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GENERALIZED CAUCHY PROBLEM FOR LINEAR PULSE DIFFERENTIAL SYSTEMS

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A generalized Cauchy problem for linear systems of differential equations with generalized pulse effects in a fixed moments of time is considered. Necessary and sufficient conditions for existence of a parametric and unique solution are obtained.

1. Statement of a problem. We consider the linear differential system

$$\dot{x} = A(t)x + f(t),\ t \in [a,b],\ t \neq \tau_i,\ i = \overline{1,p}, \tag{1}$$

$$a = \tau_0 < \tau_1 < \ldots < \tau_p < \tau_{p+1} = b,$$

where the coefficients of the system satisfy the conditions:
(H1) $A(t)$ — $(n \times n)$ matrix, which has a continuous elements in the interval $[a,b]$;
(H2) $f(t)$ is a partially continuous n-dimensional vector function, which has a breaks of the first kind in the points τ_i, i.e.

$$f(t) = f_i(t),\ t \in (\tau_{i-1}, \tau_i],\ i = \overline{1,p+1},\ f(a) = f_1(\tau_0),$$
$$f(b) = f_{p+1}(\tau_{p+1}), f_{i+1}(\tau_i) = \lim_{t \to \tau_i + 0} f(t),\ i = \overline{1,p}.$$

We seek an n-dimensional partially continuous vector-function $x(t)$, which will satisfy the system (1), the generalize condition of Cauchy

$$Dx(a) = v, \tag{2}$$

where D is a given $(s \times n)$ matrix with constant elements, v is a given column vector with s components and the generalize impulse conditions in fixed moments of time

$$M_i x(\tau_i - 0) + N_i x(\tau_i + 0) = h_i,\ i = \overline{1,p}. \tag{3}$$

The matrices M_i and N_i have the same dimension $(k \times n)$ and $h_i \in \mathrm{R}^k$.

The problem (1), (2) is considered in [4]. A lines to research of the problem (1), (3) can be found in [5]. The condition (3) when $k = n$ and generalize linear differential equations in the form

$$dx = d[A]x + df.$$

are considered in [10]. In [3] the pulse system (1), (3) is transformed in the form

$$\dot{x} = A(t)x(t) + \sum_k \delta_{ik} c_k \quad \text{(here } delta_{ik} \text{ is Dirac's function)},$$

where $f = 0$, $N_i = -M_i = E_n$ and the researches are done with the help of the theory of the distributions.

The problem of Cauchy (1), (2), when $D = E_n$ (E_n is the n^{th} order unit matrix) and impulse conditions in the form

$$\Delta x|_{t=\tau_i} = B_i x + a_i, \tag{4}$$

where $E_n + B_i$ is nonsingular $(n \times n)$ matrix with constant elements, a_i are given n-dimensional vectors, is considered in [2].

Diverse questions connected to impulse systems can be seen in [6].

In this paper the problem (1)–(3) will be researched similarly to the way in [7].

2. Modification of the problem. Let we denote by Θ $(k \times n)$-matrix of zero elements and by $x_i(t)$ — the solution of the system (1) in the intervals $[\tau_0, \tau_1]$, $]\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$, i.e.

$$x(t) = \begin{cases} x_1(t), & t \in [\tau_0, \tau_1], \\ x_i(t), & t \in]\tau_{i-1}, \tau_i]. \end{cases}$$

Then $x(a) = x_1(\tau_0)$, $x(\tau_i + 0) = x_{i+1}(\tau_i) = \lim\limits_{t \to \tau_i + 0} x_{i+1}(t)$, $i = \overline{1, p}$.

The impulse conditions (3) we rewrite in the form

$$\sum_{i=1}^{p+1} l_i x_i(\cdot) = h, \quad h \in \mathrm{R}^{pk},$$

where l_i are $(pk \times n)$-matrix operators, acting on the functions $x_1(t) : [\tau_0, \tau_1] \to \mathrm{R}^n$, $x_i(t) :]\tau_{i-1}, \tau_i] \to \mathrm{R}^n$ in the following way

$$\begin{gathered} l_1 x_1(\cdot) = [M_1 \;\; \Theta \;\cdots\; \Theta]^T x_1(\tau_1), \\ l_i x_i(\cdot) = [\Theta \cdots \Theta \; N_{i-1} \; \Theta \cdots \Theta]^T x_{i-1}(\tau_{i-1}) + [\Theta \cdots \Theta \; M_i \; \Theta \cdots \Theta]^T x_i(\tau_i), \; i = \overline{2, p}, \\ l_{p+1} x_{p+1}(\cdot) = [\Theta \;\cdots\; \Theta \;\; N_p]^T x_{p+1}(\tau_p), \; h = [h_1 \cdots h_p]^T \end{gathered} \tag{6}$$

Here $h_i \in \mathrm{R}^k$ and the $(k \times n)$ matrices N_{i-1} and M_i take up $(i-1)^{-\text{th}}$ and $i^{-\text{th}}$ blocks, respectively.

