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SENSITIVITY OF GENERAL DISCRETE ALGEBRAIC RICCATI EQUATIONS

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Local and non-local perturbation bounds for general discrete algebraic Riccati equations (GDARE) are derived using the technique proposed in [2]. The equations considered generalize the symmetric discrete-time matrix Riccati equations arising in the optimal control and filtering of linear discrete time-invariant dynamic systems [3].

1. Problem statement. Consider the GDARE

$$F(X,P) := F_1(X,P_1) + F_2(X,P_2)F_3^{-1}(X,P_3)F_4(X,P_4) = 0, \tag{1}$$

where $X \in \mathbb{R}^{n\times n}$ is the unknown matrix. Here the operators F_i are defined from

$$F_i(X,P_i) := C_i + \sum_{k=1}^{r_i} \left(A_{ik}XB_{ik}^\top + B_{ik}XA_{ik}^\top + \varepsilon_{ik}D_{ik}XD_{ik}^\top \right),\ i = 1,3,$$

with $C_1 = C_1^\top$, $C_3 = C_3^\top$, $\varepsilon_{ik} = \pm 1$ and $F_2(X,P_2) = C_2 + \sum_{k=1}^{r_2} A_{2k}XB_{2k}^\top$, $F_4(X,P_2) = F_2^\top(X^\top,P_2)$. The function $F(\cdot,P) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a symmetric fractional affine matrix operator, depending on the matrix collection $P = (P_1,P_2,P_3,P_4)$ and satisfying $F^\top(X,P) = F(X^\top,P)$. The affine operators F_i depend on the the matrix collections $P_i := (C_i, A_{i1},\ B_{i1},\ \ldots, A_{i,r_i},\ B_{i,r_i})$.

Denote by $F_Z(X,P) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ the partial Fréchet derivative of F in the corresponding matrix argument $Z \in \mathcal{P} := \{C_1, A_{11}, B_{11}, \ldots, A_{3,r_3}, B_{3,r_3}\}$, computed at the point (X,P). Suppose that equation (1) has a solution X, such that the linear operator $F_X := F_X(X,P) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is invertible.

Let the matrices from $\mathcal{P}$ be perturbed as $Z \mapsto Z + \delta Z$. Denote by $P + \delta P$ the perturbed collection P, in which each matrix $Z \in \mathcal{P}$ is replaced by $Z + \delta Z$. Then the perturbed equation is $F(Y, P + \delta P) = 0$. In general some of the coefficient matrices from $\mathcal{P}$ may not be perturbed. Denote by $\mathcal{P}^* := \{Z_1, Z_2, \ldots, Z_r\} \subset \mathcal{P}$ the set of all matrices from $\mathcal{P}$, which are perturbed.

Since the operator F_X is invertible, the perturbed equation has an unique isolated solution $Y = X + \delta X$ in the neighbourhood of X if the perturbation δP is sufficiently small. Moreover, in this case the elements of δX are analytical functions of the elements of δP.

Let $\Delta := [\Delta_1, \Delta_2, \ldots, \Delta_r]^\top := [\delta_{Z_1}, \delta_{Z_2}, \ldots, \delta_{Z_r}]^\top \in \mathbb{R}_+^r$ be the vector of non-zero absolute norm perturbations $\delta_Z := \|\delta Z\|_F$ in the data matrices $Z \in \mathcal{P}^*$.

The perturbation problem for GDARE (1) is to find a bound

$$\delta_X \le f(\Delta),\ \Delta \in \Omega \subset \mathbb{R}_+^r,$$

for the perturbation $\delta_X := \|\delta X\|_{\mathrm{F}}$. Here Ω is a certain set and f is a continuous function, non-decreasing in each of its arguments Δ_j and satisgying $f(0) = 0$. The inclusion $\Delta \in \Omega$ guarantees that the perturbed equation has an unique solution $Y = X + \delta X$ in a neighbourhood of the unperturbed solution X, such that the elements of δX are analytical functions of the elements of the matrices δZ, $Z \in \mathcal{P}^*$, provided Δ is in the interior of Ω [3, 4].

2. Sensitivity of GDARE Consider the conditioning of the GDARE (1). Denote by **Lin** the space of linear operators $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$.

