\infty \frac{D_x^-}{[\alpha]} [\tilde{x}] \leq [\tilde{x}].$$

In this way we prove that the completely continuous positive vector fields $I-P$ and $I-P_*$ are linear positive homotopic at $x_M + y_M = 2R$. Let us compute $ind(\infty, P_*; C_+(\omega))$. For this purpose, assume that there is $(\tilde{x}, \tilde{y}) \in C_+(\omega)$ for which $P_*(\tilde{x}, \tilde{y}) \leq (\tilde{x}, \tilde{y})$. Then

$$\frac{1}{\omega} \int_0^\omega \tilde{y}(s) \, ds + 1 \leq \tilde{y}(s),$$

which, after integrating at $[0,\omega]$, yields to the impossible inequality

$$[\tilde{y}]+1\leq[\tilde{y}]\,.$$

From the last conclusion and from Theorem 2*ii*), it follows $ind\,(\infty,P_*;C_+\,(\omega))=0$. Since the vector fields $I-P$ and $I-P_*$ are linear positive homotopic we have

$$ind\,(\infty,P;C_+\,(\omega))=ind\,(\infty,P_*;C_+\,(\omega))=0,$$

therefore

$$0=ind\,(\infty,P;C_+\,(\omega))\neq ind\,(0,P;C_+\,(\omega))=1.$$

Now from Theorem 2*iii*) follows that the operator P has a nontrivial fixed point in $C_+\,(\omega)$. In particular, system (1) has at least one positive ω-periodic solution. $\square$

Using similar arguments, as above in the proof of Theorem 1 one can see that the following theorem is valid

Theorem 3. *Let the coefficients α,δ be continuous ω-periodic functions with $[\alpha]>0$, $[\delta]>0$ and γ is continuous positive ω-periodic function. Let the function $p\in C^1\,[0,\infty)$ satisfies the conditions $p\,(0)=0$, $p'\,(u)>0$ for every $u\geq 0$ and let also there exist constants $B>0$, K such that $p(u)\geq K$ for $u\geq B$ and*

$$K>\frac{[\delta]}{\gamma_L D_y^-}.$$

Then system (1) has at least one positive ω-periodic solution.

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ВЪРХУ НЕАВТОНОМНАТА СИСТЕМА НА ГАУС

Светлин Георгиев Георгиев, Димитър Иванов Петров

В тази работа разглеждане неавтономна периодична система, която моделира взаимодействието между два вида от тип „хищник-жертва“. Даваме условия при които разглежданата система има положителни периодични решения.