The problem (1,2,6) can be written as follows

$$\begin{gathered} \dot{x} = A(t)x + f(t), \; t \in [a, b], \; t \neq \tau_i, \; i = \overline{1, p}, \; \tau_i \in (a, b), \; a = \tau_0, \; b = \tau_{p+1}, \\ Dx(a) = v, \\ \sum_{i=1}^{p+1} l_i x_i(\cdot) = h, \; h \in \mathrm{R}^{pk}\,. \end{gathered} \tag{7}$$

We shall call the problem (7) a generalized problem of Cauchy with pulse effects.

Let we remark that pulse conditions (4) can be written in the form (3). In (3) or (6) additional conditions for the matrices N_i and M_i are not put. This shows that we will consider also the case when in (4) $\det(E + B_i) = 0$. Consequently we can not use a fundamental matrix which is constructed in [2] for the problem (1,4).

Main results. We seek the solution $x(t)$ of the problem (1) in each interval $[\tau_0, \tau_1]$, $]\tau_{i-1}, \tau_i]$, $i = \overline{2, n}$ in the form

$$x_i(t) = X(t)X^{-1}(\tau_{i-1})c_{i-1} + \int_{\tau_{i-1}}^{t} X(t)X^{-1}(s)f_i(s)ds, \; i = \overline{1, p+1}, \; c_{i-1} \in \mathrm{R}^n, \tag{8}$$

where $X(t)$ is the normal fundamental matrix of the solutions of the system $\dot{x} = Ax$, $X(a) = E_n$.

3.1. Let the following condition hold

(H3) $\operatorname{rank} D = k_1 < \min(s, n)$.

We denote by D^+ the unique Moore-Penrose inverse matrix of the matrix D. By P_D and P_{D^*} we denote the matrix orthoprojectors $P_D : \mathrm{R}^n \to \ker(D)$ and $P_{D^*} : \mathrm{R}^s \to \ker(D^*)$, $D^* = D^T$ [8], [9], [1]. Then the $(n \times n)$ matrix P_D has a rank $d_1 = n - k_1$ and the $(s \times s)$ matrix P_{D^*} has a rank $d_2 = s - k_1$. Therefore there exist d_1 linear independent columns in P_D and d_2 linear independent rows in P_{D^*}.

We denote by $P_{D^*}^{d_2}$ $(d_2 \times s)$ matrix, which consists of arbitrary d_2 linear independent rows from the matrix P_{D^*}.

From (2) we obtain that

$$c_0 = x_1(\tau_0) = P_D\xi + D^+v, \quad \xi \in \mathrm{R}^n \tag{9}$$

if and only if

(H4) $P_{D^*}v = 0 \Longrightarrow P_{D^*}^{d_2}v = 0$.

We introduce $(pk \times (p+1)n)$ matrix $Q = [Q_1 \;\; Q_2 \;\; \cdots \;\; Q_{p+1}]$, where $Q_1 = l_1(X(\cdot)P_D)$, $Q_i = l_i(X(\cdot)X^{-1}(\tau_{i-1}))$, $i = \overline{2, p+1}$ are $(pk \times n)$ matrices, and $(p+1)n$-dimensional vector $\overline{c} = \operatorname{col}(\xi, c_1, \cdots, c_p)$.

We substitute (8) and (9) in (5) and we get algebraic system with respect to c

$$Qc = \overline{h}, \quad \overline{h} = h - l(X(\cdot))D^+v - \sum_{i=1}^{p+1} l_i\left(\int_{\tau_{i-1}}^{(\cdot)} X(\cdot)X^{-1}(s)f_i(s)ds\right). \tag{10}$$

Let the following condition be fulfilled

(H5) $\operatorname{rank} Q = k_2 < \min(pk, (p+1)n)$.