Having in mind that $F(X, P) = 0$, the perturbed equation may be written as

$$F(X + \delta X, P + \delta P) := F_X(\delta X) + \sum_{Z\in\mathcal{P}^*} F_Z(\delta Z) + G(\delta X, \delta P) = 0,$$

where $F_Z(\cdot) := F_Z(X, P)(\cdot) \in \mathbf{Lin}$ are the Fréchet derivatives of $F(X, P)$ in $Z = X$ or $Z \in \mathcal{P}^*$, evaluated at the solution X. The matrix $G(\delta X, \delta P)$ contains second and higher order terms in δX, δP.

A straightforward calculation leads to

$$\begin{aligned} F_X(Z) = {} & \sum_{k=1}^{r_1}(A_{1k} Z B_{1k}^\top + B_{1k} Z A_{1k}^\top + \varepsilon_{1k} D_{1k} Z D_{1k}^\top) \\ & + \left(\sum_{k=1}^{r_2} A_{2k} Z B_{2k}^\top\right) N \\ & - M\left(\sum_{k=1}^{r_3}(A_{3k} Z B_{3k}^\top + B_{3k} Z A_{3k}^\top + \varepsilon_{3k} D_{3k} Z D_{3k}^\top)\right) N \\ & + M\left(\sum_{k=1}^{r_2} B_{2k} Z A_{2k}^\top\right), \end{aligned} \tag{2}$$

and

$$\begin{aligned} F_{C_1}(Z) &= Z, \\ F_{A_{1k}}(Z) &= Z X B_{1k}^\top + B_{1k} X Z^\top, \\ F_{B_{1k}}(Z) &= A_{1k} X Z^\top + Z X A_{1k}^\top, \\ F_{D_{1k}}(Z) &= \varepsilon_{1k} Z X D_{1k}^\top + \varepsilon_{1k} D_{1k} X Z^\top, \\ F_{C_2}(Z) &= Z N + M Z^\top, \\ F_{A_{2k}}(Z) &= Z X B_{2k}^\top N + M B_{2k} X Z^\top, \\ F_{B_{2k}}(Z) &= A_{2k} X Z^\top N + M Z X A_{2k}^\top, \\ F_{C_3}(Z) &= -M Z N, \\ F_{A_{3k}}(Z) &= -M Z X B_{3k}^\top N - M B_{3k} X Z^\top N, \\ F_{B_{3k}}(Z) &= -M A_{3k} X Z^\top N - M Z X A_{3k}^\top N, \\ F_{D_{3k}}(Z) &= -\varepsilon_{3k} M Z X D_{3k}^\top N - \varepsilon_{3k} M D_{3k} X Z^\top N, \end{aligned} \tag{3}$$

where $M := F_2(X, P) F_3^{-1}(X, P)$, $N := F_3^{-1}(X, P) F_4(X, P)$.

Since the operator $F_X(.)$ is invertible we get

$$\delta X = -\sum_{Z \in \mathcal{P}^*} F_X^{-1} \circ F_Z(\delta Z) - F_X^{-1}(G(\delta X, \delta P)). \tag{4}$$

The relation (4) gives

$$\delta_X \leq \sum_{Z \in \mathcal{P}^*} K_Z \delta_Z + O(\|\Delta\|^2),\ \Delta \to 0, \tag{5}$$

where the quantities $K_Z := \left\|F_X^{-1} \circ F_Z\right\|,\ Z \in \mathcal{P}^*$, are the *absolute individual condition numbers* of GDARE (1). Here $\|.\|$ is the induced norm in the space **Lin**.

