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POSITIVE DEFINITE SOLUTIONS OF A NONLINEAR MATRIX EQUATION

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In this paper we investigate the positive definite solutions of a special nonlinear matrix equation. Sufficient conditions for existence positive definite solutions of the described equation are derived. Two algorithms for numerical computing of these solutions are given.

1. Introduction. In this paper we investigate the positive definite solutions of the matrix equation

$$X + A^*\sqrt{X^{-1}}A = I, \tag{1}$$

where I is the $n \times n$ identity matrix and A is a $n \times n$ invertible matrix.

We have to solve linear systems in many physical applications of the form $Mx = f$ [2] where M has the form $M = \begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$.

We can use the equation (1) for solving the above system. This system can be solved when we use the matrix equation $X + A^*X^{-1}A = I$, which is considered by Engwerda in [3, 4]. Such approach for solving the system $Mx = f$ is considered in [5].

In this paper we propose two iterative methods which are convergent to different positive definite solutions of the equation (1). The first method generalizes the method described in El-Sayed's PhD thesis [1]. The proposed modification extends the set of matrices A for wich the method is convergent. In this paper we consider new method which is convergent to other solution of (1). That is the second method in this paper. The rate of convergent of these methods depend on parameters α and β. We carry out numerical experiments with these methods and results are given.

We use the following well known results for matrices A and B for s.t. $AB = BA$

(i) $A(B \pm I) = (B \pm I)A$;

(ii) If B is invertible then $AB^{-1} = B^{-1}A$;

(iii) If $A > B > 0$ then $A^2 > B^2$;

2. Solutions of the matrix equation. The following theorem is proved in the PhD thesis [1]

Theorem 1. *If* $A^*A < \frac{1}{2\sqrt{2}}I$ *then the equation* (1) *has a positive definite solution.*

Consider the iterative method

$$X_0 = \beta I, \quad X_{k+1} = I - A^*\sqrt{X_k^{-1}}A, \quad k = 0, 1, 2, \dots. \tag{2}$$

Further on we shall use $\|A\|$ to denote the spectral norm of the matrix A and $\|A\| = \sqrt{\max_i \lambda_i}$ where λ_i are the eigenvalues of AA^*.

We extend the Theorem 1

Theorem 2. *If there are numbers* α *and* β *such that* $\frac{1}{3} \le \beta < \alpha \le 1$ *and the matrix* A *satisfies inequalities*

$$\sqrt{\alpha}(1-\alpha)I < A^*A < \sqrt{\beta}(1-\beta)I. \tag{3}$$

then the matrix equation (1) *has a positive definite solution* X *such that* $\beta I < X < \alpha I$ *and* $\|X_k - X\| < q^k(\alpha - \beta)$, *where* X_k *is a matrix of the sequence* (2) *and* $q = \frac{\|A\|^2}{2\beta\sqrt{\beta}} < 1$.

Proof. We shall prove that the matrix sequence $\{X_k\}$ from (2) is monotone increasing and bounded from above. We prove that by induction. We have $X_1 = I - A^*\sqrt{(\beta I)^{-1}}A$. According to (3) we receive

$$X_0 = I + (\beta - 1)I < I - \frac{1}{\sqrt{\beta}}A^*A = X_1, \quad X_0^{-1} > X_1^{-1}.$$

We assume $X_k > X_{k-1}$.

Compute $X_{k+1} = I - A^*\sqrt{X_k^{-1}}A > I - A^*\sqrt{X_{k-1}^{-1}}A = X_k$.

Consequently $X_{k+1} > X_k$ where $k = 0, 1, 2 \dots$.

Obviously $X_0 = \beta I < \alpha I$. Assume that $X_k < \alpha I$.

We obtain $X_{k+1} = I - A^*\sqrt{X_k^{-1}}A < I - \frac{1}{\sqrt{\alpha}}A^*A < \alpha I$.

Hence the matrix sequence $\{X_k\}$ converges and its limit X is a positive definite solution of the equation (1). Since $\beta I < X_k < \alpha I$ for $k = 1, 2, \dots$ and from (3) we obtain $\beta I < X < \alpha I \le I$.

Consider the spectral norm of the matrix $X_k - X$.

$$\begin{aligned}
\|X_k - X\| &= \|A^*(\sqrt{X_{k-1}^{-1}} - \sqrt{X^{-1}})A\| \\
&= \|A^*\sqrt{X_{k-1}^{-1}}\left(\sqrt{X} - \sqrt{X_{k-1}}\right)\sqrt{X^{-1}}A\| \\
&\le \|A\|^2 \, \|\sqrt{X^{-1}}\| \, \|\sqrt{X_{k-1}^{-1}}\| \, \|\sqrt{X_{k-1}} - \sqrt{X}\|.
\end{aligned}$$

Using the theorem 8.5.2 [6] as in [5] we obtain

$$\sqrt{X_{k-1}} - \sqrt{X} = \int_0^\infty e^{-\sqrt{X_{k-1}}t}(X_{k-1} - X)e^{-\sqrt{X}t}dt.$$