In this case, the solution of the system (10) is

$$c = P_Q^r\eta + Q^+\overline{\overline{h}}, \quad \eta \in \mathrm{R}^r \tag{11}$$

if and only if

(H6) $P_{Q^*}\overline{\overline{h}} = 0 \Longrightarrow P_{Q^*}^d\overline{\overline{h}} = 0$,

where $r = (p+1)n - k_2$, $d = pk - k_2$. The matrix P_Q^r is formed by r linear independent columns of the matrix P_Q.

The solution (11) can be written in the form

$$\xi_0 = [P_Q^r]_{n_1}\eta + [Q^+\overline{h}]_{n_1}, \quad c_i = [P_Q^r]_{n_{i+1}}\eta + [Q^+\overline{h}]_{n_{i+1}}, \quad i = \overline{1, p}, \tag{12}$$

where $[Q^+\overline{h}]_{n_i}$, $i = \overline{1, p+1}$ are sequential n-dimensional components of $n(p+1)$-dimensional vector $Q^+\overline{h}$, $n_1 + n_2 + \cdots + n_{p+1} = n(p+1)$, and $[P_Q^r]_{n_i}$ are $(n_i \times r)$ matrices obtained from P_Q^r.

Keeping in mind (9) we obtain from (8) for the solution $x_i(t)$

$$\begin{aligned} x_1(t) &= X(t)P_D[P_Q^r]_{n_1}\eta + X(t)P_D[Q^+\overline{h}]_{n_1} + X(t)D^+v \\ &\quad + \int_a^t X(t)X^{-1}(s)f_1(s)ds, \; t \in [a, \tau_1], \\ x_i(t) &= X_{n_i}(t)\eta + X(t)X^{-1}(\tau_{i-1})[Q^+\overline{h}]_{n_i} \\ &\quad + \int_{\tau_{i-1}}^t X(t)X^{-1}(s)f_i(s)ds, \; t \in]\tau_{i-1}, \tau_i], \end{aligned} \tag{13}$$

where $X_{n_i}(t) = X(t)X^{-1}(\tau_{i-1})[P_Q^r]_{n_i}$, $i = \overline{2, p+1}$.

Thus we proof the following theorem.

Theorem 1. *Let the conditions* (H1)–(H3) *and* (H5) *be fulfilled. Initial value problem with impulse effects* (7) *has a r-parametric solution, which in the intervals* $[\tau_0, \tau_1]$, $]\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$ *has the representation* (13) *if and only if* v, h *and* $f(t)$ *satisfy the conditions* (H4) *and* (H6).

We assume that Q has a full rank and instead of (H5) the following condition is fulfilled

(H7) $\operatorname{rank} Q = pk$ $(pk < (p+1)n)$.

Hence $d = 0$ and $P_{Q^*} = 0$. This means that the condition for solvability (H6) is always fulfilled. The solution of the system (10) has the same representation from (12), moreover $r = (p+1)n - pk$.

Corollary 1. *Let the condition* (H1)–(H3) *and* (H7) *be fulfilled. Then the problem* (7) *has a r-parametric solution in the form* (13) *if and only if* v *satisfies* (H4), *for each function* $f(t) \in C([a, b] \setminus \tau_i)$ *and each vector* $h \in \mathrm{R}^{pk}$.

Let the following condition be fulfilled

(H8) $\operatorname{rank} Q = k_2 = (p+1)n$, $(pk > (p+1)n)$.

Then $r = 0$ and $P_Q = 0$. The system (10) has an unique solution $c = Q^+\overline{h}$ if and only if the condition (H6) is fulfilled ($d = pk - (p+1)n$). The components of the vector c take the form $\xi = [Q^+\overline{h}]_{n_1}$, $c_i = [Q^+\overline{h}]_{n_{i+1}}$, and the pulse problem (7) has an unique solution

$$
\begin{gathered}
x_1(t) = X(t)P_D[Q^+\overline{h}]_{n_1} + X(t)D^+v + \textstyle\int_a^t X(t)X^{-1}(s)f_1(s)ds,\ t \in [a, \tau_1], \\
x_i(t) = X(t)X^{-1}(\tau_{i-1})[Q^+\overline{h}]_{n_i} + \int_{\tau_i}^t X(t)X^{-1}(s)f_i(s)ds,\ t \in]\tau_{i-1}, \tau_i],\ i = \overline{2, p+1}.
\end{gathered}
\tag{14}
$$

Corollary 2. *Let the conditions* (H1)–(H3) *and* (H8) *be fulfilled. The problem* (7) *has an unique solution in the form* (14) *if and only if* v, h *and* $f(t)$ *satisfy* (H4)–(H6).