Denote by $L_Z \in \mathbb{R}^{n^2 \times n^2}$ the matrix representation of the operator $F_Z(\cdot) \in$ **Lin**. We have

$$\begin{aligned} L_X &= \sum_{k=1}^{r_1} (B_{1k} \otimes A_{1k} + A_{1k} \otimes B_{1k} + \varepsilon_{1k} D_{1k} \otimes D_{1k}) \\ &\quad + \sum_{k=1}^{r_2} ((B_{2k}^\top N)^\top \otimes A_{2k} + A_{2k} \otimes (M B_{2k})) \\ &\quad - \sum_{k=1}^{r_3} ((B_{3k}^\top N)^\top \otimes (M A_{3k}) + (A_{3k}^\top N)^\top \otimes (M B_{3k}) \\ &\quad + \varepsilon_{3k} (D_{3k}^\top N) \otimes (M D_{3k})), \end{aligned} \tag{6}$$

and

$$\begin{aligned} L_{C_1} &= I_n, \\ L_{A_{1k}} &= (X B_{1k}^\top)^\top \otimes I_n + (I_n \otimes (B_{1k} X))\Pi_n, \\ L_{B_{1k}} &= (X A_{1k}^\top)^\top \otimes I_n + (I_n \otimes (A_{1k} X)\Pi_n, \\ L_{D_{1k}} &= \varepsilon_{1k} (X D_{1k}^\top)^\top \otimes I_n + \varepsilon_{1k} (I_n \otimes (D_{1k} X))\Pi_n, \\ L_{C_2} &= N^\top \otimes I_n + (I_n \otimes M)\Pi_n, \\ L_{A_{2k}} &= (X B_{2k}^\top N)^\top \otimes I_n + (I_n \otimes M B_{2k} X)\Pi_n, \\ L_{B_{2k}} &= (N^\top \otimes (A_{2k} X))\Pi_n + (X A_{2k}^\top)^\top \otimes M, \\ L_{C_3} &= -N^\top \otimes M, \\ L_{A_{3k}} &= -(X B_{3k}^\top N)^\top \otimes M - (N^\top \otimes (M B_{3k} X))\Pi_n, \\ L_{B_{3k}} &= -(N^\top \otimes (M A_{3k} X))\Pi_n - (X A_{3k}^\top N)^\top \otimes M, \\ L_{D_{3k}} &= -\varepsilon_{3k} ((X D_{3k}^\top N)^\top \otimes M) - \varepsilon_{3k} (N^\top \otimes (M D_{3k} X))\Pi_n, \end{aligned}$$

where Π_n is vec-permutation matrix. Having in mind the above expressions, the absolute condition numbers are calculated from $K_Z = \|L_X^{-1} L_Z\|_2,\ Z \in \mathcal{P}^*$.

Local bounds of the type considered above are usually used neglecting terms of order $O(\|\Delta\|^2)$, i.e. they are valid only for $\Delta \to 0$. This disadvantage of the local estimates may be overcome using the techniques of non-linear perturbation analysis, presented below.

The perturbed quantities $F_i(X + \delta X, P_i + \delta P_i)$ are expressed as follows. We have

$$\widetilde{F}_i := F_i(X + \delta X, P_i + \delta P_i) = F_i + E_i(\delta X, \delta P_i),$$

where $F_i := F_i(X, P_i)$ and

$$E_i(\delta X, \delta P_i) := L_i(\delta X) + K_i(\delta P_i) + Q_i(\delta X, \delta P_i).$$

Here the operators $L_i(\cdot) \in \mathbf{Lin}$ are defined as

$$L_i(Z) := \sum_{k=1}^{r_i}(A_{ik} Z B_{ik}^\top + B_{ik} Z A_{ik}^\top + \varepsilon_{ik} D_{ik} Z D_{ik}^\top), \quad i = 1, 3,$$

$$L_2(Z) := \sum_{k=1}^{r_2} A_{2k} Z B_{2k}^\top, \ L_4(Z) := \sum_{k=1}^{r_2} B_{2k} Z A_{2k}^\top,$$

the term

$$\begin{aligned} K_i(\delta P_i) \, &:= \, \delta C_i + \sum_{k=1}^{r_i}(\delta A_{ik} X B_{ik}^\top + A_{ik} X \delta B_{ik}^\top + \delta B_{ik} X A_{ik}^\top + B_{ik} X \delta A_{ik}^\top \\ &\quad + \varepsilon_{ik}(\delta D_{ik} X D_{ik}^\top + D_{ik} X \delta D_{ik}^\top)), \quad i = 1, 3 \\ K_2(\delta P_2) &:= \delta C_2 + \sum_{k=1}^{r_2}(\delta A_{2k} X B_{2k}^\top + A_{2k} X \delta B_{2k}^\top) \\ K_4(\delta P_4) &:= \delta C_2^\top + \sum_{k=1}^{r_2}(\delta B_{2k} X A_{2k}^\top + B_{2k} X \delta A_{2k}^\top) \end{aligned}$$

