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CRITICAL POINT THEOREMS FOR LOCALLY LIPSCHITZ FUNCTIONALS AND APPLICATIONS TO FOURTH ORDER PROBLEMS

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In this work we prove existence results for semilinear beam equations with discontinuous nonlinearities arising in elasticity theory. The proofs are based on critical point theory for locally Lipschitz functionals.

1. Introduction. In this paper we consider some elements of critical point theory for locally Lipschitz functionals. It is well known that generalized gradients can be defined for such functionals, Clarke [6]. We reformulate the well known Ekeland's variational principle in terms of directional derivatives. Next, we introduce a Palais–Smale type condition (PS^1) (which implies the condition introduced by Chang [5] and formulate a minimization theorem, some coercivity results and theorems of mountain–pass type.

As an application we consider the existence of weak solutions of the fourth order problem:

$$(P): \begin{cases} u''''(x) + \xi(x) = 0, & \\ \xi(x) \in \partial j(x, u(x)), & \text{a.e. in } [0,1], \\ u''(0) = u''(1) = 0, & \\ u'''(0) = u'''(1) = 0, & \end{cases}$$

which describes the vibrations of an elastic beam with free ends and discontinuous forcing term. Here $j : \mathbb{R} \times R \to R$ is a function measurable in x and locally Lipschitz in u and ∂j denotes its Clarke derivative.

The problem (P) with j differentiable in u and nonlinear terms in boundary conditions is considered by M. Grossinho & T. Ma [7] using variational methods for differentiable functionals. The problem (P) can be formulated in terms of hemivariational inequalities introduced by P. D. Panagiotopoulos [8].

2. Critical point theory for locally Lipschitz functionals . Let X be a Banach space, X^* its dual space, $\|.\|$ the norm in X. Let $< p, x >$ for $p \in X^*, x \in X$ denote the duality bracket between X and X^*. Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional, i.e. for each $x \in X$ there exists a neighbourhood $N(x)$ of x and a constant K depending on $N(x)$ such that

$$|f(y_1) - f(y_2)| \le K\|y_1 - y_2\|, \quad \forall y_1, \forall y_2 \in N(x).$$

Denote by $LL(X)$ the space of locally Lipschitz functionals over X. For each $v \in X$ consider the directional derivative defined by

$$f^0(x;v) = \inf_{\delta>0} \sup_{\|y-x\|<\delta, 0<h<\delta} \frac{f(y+hv)-f(y)}{h}.$$

Basic properties of $f^0(x;v)$ are considered in Aubin [2], [3]. Recall that the function

$$\lambda_f(x) = \min\{\|p\|_* :< p, v >\leq f^0(x;v), \forall v \in X\}$$

is well defined and lower–semicontinuous. $\partial f(x)$ denotes the generalized gradient due to F. Clarke of f at x

$$p \in \partial f(x) \Longleftrightarrow < p, v >\leq f^0(x;v), \quad \forall v \in X.$$

We reformulate Ekeland's variational principle [3] for locally Lipschitz functionals as

Theorem 1. *Let $f \in LL(X)$ be bounded from below and $x_0 \in D(f)$. Then there exists $y_0 \in X$:*

$$\begin{array}{ll} i^*) & f(y_0) + \varepsilon\|x_0 - y_0\| \leq f(x_0), \\ ii^*) & 0 \leq f^0(y_0;v) + \varepsilon\|v\|, \quad \forall v \in X. \end{array}$$

Recall that x_0 is a critical point of f if

$$0 \leq f^0(x_0;v), \quad \forall v \in X,$$

that is, if $0 \in \partial f(x_0)$. The following Palais–Smale (PS) condition is introduced by Chang [5]

Definition 1. *The functional $f \in LL(X)$ satisfies (PS^0) condition if whenever $\{x_n\} \subset X$ is such that*

$$\begin{array}{ll} (j) & |f(x_n)| \quad \text{is bounded,} \\ (jj) & \lambda_f(x_n) = \min\{\|p\|_* : p \in \partial f(x_n)\} \longrightarrow 0, \end{array}$$

then $\{x_n\}$ possesses a convergent subsequence.