Remark 1. *Let we assume that* $pk = (p+1)n$. *Then* Q *is a quadratic matrix. If* $\operatorname{rank} Q < pk = (p+1)n$, *then* $d = r$ *and we can reason analogous as above. If* $\operatorname{rank} Q = pk = (p+1)n$, *then* $Q^+ = Q^{-1}$ *and the system* (10) *has an unique solution* $c = Q^{-1}\overline{h}$ *for every* $\overline{h}$. *The solution of the problem* (7) *has the representation* (14).

3.2. Let we consider the case when D has a full rank. Let $\operatorname{rank} D = s$, $(n > s)$. Then $P_{D^*} = 0$, i.e. condition (H4) is always fulfilled. The expression (9) and the equation (10) do not change. Consequently all calculations further do not change. Condition (H4) does not appear in analogous to Theorem 1 contention.

We assume that the following condition is fulfilled

(H9) $\operatorname{rank} D = n$, $(n < s)$.

Then $P_D = 0$, $d_2 = s - n$ and $c_0 = D^+v$, i.e. c_0 does not depend on an arbitrary constant vector. The system (10) is changed as following

$$
\overline{Q}\overline{c} = \overline{h},
\tag{15}
$$

where $\overline{Q} = [Q_2,\ Q_3, \cdots, Q_{p+1}]$, $\overline{c} = \operatorname{col}(c_1,\ c_2, \cdots, c_p)$, $\overline{Q}$ — $(pk \times pn)$ matrix, $\overline{c} \in \mathrm{R}^{pn}$, $\overline{h}$ — expression indicated above.

If the following condition is fulfilled

(H10) $\operatorname{rank}\overline{Q} = k_3 < \min(pk, pn)$,

then we denote by $\overline{r} = pn - k_3$, $\overline{d} = pk - k_3$. The condition for existence of the solution of the system (15) has the form

(H11) $P_{\overline{Q}^*}\overline{h} = 0 \Longrightarrow P_{\overline{Q}^*}^{\overline{d}}\overline{h} = 0$,

and the solution is

$$\overline{c} = P_{\overline{Q}}^{\overline{r}}\overline{\eta} + \overline{Q}^{+}\overline{h}, \quad \overline{\eta} \in \mathrm{R}^{\overline{r}}$$

or in the co-ordinate record

$$c_i = \left[P_{\overline{Q}}^{\overline{r}}\right]_{n_i}\overline{\eta} + \left[\overline{Q}^{+}\overline{h}\right]_{n_i}, \quad i = \overline{1,p}.$$

For the solution $x_i(t)$ of the system (7) we find

$$x_1(t) = X(t)D^{+}v + \int_{\tau_0}^{t} X(t)X^{-1}(s)f_1(s)ds, \quad t \in [\tau_0, \tau_1], \tag{16}$$

$$x_i(t) = X_{n_i}(t)\overline{\eta} + X(t)X^{-1}(\tau_{i-1})\left[\overline{Q}^{+}\overline{h}\right]_{n_i} + \int_{\tau_i}^{t} X(t)X^{-1}(s)f_i(s)ds, \ t \in]\tau_{i-1}, \tau_i], \tag{17}$$

where $X_{n_i} = X(t)X^{-1}(\tau_{i-1})[P_{\overline{Q}}^{r}]_{n_i}$, $i = \overline{2, p+1}$.

Theorem 2. *Let the condition* (H1), (H2), (H9) *and* (H10) *be fulfilled. Then initial value problem with impulse effects* (7) *in the intervals* $]\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$ *has r-parametric solution in the form* (17)*, but in the interval* $[\tau_0, \tau_1]$ *has an unique solution, which has the representation* (16) *if and only if* v, h *and* $f(t)$ *satisfy* (H11).

Corollary 3. *Let the conditions* (H1), (H2), (H7) *be fulfilled and* $\operatorname{rank}\overline{Q} = pk$, *(*$k < n$*). Then the problem* (7) *has a r-parametric solution in the form* (16) *in the intervals* $]\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$ *and an unique solution in the form* (17) *in the interval* $[\tau_0, \tau_1]$*, for each function* $f(t) \in C([a, b] \setminus \tau_i)$*, for each* $h \in \mathrm{R}^{pk}$ *and for each* $v \in \mathrm{R}^{s}$*.*

Obviously, in this case $\overline{d} = 0$ and $P_{\overline{Q}^*} = 0$, i.e. the condition (H11) is always real. The solution of the system (15) $\overline{c}$ has the same form as above.