contains the first order perturbations in P_i and $Q_i(\cdot, \delta P_i)$ is the affine operator

$$\begin{aligned} Q_i(Z, \delta P_i) \, &:= \, \sum_{k=1}^{r_i}(\delta A_{ik} Z B_{ik}^\top + A_{ik} Z \delta B_{ik}^\top + \delta A_{ik}(X + Z)\delta B_{ik}^\top \\ &\quad + \delta B_{ik} Z A_{ik}^\top + B_{ik} Z \delta A_{ik}^\top + \delta B_{ik}(X + Z)\delta A_{ik}^\top \\ &\quad + \varepsilon_{ik}(\delta D_{ik} Z D_{ik}^\top + D_{ik} Z \delta D_{ik}^\top + \delta D_{ik}(X + Z)\delta D_{ik}^\top)), \quad i = 1, 3, \\ Q_2(Z, \delta P_2) &:= \sum_{k=1}^{r_2}(\delta A_{2k} Z B_{2k}^\top + A_{2k} Z \delta B_{2k}^\top + \delta A_{2k}(X + Z)\delta B_{2k}^\top) \\ Q_4(Z, \delta P_4) &:= \sum_{k=1}^{r_2}(\delta B_{2k} Z A_{2k}^\top + B_{2k} Z \delta A_{2k}^\top + \delta B_{2k}(X + Z)\delta A_{2k}^\top). \end{aligned}$$

Thus the expression $Q_i(\delta X, \delta P_i)$ contains the second and third order terms in δX and δP_i. The perturbed equation $\widetilde{F}_1 + \widetilde{F}_2 \widetilde{F}_3^{-1} \widetilde{F}_4 = 0$ may be written as

$$\widetilde{F}(\mathcal{E}) := F_1 + \mathcal{E}_1 + (F_2 + \mathcal{E}_2)(F_3 + \mathcal{E}_3)^{-1}(F_4 + \mathcal{E}_4) = 0, \tag{7}$$

where $\mathcal{E} := (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$ and $\mathcal{E}_i := E_i(\delta X, \delta P_i)$. We may represent $\widetilde{F}(\mathcal{E})$ as the sum of the first order terms $L(\mathcal{E})$ plus the second and higher order terms $Q(\mathcal{E})$ in $\mathcal{E}$, $\widetilde{F}(\mathcal{E}) = L(\mathcal{E}) + Q(\mathcal{E})$. We have $\widetilde{F}_{\mathcal{E}_1}(0)(Z) = Z$, $\widetilde{F}_{\mathcal{E}_2}(0)(Z) = ZN$, $\widetilde{F}_{\mathcal{E}_3}(0)(Z) = -MZN$, $\widetilde{F}_{\mathcal{E}_4}(0)(Z) = MZ$. Hence

$$L(\mathcal{E}) = \mathcal{E}_1 + \mathcal{E}_2 N - M\mathcal{E}_3 N + M\mathcal{E}_4. \tag{8}$$

The expression for $Q(\mathcal{E})$ is

$$Q(\mathcal{E}) \, = \mathcal{E}_2(F_3 + \mathcal{E}_3)^{-1}\mathcal{E}_4 - \mathcal{E}_2(F_3 + \mathcal{E}_3)^{-1}\mathcal{E}_3 N \tag{9}$$

$$
\begin{aligned}
&+ M\mathcal{E}_3(F_3+\mathcal{E}_3)^{-1}\mathcal{E}_3 N - M\mathcal{E}_3(F_3+\mathcal{E}_3)^{-1}\mathcal{E}_4 \\
&= (\mathcal{E}_2 - M\mathcal{E}_3)(F_3+\mathcal{E}_3)^{-1}(\mathcal{E}_4 - \mathcal{E}_3 N).
\end{aligned}
$$