We formulate another (PS) type condition

Definition 2. *The functional $f \in LL(X)$ satisfies (PS^1) condition if whenever $\{x_n\} \subset X$ is such that:*

$$\begin{array}{ll} (j^*) & |f(x_n)| \quad \text{is bounded,} \\ (jj^*) & \forall \varepsilon > 0, \exists n_0, \forall v \in X : n > n_0 \Rightarrow 0 \leq f^0(x_n;v) + \varepsilon||v||, \end{array}$$

then $\{x_n\}$ possesses a convergent subsequence.

Lemma 1. *If $f \in LL(X)$ and a sequence $\{x_n\} \subset X$ satisfies condition (jj) then it satisfies condition (jj^*).*

Theorem 2. *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a locally Lipschitz functional which is bounded from below and satisfies (PS^1) condition. Then there exists x_0 such that $f(x_0) = \inf f(x)$ and x_0 is a critical point i.e.: $0 \leq f^0(x_0;v), \quad \forall v \in X$.*

Theorem 3. *Let $f \in LL(X)$ be a function which is bounded from below and $x_0 \in X$*

be given such that $f(x_0) \le \inf f + \varepsilon$. *Then for every* $\lambda > 0$ *there exists* $y_0 \in X$*:*

$$\begin{aligned} &1) \qquad f(y_0) \le f(x_0), \\ &2) \qquad \|x_0 - y_0\| \le \frac{1}{\lambda}, \\ &3) \quad 0 \le f^0(y_0; v) + \lambda\varepsilon\|v\|, \quad \forall v \in X. \end{aligned}$$

We say that $f \in LL(X)$ is coercive if $f(x) \to +\infty$ as $\|x\| \to \infty$.

Theorem 4. *Let* X *be a Banach space and* $f \in LL(X)$ *be a functional satisfying* (PS^1) *condition. If* f *is bounded from below, then* f *is coercive.*

This result is an extension of those proved by Caklovic, Li & Willem [4] for differentiable functionals.

Proof. Suppose that the conclusion is not true and $c = \liminf_{\|x\|\to\infty} f(x)$ is finite. Then for $\varepsilon = 1/n$ there exists x_n such that $\|x_n\| \ge 2n$ and

$$f(x_n) \le c + \frac{1}{n} = \inf f + (c + \frac{1}{n} - \inf f).$$

By Theorem 3 there exists $y_n \in X$ such that

$$f(y_n) \le f(x_n),$$

$$\|x_n - y_n\| \le n,$$

$$0 \le f^0(y_n; v) + \frac{1}{n}(c + \frac{1}{n} - \inf f)\|v\|, \quad \forall v \in X.$$

We have $\|y_n\| \ge \|x_n\| - ||y_n - x_n|| \ge 2n - n = n$, i.e. $\lim_{n\to\infty} \|y_n\| = \infty$ and $|f(y_n)|$ is bounded. Let $\varepsilon > 0$ and n_0 be such that if $n > n_0$

$$0 < \frac{1}{n}(c + \frac{1}{n} - \inf f) < \varepsilon.$$

For $n > n_0$ we have

$$0 \le \ f(y_n; v) + \varepsilon||v||, \quad \forall v \in X,$$

and by (PS^1) condition there exists a convergent subsequence of $\{y_n\}$, which is a contradiction. Then $c = +\infty$. □

Next we give a generalization of the mountain–pass theorem for locally Lipschitz functions due to Chang [5]. Following ideas developed in Aubin & Ekeland [3] we prove

Theorem 5. *Let* $f \in LL(X)$ *be a functional satisfying* $\left(PS^1\right)$ *condition. Suppose that there exist* $\rho > 0$ *and* $e \in X$ *such that*

$$\begin{aligned} &1) \quad m(\rho) = \inf\{f(x) : \|x\| = \rho\} > f(0), \\ &2) \qquad \|e\| > \rho, \quad f(e) < m(\rho), \end{aligned}$$

Then there exists x_0 *such that*

$$f(x_0) \ge m(\rho), \quad 0 \in \partial f(x_0).$$

We prove also a version of Mountain–pass theorem based on Ekeland's variational principle.