Corollary 4. *Let the conditions* (H1), (H2), (H9) *be fulfilled and* $\operatorname{rank}\overline{Q} = pn$, *(*$k > n$*). The problem (7) has an unique solution*

$$x(t) = \begin{cases} X(t)D^{+}v + \int_{\tau_0}^{t} X(t)X^{-1}(s)f_1(s)ds, & t \in [\tau_0, \tau_1], \\ X(t)X^{-1}(\tau_{i-1})\left[\overline{Q}^{+}\overline{\overline{h}}\right]_{n_i} + \displaystyle\int_{\tau_i}^{t} X(t)X^{-1}(s)f_i(s)ds, & t \in]\tau_{i-1}, \tau_i], \ i = \overline{2, p+1}, \end{cases} \tag{18}$$

if and only if v, h *and* $f(t)$ *satisfy the condition (H11).*

Remark 2. *Let we assume that* $k = n$*. Then* $\overline{Q}$ *is a quadratic matrix and if* $\operatorname{rank}\overline{Q} = pk = pn$ *then* $\overline{Q}^{+} = \overline{Q}^{-1}$ *and the system* (15) *has an unique solution* $\overline{c} = \overline{Q}^{-1}\overline{h}$*. The solution of the problem* (7) *has the representation* (18).

Example. Let us consider the generalized Cauchy's problem with two pulse effects in the points $\tau_1 = \dfrac{1}{4}$, $\tau_2 = \dfrac{1}{2}$

$$\begin{gathered}\dot{x} = Ax + f(t), \quad t \in [0,1] \setminus \{\tau_1, \tau_2\},\\ Dx(0) = v,\\ M_1 x(\tau_1 - 0) + N_1 x(\tau_1 + 0) = h_1,\\ M_2 x(\tau_2 - 0) + N_2 x(\tau_2 + 0) = h_2,\end{gathered}$$

where

$$x(t) \in \mathrm{R}^2, \quad f(t) = [f_1(t)\ f_2(t)]^T, \quad h_1 = h_2 = [1\ 1]^T,$$

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad f_1(t) = f_2(t) = \begin{cases} \exp(2t), & t \in \left[0, \dfrac{1}{4}\right], \\ 0, & t \in \left(\dfrac{1}{4}, \dfrac{1}{2}\right], \\ \exp(t), & t \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$

$$v \text{ is an arbitrary constant}, \quad D = [1\ 1]^T.$$

We denote by $x_i(t)$, $i = 1, 2, 3$ the solution of the impulse problem in the intervals $\left[0, \dfrac{1}{4}\right]$, $\left(\dfrac{1}{4}, \dfrac{1}{2}\right]$, $\left(\dfrac{1}{2}, 1\right]$, respectively. We introduce the generalize conditions (5) $\sum\limits_{i=1}^{3} l_i x_i(\cdot) = h$, where

$$l_1 x_1(\cdot) = \begin{bmatrix} -2 & 1 \\ -2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_1\left(\frac{1}{4}\right), \quad l_2 x_2(\cdot) = \begin{bmatrix} -2 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_2\left(\frac{1}{4}\right) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 1 \\ 0 & 0 \end{bmatrix} x_2\left(\frac{1}{2}\right),$$

$$l_3 x_3(\cdot) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} x_3\left(\frac{1}{2}\right), \quad h = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We find the normal fundamental matrix $X(t)$ and its inverse matrix $X^{-1}(t)$

$$X(t) = \begin{bmatrix} 2e^{2t} - e^t & e^t - e^{2t} \\ 2e^{2t} - 2e^t & 2e^t - e^{2t} \end{bmatrix}, X^{-1} = \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix}.$$