Next we shall give an estimate for $\varphi(\mathcal{E}) := \left\|F_X^{-1}(Q(\mathcal{E}))\right\|_{\mathrm{F}}$. We have

$$
\varphi(\mathcal{E}) \le \left\|L_X^{-1}\right\|_2 \frac{\|\mathcal{E}_2 - M\mathcal{E}_3\|_2 \left\|F_3^{-1}(\mathcal{E}_4 - \mathcal{E}_3 N)\right\|_2}{1 - \left\|F_3^{-1}\mathcal{E}_3\right\|_{\mathrm{F}}}. \tag{10}
$$

The perturbed equation may be rewritten as an equivalent operator equation,

$$
\delta X = \Phi(\delta X, \delta P) := \Phi_1(\delta P) + \Phi_2(\delta X, \delta P) + \Psi(\delta X, \delta P), \tag{11}
$$

where

$$
\begin{aligned}
\Phi_1(\delta P) &:= -F_X^{-1}(K_1(\delta P_1) + K_2(\delta P_2)N \\
&\qquad - MK_3(\delta P_3)N + MK_4(\delta P_4)), \\
\Phi_2(Z, \delta P) &:= -F_X^{-1}(Q_1(Z,\delta P_1) + Q_2(Z,\delta P_2)N \\
&\qquad - MQ_3(Z,\delta P_3)N + MQ_4(Z,\delta P_4)), \\
\Psi(Z,\delta P) &:= -F_X^{-1}((E_2(Z,\delta P_2) - ME_3(Z,\delta P_3)) \\
&\qquad \times (F_3 + E_3(Z,\delta P_3))^{-1}(E_4(Z,\delta P_4) - E_3(Z,\delta P_3)N)).
\end{aligned} \tag{12}
$$

Let $\|Z\|_{\mathrm{F}} \le \rho$. After some computations we get

$$
\begin{aligned}
\|E_2(Z,\delta P_2) - ME_3(Z,\delta P_3)\|_2 &\le \alpha_2(\Delta) + \beta_2(\Delta)\rho, \\
\left\|F_3^{-1}E_3(Z,\delta P_3)\right\|_{\mathrm{F}} &\le \alpha_3(\Delta) + \beta_3(\Delta)\rho, \\
\left\|F_3^{-1}(E_4(Z,\delta P_4) - E_3(Z,\delta P_3)N)\right\|_2 &\le \alpha_4(\Delta) + \beta_4(\Delta)\rho,
\end{aligned} \tag{13}
$$

where, for $i = 2,3,4$, $\alpha_i(\Delta) := \alpha_{i1}(\Delta) + \alpha_{i2}(\Delta)$, $\beta_i(\Delta) := \beta_{i0}(\Delta) + \beta_{i1}(\Delta) + \beta_{i2}(\Delta)$. The quantities $\alpha_{ij}(\Delta) = O(\|\Delta\|^j)$; $\beta_{ik}(\Delta) = O(\|\Delta\|^k)$, $\Delta \to 0$, are determined as follows.

Case $i = 2$:

$$
\begin{aligned}
\alpha_{21}(\Delta) &:= \delta_{C_2} + \sum_{k=1}^{r_2}(\|XB_{2k}^\top\|_2\delta_{A_{2k}} + \|A_{2k}X\|_2\delta_{B_{2k}}) \\
&\quad + \|M\|_2\delta_{C_3} + \sum_{k=1}^{r_3}\big(\|M\|_2\|XB_{3k}^\top\|_2\delta_{A_{3k}} + \|MA_{3k}X\|_2\delta_{B_{3k}} \\
&\quad + \|M\|_2\|XA_{3k}^\top\|_2\delta_{B_{3k}} + \|MB_{3k}X\|_2\delta_{A_{3k}} \\
&\quad + \|M\|_2\|XD_{3k}^\top\|_2\delta_{D_{3k}} + \|MD_{3k}X\|_2\delta_{D_{3k}}\big), \\
\alpha_{22}(\Delta) &:= \|X\|_2\beta_{22}(\Delta), \\
\beta_{20}(\Delta) &:= \Bigg\|\sum_{k=1}^{r_2} B_{2k}\otimes A_{2k} \\
&\quad + \sum_{k=1}^{r_3}(B_{3k}\otimes(MA_{3k}) + A_{3k}\otimes(MB_{3k}) + \varepsilon_{3k}D_{3k}\otimes(MD_{3k}))\Bigg\|_2, \\
\beta_{21}(\Delta) &:= \sum_{k=1}^{r_2}(\|B_{2k}\|_2\delta_{A_{2k}} + \|A_{2k}\|_2\delta_{B_{2k}})
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&+ \sum_{k=1}^{r_3} (\|M\|_2 \|B_{3k}\|_2 \delta_{A_{3k}} + \|MA_{3k}\|_2 \delta_{B_{3k}} \\
&+ \|M\|_2 \|A_{3k}\|_2 \delta_{B_{3k}} + \|MB_{3k}\|_2 \delta_{A_{3k}} \\
&+ \|M\|_2 \|D_{3k}\|_2 \delta_{D_{3k}} + \|MD_{3k}\|_2 \delta_{D_{3k}}) , \\
\beta_{22}(\Delta) := &\sum_{k=1}^{r_2} \delta_{A_{2k}} \delta_{B_{2k}} + \|M\|_2 \sum_{k=1}^{r_3} \left(2\delta_{A_{3k}} \delta_{B_{3k}} + \delta_{D_{3k}}^2 \right) .
\end{aligned}
$$

Case $i = 3$:

$$
\begin{aligned}
\text{(15)}\quad \alpha_{31}(\Delta) := &\left\| F_3^{-1} \right\|_2 \delta_{C_3} \\
&+ \sum_{k=1}^{r_3} \left(\left\| F_3^{-1} \right\|_2 \|XB_{3k}^\top\|_2 \delta_{A_{3k}} + \left\| F_3^{-1} A_{3k} X \right\|_2 \delta_{B_{3k}} \right. \\
&+ \left\| F_3^{-1} \right\|_2 \|XA_{3k}^\top\|_2 \delta_{B_{3k}} + \left\| F_3^{-1} B_{3k} X \right\|_2 \delta_{A_{3k}} \\
&\left. + \left\| F_3^{-1} \right\|_2 \|XD_{3k}^\top\|_2 \delta_{D_{3k}} + \left\| F_3^{-1} D_{3k} X \right\|_2 \delta_{D_{3k}} \right) , \\
\alpha_{32}(\Delta) := &\|X\|_2 \beta_{32}(\Delta), \\
\beta_{30}(\Delta) := &\left\| \sum_{k=1}^{r_3} \left(B_{3k} \otimes \left(F_3^{-1} A_{3k} \right) + A_{3k} \otimes \left(F_3^{-1} B_{3k} \right) + \varepsilon_{3k} D_{3k} \otimes \left(F_3^{-1} D_{3k} \right) \right) \right\|_2 , \\
\beta_{31}(\Delta) := &\sum_{k=1}^{r_3} \left(\left\| F_3^{-1} \right\|_2 \|B_{3k}\|_2 \delta_{A_{3k}} + \left\| F_3^{-1} A_{3k} \right\|_2 \delta_{B_{3k}} \right. \\
&+ \left\| F_3^{-1} \right\|_2 \|A_{3k}\|_2 \delta_{B_{3k}} + \left\| F_3^{-1} B_{3k} \right\|_2 \delta_{A_{3k}} \\
&\left. + \left\| F_3^{-1} \right\|_2 \|D_{3k}\|_2 \delta_{D_{3k}} + \left\| F_3^{-1} D_{3k} \right\|_2 \delta_{D_{3k}} \right) , \\
\beta_{32}(\Delta) := &\left\| F_3^{-1} \right\|_2 \sum_{k=1}^{r_3} \left(2\delta_{A_{3k}} \delta_{B_{3k}} + \delta_{D_{3k}}^2 \right) .
\end{aligned}
$$