Theorem 6. *Let* $f \in LL(X)$ *satisfy* $\left(PS^1\right)$ *condition. Suppose that* f *has two local minima. Then* f *has at least one more critical point.*

Proof. Without loss of generality let 0 and $e \neq 0$ be two points of local minima, $c_0 = f(0), c_1 = f(e), c_0 \geq c_1$. Let ε be such that $0 < \varepsilon < ||e||$ and $f(x) \geq f(0)$, if $||x|| \leq \varepsilon$. We have the following alternative:

i) there exists $\rho \in (0, \varepsilon)$ such that $b = \inf\{f(x) : ||x|| = \rho\} > c_0$,

or

ii) for every $\rho \in (0, \varepsilon)$, $\inf\{f(x) : ||x|| = \rho\} = c_0$.

If i) holds the assertion follows by Mountain- pass theorem, [5].

Let ii) holds and take ρ and R such that $0 < \rho < R < \varepsilon$. Let $\{x_n\}$ be a minimizing sequence, that is, a sequence satisfying $||x_n|| = \rho, \quad f(x_n) \to c_0 = f(0) = \inf\{f(x) : ||x|| = \rho\}$ and $f(x_n) \leq c_0 + \dfrac{1}{n}$.

Define

$$\bar{f}(x) = \begin{cases} f\left(R\dfrac{x}{||x||}\right), & ||x|| \geq R, \\ f(x), & ||x|| \leq R. \end{cases}$$

By Theorem 3, applied to $\bar{f}(x)$, there exists $y_n \in X$ such that

$$\begin{aligned} \bar{f}(y_n) &\leq \bar{f}(x_n), \quad \|x_n - y_n\| \leq \frac{1}{\sqrt{n}}, \\ 0 &\leq \bar{f}^0(y_n; v) + \frac{1}{\sqrt{n}}\|v\|, \quad \forall v \in X. \end{aligned} \tag{1}$$

As $\bar{f}(x_n) = f(x_n) \to c_0$ by (PS^1) condition there exists a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y$ and $||y|| = \rho$. By upper semicontinuity of $\bar{f}^0(.,.)$, taking a limit in (1) we obtain$0 \leq \bar{f}^0(y; v), \quad \forall v \in X$.As $||y|| = \rho < R, \quad \bar{f}^0(y; v) = f^0(y; v)$ and therefore $0 \in \partial f(y)$. Note that we get a critical point y, $||y|| = \rho$ for every $\rho \in (0, \varepsilon)$. □

3. Existence results for a fourth order equation with discontinuous nonlinearities . Let us consider at first the linear eigenvalue problem

$$(L) : \begin{cases} y''''(x) = \lambda y(x), \\ y''(0) = y''(1) = 0, \\ y'''(0) = y'''(1) = 0. \end{cases}$$

Problem (L) has a sequence of eigenvalues $\lambda_k, \ k \geq -1$, such that $\lambda_{-1} = \lambda_0 = 0$ and $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$. The first positive eigenvalue is $\lambda_1 \approx 500.55$. Denote by $\psi_{-1}, \psi_0, \psi_1, \ldots, \psi_n, \ldots$ the corresponding eigenfunctions. The eigenfunctions corresponding to $\lambda_{-1} = \lambda_0 = 0$ are $\psi_{-1} = 1$ and $\psi_0 = x - \dfrac{1}{2}$.

Let $V = H^2(0,1) \subset E = L^2(0,1)$ be the usual Sobolev space with norm $||u||^2 = ||u''||_2^2 + ||u||_2^2$, where $||.||_2$ denotes the $E-$ norm.

The eigenfunctions $\{\psi_j : j = -1, 0, 1, \ldots\}$ form an orthogonal basis both for V and E. Therefore $V = X \oplus Y$, where $X = sp\{1, x\}, \quad Y = X^{\perp}$. We use the notation $u(x) = \bar{u}(x) + \tilde{u}(x), \quad \bar{u} \in X, \tilde{u} \in Y$. By the variational characterization of λ_1

$$\int_0^1 (y''(x))^2 dx \geq \lambda_1 \int_0^1 y^2(x) dx, \quad \forall\, y \in Y. \tag{2}$$

Let us consider the problem:

$$(P):\begin{cases} u''''(x)+\xi(x)=0, & \\ \xi(x)\in\partial j(x,u(x)), & \text{a.e. in } [0,1], \\ u''(0)=u''(1)=0, & \\ u'''(0)=u'''(1)=0, & \end{cases}$$

which describes the vibrations of an elastic beam with free ends and discontinuous forcing term. We assume that the function $j:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, with generalized gradient ∂j, satisfies the following conditions

(J_1) the function $x\to j(x,u)$ is measurable for each $u\in\mathbb{R}$.