For sequence (2) we have $X_k > \beta I$, $\sqrt{X_k^{-1}} < \frac{1}{\sqrt{\beta}} I$, $\sqrt{X^{-1}} < \frac{1}{\sqrt{\beta}} I$. Then

$$\begin{aligned} \|X_k - X\| &< \frac{1}{\beta}\|A\|^2\|X_{k-1} - X\| \int_0^\infty \|e^{-\sqrt{X_{k-1}}t}\| \, \|e^{-\sqrt{X}t}\| \, dt \\ &< \frac{1}{\beta}\|A\|^2\|X_{k-1} - X\| \int_0^\infty e^{-2\sqrt{\beta}t} \, dt \\ &= \frac{1}{2\beta\sqrt{\beta}} \, \|A\|^2\|X_{k-1} - X\|. \end{aligned}$$

Consequently $\|X_k - X\| < q^k\|X_0 - X\| \leq q^k(\alpha - \beta)$.

Since $A^*A < \sqrt{\beta}(1-\beta)I$ then $\|A\|^2 < \sqrt{\beta}(1-\beta) \leq 2\beta\sqrt{\beta}$ where $\frac{1}{3} \leq \beta < 1$. Consequently $q = \frac{\|A\|^2}{2\beta\sqrt{\beta}} < 1$.

Remark 1. *If we consider the iterative method* (2) *where* $X_0 = \alpha I$ *we shall receive the monotone decreasing and bounded from bellow matrix sequence* $\{X_k\}$*. It is easy to show that both iterative method where* $X_0 = \alpha I$ *and* $X_0 = \beta I$ *are convergent to common limit which is the positive definite solution of the equation* (1)*. The initial condition* $X_0 = I$ *is used in* [1].

Colorary 1. *If* $A^*A < \frac{2}{3\sqrt{3}} I$*. Then the matrix equation* (1) *has a positive definite solution.*

Proof. Consider the function $f(x) = \sqrt{x}(1-x)$ where $\frac{1}{3} \leq x \leq 1$. We obtain $\max_x f(x) = f(\frac{1}{3}) = \frac{2}{3\sqrt{3}}$, $f(1) = 0$. Hence there are α and β which are from the theorem (2).

Consider second iterative method for computing a positive definite solution of the eqaution (1).

$$Y_0 = \beta I, \quad Y_{k+1} = \left[A(I - Y_k)^{-1}A^*\right]^2, \quad k = 0, 1, 2, \ldots. \tag{4}$$

Theorem 3. *Let* A *be a normal matrix and there are numbers* α *and* β *such that* $0 < \alpha < \beta \leq \frac{1}{3}$ *and for the matrix* A *are satisfied inequalities* (3) *of theorem 2. Then the equation* (1) *has a positive definite solution* Y *such that* $\alpha I < Y < \beta I$ *and* $\|Y_k - Y\| < q^{2k}(\beta - \alpha)$*, where* Y_k *is of the iterative method* (4) *and* $q = \sqrt{\frac{2}{1-\beta}\frac{\|A\|^2}{1-\beta}} < 1$.

Proof. We introduce $P_k = A(I - Y_k)^{-1}A^*$. We shall prove that $A^*AP_k = P_kA^*A$ and $P_kP_{k+1} = P_{k+1}P_k$, $k = 0, 1, 2 \ldots$. We omit the proof here. The reader can prove the theorem using statements `(i)` and `(ii)`.

We shall prove that the matrix sequence $\{Y_k\}$ is monotone decreasing and bounded from bellow.

$$Y_1 = \left[\frac{AA^*}{1-\beta}\right]^2 < \left[\frac{\sqrt{\beta}(1-\beta)}{1-\beta}\right]^2 I = Y_0.$$

Assume that $Y_k < Y_{k-1}$.

$$Y_{k+1} = \left(A(I-Y_k)^{-1}A^*\right)^2 < \left(A(I-Y_{k-1})^{-1}A^*\right)^2 = Y_k$$

Consequently $Y_{k+1} < Y_k$ for $k = 0, 1, 2, \ldots$.

We have $Y_0 = \beta I > \alpha I$. We assume that $Y_k > \alpha I$.

$$Y_{k+1} = \left(A(I-Y_k)^{-1}A^*\right)^2 > \left(A(I-\alpha I)^{-1}A^*\right)^2 > \alpha I.$$

Hence the sequence $\{Y_k\}$ is convergent to the positive definite matrix Y, which is a solution of the equation (1). From $\alpha I < Y_k < \beta I$ for $k = 1, 2, \ldots$ and from the condition (3) follow $\alpha I < Y < \beta I$.