Case $i = 4$:

$$
\begin{aligned}
\text{(16)}\quad \alpha_{41}(\Delta) := &\left\| F_3^{-1} \right\|_2 \delta_{C_2} \\
&+ \sum_{k=1}^{r_2} \left(\left\| F_3^{-1} \right\|_2 \|XA_{2k}^\top\|_2 \delta_{B_{2k}} + \|F_3^{-1} B_{2k} X\|_2 \delta_{A_{2k}} \right) \\
&+ \left\| F_3^{-1} \right\|_2 \|N\|_2 \delta_{C_3} \\
&+ \sum_{k=1}^{r_3} \left(\left\| F_3^{-1} \right\|_2 \|XB_{3k}^\top N\|_2 \delta_{A_{3k}} + \|F_3^{-1} A_{3k} X\|_2 \|N\|_2 \delta_{B_{3k}} \right. \\
&+ \left\| F_3^{-1} \right\|_2 \|XA_{3k}^\top N\|_2 \delta_{B_{3k}} + \|F_3^{-1} B_{3k} X\|_2 \|N\|_2 \delta_{A_{3k}} \\
&\left. + \left\| F_3^{-1} \right\|_2 \|XD_{3k}^\top N\|_2 \delta_{D_{3k}} + \|F_3^{-1} D_{3k} X\|_2 \|N\|_2 \delta_{D_{3k}} \right) , \\
\alpha_{42}(\Delta) := &\|X\|_2 \beta_{42}(\Delta),
\end{aligned}
$$

$$
\begin{aligned}
\beta_{40}(\Delta) \;:=&\; \Bigg\|\sum_{k=1}^{r_2} A_{2k}\otimes (F_3^{-1}B_{2k}) + \sum_{k=1}^{r_3}\big((B_{3k}^{\top}N)^{\top}\otimes (F_3^{-1}A_{3k})\\
&+\;(A_{3k}^{\top}N)^{\top}\otimes (F_3^{-1}B_{3k}) + \varepsilon_{3k}(D_{3k}^{\top}N)^{\top}\otimes (F_3^{-1}D_{3k})\big)\Bigg\|_2,\\
\beta_{41}(\Delta) \;:=&\; \sum_{k=1}^{r_2}\left(\left\|F_3^{-1}\right\|_2\|A_{2k}^{\top}\|_2\delta_{B_{2k}} + \left\|F_3^{-1}B_{2k}\right\|_2\delta_{A_{2k}}\right)\\
&+\sum_{k=1}^{r_3}\Big(\left\|F_3^{-1}\right\|_2\|B_{3k}^{\top}N\|_2\delta_{A_{3k}} + \left\|F_3^{-1}A_{3k}\right\|_2\|N\|_2\delta_{B_{3k}}\\
&+\left\|F_3^{-1}\right\|_2\|A_{3k}^{\top}N\|_2\delta_{B_{3k}} + \left\|F_3^{-1}B_{3k}\right\|_2\|N\|_2\delta_{A_{3k}}\\
&+\;\left\|F_3^{-1}\right\|_2\|D_{3k}^{\top}N\|_2\delta_{D_{3k}} + \left\|F_3^{-1}D_{3k}\right\|_2\|N\|_2\delta_{D_{3k}}\Big),\\
\beta_{42}(\Delta) \;:=&\; \left\|F_3^{-1}\right\|_2\left(\sum_{k=1}^{r_2}\delta_{A_{2k}}\delta_{B_{2k}} + \|N\|_2\sum_{k=1}^{r_3}\left(2\delta_{A_{3k}}\delta_{B_{3k}} + \delta_{D_{3k}}^2\right)\right).
\end{aligned}
$$

It follows from (13)–(16) that

$$
\begin{aligned}
\|\Phi_1(\delta P) + \Phi_2(Z,\delta P)\|_{\mathrm{F}} \;\le&\; a_0(\Delta) + a_1(\Delta)\rho,\\
\|\Psi(Z,\delta P)\|_{\mathrm{F}} \;\le&\; \frac{b_0(\Delta) + b_1(\Delta)\rho + b_2(\Delta)\rho^2}{1-\alpha_3(\Delta)-\beta_3(\Delta)\rho},
\end{aligned}
\tag{17}
$$

provided that $\rho < (1-\alpha_3(\Delta))/\beta_3(\Delta)$. Here

$$
\begin{aligned}
a_0(\Delta) \;:=&\; a_{01}(\Delta) + a_{02}(\Delta) := \mathrm{est}(\Delta) + \|X\|_2 a_{12}(\Delta),\\
a_1(\Delta) \;:=&\; a_{11}(\Delta) + a_{12}(\Delta),
\end{aligned}
$$