(J_2) there exists $k\in E$

$$|j(x,u_1)-j(x,u_2)|\le k(x)|u_1-u_2|,\quad \forall u_1,u_2\in\mathbb{R}.$$

The problem (P) can be formulated in terms of hemivariational inequalities, introduced by P. D. Panagiotopoulos [8] as follows

Definition 3. *The function $u\in V$ is said a "strong solution" of (P) if there exists $\xi\in E$ such that*

$$\int_0^1 u''(x)v''(x)\,dx+\int_0^1 \xi(x)v(x)\,dx=0,\quad \forall v\in C_*^\infty[0,1], \tag{3}$$

$$\xi(x)\in\partial j(x,u(x)),\quad \textit{a.e. in } [0,1] \tag{4}$$

Here $C_*^\infty[0,1]=\{v\in C^\infty[0,1]: v''(0)=v''(1)=v'''(0)=v'''(1)=0\}$. Note that $C_*^\infty[0,1]$ is dense in $V\subset E$, and for $u,v\in C_*^\infty[0,1]$:

$$\int_0^1 u''''(x)v(x)\,dx=\int_0^1 u''(x)v''(x)\,dx$$

Definition 4. *The function $u\in V$ is said a "weak solution" of (P) if*

$$\int_0^1 u''(x)v''(x)\,dx+\int_0^1 j^0(x,u(x),v(x))dx\ge 0,\quad \forall v\in V. \tag{5}$$

The inequality (5) is said a hemivariational inequality. Using a standard way developed in Adly & Goeleven [1], Panagiotopoulos [8] one can prove

Proposition 1. *If u is a "strong solution" of (P), then is a "weak solution" of (P).*

Now the problem of finding strong solutions of (P) reduces to finding critical points of the functional

$$f(u)=\frac{1}{2}\int_0^1 |u''(x)|^2dx+J|_V(u),$$

where $J : E \to \mathbb{R}$ is defined by

$$J(u) = \int_0^1 j(x, u(x))dx.$$

By a result of Chang [5]

$$\partial J|_V(u) \subset \partial J(u) = \{w \in E : J^0(u; v) \geq \int_0^1 w(x)v(x)\,dx,\ \forall v \in E\}.$$

Proposition 2. *If $0 \in \partial f(u)$ then u is a "strong solution" of (P).*

We consider an additional assumption

(C_1) $j(x,u) \geq l(x)$, a.e. $x \in (0,1)$, $l(x) \in L^1(0,1)$,
$j(x,u) \to \infty$ as $|u| \to \infty$.

Theorem 7. *Suppose that j satisfies assumptions (J_1), (J_2) and (C_1). Then the problem (P) admits at least one solution $u \in V$, that minimizes the functional f.*

Proof. Applying minimization Theorem 2,we show that there exists $u \in V$ such that $0 \in \partial f(u)$. Then, by Propositions 1 and 2, u is a weak solution of (P) and the result is proved. □

Let (C_1) hold. As, for $u = \bar{u} + \tilde{u} \in X \oplus Y$,

$$f(u) \geq \frac{1}{2}||u''||_2^2 - ||l||_1 = \frac{1}{2}||\tilde{u}''||_2^2 - ||l||_1, \tag{6}$$

then f is bounded from below. We show that f satisfies (PS^1) condition and then apply Theorem 2. It follows by Theorem 4 that f is also coercive.

Let $u_n = \bar{u}_n + \tilde{u}_n$ be such that $|f(u_n)|$ is bounded and for every $\varepsilon > 0$ and there exists n_0 such that for $n > n_0$

$$0 \leq f^0(u_n; v) + \varepsilon||v||, \quad \forall v \in V.$$

By (6), $||\tilde{u}_n''||_2$ is bounded and by (2), $\{\tilde{u}_n\}$ is also bounded in V, that is, there exists $M > 0$ such that $||\tilde{u}_n|| \leq M$.