The matrix Q has the representation $Q = [Q_1\ Q_2\ Q_3]$ from (10), where the matrices Q_1, Q_2, Q_3 have the form

$$Q_1 = -\frac{\sqrt{e}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -2 & 1 \\ 0 & 0 \\ -2e\sqrt{e} & e\sqrt{e} \\ 0 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Sequently we find

$$Q^{+}=\frac{1}{5e(1+e^{3})}\begin{bmatrix} -\frac{5}{2}e^{3}\sqrt{e} & -5\left(\sqrt{e}+\frac{\sqrt{e}}{2}e^{3}\right) & \frac{5}{2}e^{2} & -\frac{5}{2}e^{2} \\ -\frac{5}{2}e^{3}\sqrt{e} & -5\left(\sqrt{e}+\frac{\sqrt{e}}{2}e^{3}\right) & \frac{5}{2}e^{2} & -\frac{5}{2}e^{2} \\ -2e & 2e & -2e^{2}\sqrt{e} & 2e^{2}\sqrt{e} \\ e & -e & e^{2}\sqrt{e} & -e^{2}\sqrt{e} \\ \frac{5}{4}e^{2}\sqrt{e} & -\frac{5}{4}e^{2}\sqrt{e} & -\frac{5}{4}e & -5\left(\frac{e}{4}+\frac{e^{4}}{2}\right) \\ -\frac{5}{4}e^{2}\sqrt{e} & \frac{5}{4}e^{2}\sqrt{e} & \frac{5}{4}e & 5\left(\frac{e}{4}+\frac{e^{4}}{2}\right) \end{bmatrix},$$

$$P_{Q}=\frac{1}{10}\begin{bmatrix} 5 & -5 & 0 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 5 & 5 \end{bmatrix},$$

$$P_{Q^{*}}=\frac{1}{2(1+e^{3})}\begin{bmatrix} e^{3} & -e^{3} & -e\sqrt{e} & e\sqrt{e} \\ -e^{3} & e^{3} & e\sqrt{e} & -e\sqrt{e} \\ -e\sqrt{e} & e\sqrt{e} & 1 & -1 \\ e\sqrt{e} & -e\sqrt{e} & -1 & 1 \end{bmatrix}, \quad \overline{h}=\begin{bmatrix} 1+\frac{\sqrt{e}}{4}-\frac{3}{2}v\sqrt{e} \\ 1+\frac{\sqrt{e}}{4}-\frac{3}{2}v\sqrt{e} \\ 1 \\ 1 \end{bmatrix}.$$

Since $\operatorname{rank} D=1$ and $\operatorname{rank} Q=3$, then

$$D^{+}=\frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad P_{D}=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P_{D^{*}}=0$$

and $\operatorname{rank} P_{Q^{*}}=1$. According to item 3.2 $d_1=1$, the conditions (H4), (H6) are fulfilled, $r=3$, $d=1$.

For the solution of the problem from (14) we find

$$x_{1}(t)=X(t)P_{D}\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\eta+X(t)P_{D}\begin{bmatrix} -\left(\frac{1}{4}+\frac{\sqrt{e}}{e}-\frac{3}{2}v\right) \\ -\left(\frac{1}{4}+\frac{\sqrt{e}}{e}-\frac{3}{2}v\right) \end{bmatrix}$$

$$+X(t)\frac{1}{2}v\begin{bmatrix} -1 \\ 1 \end{bmatrix}+\begin{bmatrix} te^{2t} \\ te^{2t} \end{bmatrix}, \quad t\in\left[0,\frac{1}{4}\right], \quad \eta\in \mathrm{R}^{3},$$

$$x_{2}(t)=X(t)X^{-1}\left(\frac{1}{4}\right)\cdot\frac{1}{10}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}\eta, \quad t\in\left(\frac{1}{4},\frac{1}{2}\right],$$

$$x_3(t) = X(t)X^{-1}\left(\frac{1}{2}\right)\cdot\frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\eta + X(t)X^{-1}\left(\frac{1}{2}\right)\begin{bmatrix} -\dfrac{1}{2} \\ \dfrac{1}{2} \end{bmatrix} + \begin{bmatrix} e^{2t-\frac{1}{2}} - e^t \\ e^{2t-\frac{1}{2}} - e^t \end{bmatrix},\ t \in \left(\frac{1}{2}, 1\right].$$

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ОБОБЩЕНА ЗАДАЧА НА КОШИ ЗА ЛИНЕЙНИ ИМПУЛСНИ ДИФЕРЕНЦИАЛНИ СИСТЕМИ

Людмил Иванов Каранджулов

В работата се разглежда обощена задача на Коши за линейни системи от диференциални уравнения с обощени импулсни условия във фиксирани моменти от времето. Получени са необходими и достатъчни условия за съществуване на параметрично и единствено решение.