Consider the spectral norm of the matrix $Y_k - Y$.

$$\begin{aligned} \|Y_k - Y\| &= \| \left(A(I-Y_{k-1})^{-1}A^*\right)^2 - \left(A(I-Y)^{-1}A^*\right)^2 \| \\ &\leq \|A(I-Y_{k-1})^{-1}A^* - A(I-Y)^{-1}A^*\| \times \\ &\quad \times \left(\|A(I-Y_{k-1})^{-1}A^*\| + \|A(I-Y)^{-1}A^*\|\right). \end{aligned}$$

It is well known $(I-Y_{k-1})^{-1} < \dfrac{1}{1-\beta}I, \quad (I-Y)^{-1} < \dfrac{1}{1-\beta}I$. We receive

$$\begin{aligned} \|Y_k - Y\| &\leq \|A\left((I-Y_{k-1})^{-1} - (I-Y)^{-1}\right)A^*\| \, \|A\|^2 \, \frac{2}{1-\beta} \\ &< \|A\|^4 \, \frac{2}{1-\beta} \, \frac{1}{(1-\beta)^2}\|Y_{k-1} - Y\| \\ &< \left[\sqrt{\frac{2}{1-\beta}}\frac{\|A\|^2}{1-\beta}\right]^{2k} \|Y_0 - Y\| \leq q^{2k}(\beta - \alpha), \end{aligned}$$

where $q = \sqrt{\dfrac{2}{1-\beta}}\dfrac{\|A\|^2}{1-\beta}$. Obviously $q < 1$ if $AA^* < \sqrt{\beta}(1-\beta)I$ and $0 < \beta \leq \dfrac{1}{3}$.

Remark 3. *If in Theorem* 3 *we consider the iterative method* (4) *where* $Y_0 = \alpha I$ *we receive the monotone increasing and bounded matrix sequence* $\{Y_k\}$. *Both iterative methods where* $Y_0 = \alpha I$ *and* $Y_0 = \beta I$ *are convergent to a common positive definite limit of the equation* (1).

3. Numerical experiments. Numerical experiments were made for computing positive definite solutions of the equation (1). We carry out experiments for different matrices A and different values n. We denote $\varepsilon(Z) = \|Z + A^T\sqrt{Z^{-1}}A - I\|_\infty$, where $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$. We use two iterative methods. X_k are the matrices obtained from the iterative method (2) where $X_0 = \gamma I$ and Y_k are the matrices obtained from the iterative method (4) where $Y_0 = \eta I$.

Let p_γ be the smallest number k for which $\varepsilon(X_k) \leq 10^{-7}$ and q_η is the smallest number k for which $\varepsilon(Y_k) \leq 10^{-7}$.

Results are given in the following table.

Example. Let A be of the form

$$A = (a_{ij}) = \begin{cases} a_{ij} = \dfrac{3i}{5n} & i = j \\ a_{ij} = \dfrac{|i-j|}{n^3} & i \neq j \end{cases}$$

Table 1.

n	$\|A\|$	γ	p_γ	$\varepsilon(X_{p_\gamma})$	η	q_η	$\varepsilon(Y_{q_\eta})$
5	0.3673	0.987	27	$7.6261E-8$	0.225	17	$7.6365E-8$
		0.456	14	$6.9832E-8$			
10	0.361	0.997	23	$9.3653E-8$	0.208	14	$6.4504E-8$
		0.477	14	$7.8858E-8$			
15	0.360	0.999	23	$7.4345E-8$	0.207	15	$6.4581E-8$
		0.48	14	$3.4811E-8$			
20	0.3601	1	23	$6.9267E-8$	0.206	14	$5.8320E-8$
		0.48	14	$9.1828E-8$			
25	0.3601	1	23	$6.7052E-8$	0.206	14	$7.9954E-8$
		0.48	15	$6.6286E-8$			
50	0.36	1	23	$6.3875E-8$	0.206	16	$5.6940E-8$
		0.48	17	$4.7086E-8$			

For different values of γ are received different number of iterations which are necessary for computations. For first value of γ the iterative method (2) is monotone decreasing matrix sequence and for the second value we obtain the monotone increasing sequence. The matrix A from above example does not satisfy the sufficient condition for existence of solution of the equation $X + A^*X^{-1}A = I$ [3, 4]. The system $Mx = f$ is solved as we use the equation (1).

4. Conclusion. The considered equation (1) has a solution where $A^*A < \dfrac{2}{3\sqrt{3}}I$. In case of normal matrix A there are at least two positive definite solutions X and Y which are obtained from the iterative methods (2) and (4) respectively. We have $0 < Y < \dfrac{1}{3}I < X < I$.

From above numerical experiments we can seen effectiveness of proposed iterative methods.

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ПОЛОЖИТЕЛНО ОПРЕДЕЛЕНИ РЕШЕНИЯ НА ЕДНО НЕЛИНЕЙНО МАТРИЧНО УРАВНЕНИЕ

Вежди Исмаилов Хасанов

В тази статия изследваме положително определените решения на едно нелинейно матрично уравнение. Изведени са достатъчни условия за съществуване на положително определени решения на разглежданото уравнение. Дадени са два алгоритъма за числено пресмятане на тези решения.