Let us now check that $\{\bar{u}_n\}$ is bounded. Suppose, by contradiction, that, for a subsequence, $||\bar{u}_n|| \to \infty$. Then

$$|u_n(x)| \geq |\bar{u}_n(x)| - |\tilde{u}_n(x)| \geq |a_n x + b_n| - \alpha M \to \infty,$$

except at most for one point in $(0,1)$. Here α is the imbedding constant of $H^2(0,1)$ in $C^0[0,1]$, that is, for all $w \in H^2(0,1)$

$$|w(x)| \leq \alpha||w||.$$

Then, by (C_1), $j(x, u_n(x)) \to \infty$ for a.e. $x \in (0,1)$. Using then Fatou's lemma and the fact that

$$f(u_n) \geq \int_0^1 j(x, u_n(x))dx,$$

we obtain a contradiction. Thus $\{u_n\}$ is bounded in V. Passing to a subsequence, if necessary, we assume that $u_n \rightharpoonup u_0$ weakly in V and show that $u_n \to u_0$ strongly in V.

As $u_n \rightharpoonup u_0$ in V taking a subsequence denoted again by $\{u_n\}$ we assume that $u_n \to u_0$ in $C[0,1]$ and $u_n \to u_0$ in $L^2(0,1)$. Let $M>0$ be such that $||u_n|| \le M, ||u_0|| \le M$. For $\varepsilon > 0$, there exists n_1 such that for $n > n_1$

$$0 \le \ f^0(u_n; v) + \frac{\varepsilon}{4M}||v||, \quad \forall v \in V,$$

which means that

$$0 \le \int_0^1 u_n'' v'' dx + \int_0^1 j^0(x, u_n(x); v(x)) dx + \frac{\varepsilon}{4M}||v||. \tag{7}$$

Taking $v = u_0 - u_n$ in (7) we have

$$0 \quad \le \quad \int_0^1 u_n''(u_0 - u_n)'' \, dx + \int_0^1 j^0(x, u_n; u_0 - u_n) dx + \frac{\varepsilon}{4M}||u_0 - u_n||.$$

By (J_2) there exists $k_1 > 0$ such that

$$\int_0^1 j^0(x, u_n; u_0 - u_n) dx \le k_1 ||u_0 - u_n||_2$$

Then

$$||u_0'' - u_n''||_2^2 \le \frac{\varepsilon}{2} + k_1||u_0 - u_n||_2 + \int_0^1 u_0''(u_0 - u_n)'' \, dx \tag{8}$$

As $u_n \rightharpoonup u_0$ in V and $u_n \to u_0$ in $L^2(0,1)$ there exists n_2 such that for $n > n_2$

$$k_1||u_0 - u_n||_2 + \int_0^1 u_0''(u_0 - u_n)'' \, dx < \frac{\varepsilon}{2}.$$

Then for $n > \max(n_1, n_2)$ by (8) we have

$$||u_0'' - u_n''||_2^2 < \varepsilon.$$

So $u_n \to u_0$ in V which proves (PS^1) condition.

Next result concerns the existence of multiple solutions of (P). We suppose that the following conditions hold:

(J_3) $\quad j(x,0) = 0, \quad \exists \mu > 0 : \lim_{u \to 0} \dfrac{j(x,u)}{u^2} = \mu$ uniformly a.e. $x \in (0,1)$.

(C_2) $\quad \exists (a,b) \ne (0,0) : \int_0^1 j(x, ax+b) dx < 0.$

Applying Theorems 5 and 7, we have

Theorem 8. *Suppose $j(x,u)$ satisfies conditions $(J_1)-(J_3),(C_1),(C_2)$. Then there exist at least two nontrivial weak solutions of the problem (P).*

An example of a function $j = j(u)$ satisfying conditions $(J_1)-(J_3),(C_1)$ and (C_2) is the following one

$$j_0(u) = \begin{cases} -2u-5 & u \le -2, \\ 2u+3 & -2 \le u \le -1, \\ u^2 & -1 \le u \le 1, \\ 2u-1 & 1 \le u. \end{cases}$$

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ТЕОРЕМИ ЗА КРИТИЧНИ ТОЧКИ НА ЛОКАЛНО ЛИПШИЦОВИ ФУНКЦИОНАЛИ И ПРИЛОЖЕНИЯ КЪМ ЗАДАЧИ ОТ ЧЕТВЪРТИ РЕД

Мариа до Розарио Грозиньо, Степан Агоп Терзиян

Доказани са теореми за съществуване на решения на полулинейни уравнения от четвърти ред с прекъснати нелинейности от теорията на еластичността. Доказателствата се основават на теореми за критични точки за локално липшицови функционали.