$$
\begin{aligned}
a_{11}(\Delta) \;:=&\; \sum_{k=1}^{r_1}\left(\left\|L_X^{-1}(B_{1k}\otimes I_n)\right\|_2 + \left\|L_X^{-1}(I_n\otimes B_{1k})\Pi_n\right\|_2\right)\delta_{A_{1k}}\\
&+\sum_{k=1}^{r_1}\left(\left\|L_X^{-1}(A_{1k}\otimes I_n)\right\|_2 + \left\|L_X^{-1}(I_n\otimes A_{1k})\Pi_n\right\|_2\right)\delta_{B_{1k}}\\
&+\sum_{k=1}^{r_1}\left(\left\|L_X^{-1}(D_{1k}\otimes I_n)\right\|_2 + \left\|L_X^{-1}(I_n\otimes D_{1k})\Pi_n\right\|_2\right)\delta_{D_{1k}}\\
&+\sum_{k=1}^{r_2}\left(\left\|L_X^{-1}\left((B_{2k}^{\top}N)^{\top}\otimes I_n\right)\right\|_2 + \left\|L_X^{-1}(I_n\otimes (MB_{2k}))\Pi_n\right\|_2\right)\delta_{A_{2k}}\\
&+\sum_{k=1}^{r_2}\left(\left\|L_X^{-1}\left(N^{\top}\otimes A_{2k}\right)\Pi_n\right\|_2 + \left\|L_X^{-1}(A_{2k}\otimes M)\right\|_2\right)\delta_{B_{2k}}\\
&+\sum_{k=1}^{r_3}\left(\left\|L_X^{-1}\left((B_{3k}^{\top}N)^{\top}\otimes M\right)\right\|_2 + \left\|L_X^{-1}\left(N^{\top}\otimes (MB_{3k})\right)\Pi_n\right\|_2\right)\delta_{A_{3k}}\\
&+\sum_{k=1}^{r_3}\left(\left\|L_X^{-1}\left(N^{\top}\otimes (MA_{3k})\right)\Pi_n\right\|_2 + \left\|L_X^{-1}\left((A_{3k}^{\top}N)^{\top}\otimes M)\right)\right\|_2\right)\delta_{B_{3k}}\\
&+\sum_{k=1}^{r_3}\left(\left\|L_X^{-1}\left((D_{3k}^{\top}N)^{\top}\otimes M\right)\right\|_2 + \left\|L_X^{-1}\left(\left(N^{\top}\otimes (MD_{3k})\right)\Pi_n\right)\right\|_2\right)\delta_{D_{3k}}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
a_{12}(\Delta) := & \sum_{k=1}^{r_1} \left\|L_X^{-1}\right\|_2 \left(2\delta_{A_{1k}}\delta_{B_{1k}} + \delta_{D_{2k}}^2\right) \\
& + \sum_{k=1}^{r_2} \left\|L_X^{-1}\left(N^\top \otimes I_n\right)\right\|_2 \delta_{A_{2k}}\delta_{B_{2k}} \\
& + \sum_{k=1}^{r_3} \left\|L_X^{-1}(N^\top \otimes M)\right\|_2 \left(2\delta_{A_{3k}}\delta_{B_{3k}} + \delta_{D_{3k}}^2\right) \\
& + \sum_{k=1}^{r_2} \left\|L_X^{-1}(I_n \otimes M)\right\|_2 \delta_{A_{2k}}\delta_{B_{2k}}
\end{aligned}
$$

and

(19)
$$
\begin{aligned}
b_0(\Delta) &:= \left\|L_X^{-1}\right\|_2 \alpha_2(\Delta)\alpha_4(\Delta), \\
b_1(\Delta) &:= \left\|L_X^{-1}\right\|_2 (\alpha_2(\Delta)\beta_4(\Delta) + \alpha_4(\Delta)\beta_2(\Delta)), \\
b_2(\Delta) &:= \left\|L_X^{-1}\right\|_2 \beta_2(\Delta)\beta_4(\Delta).
\end{aligned}
$$

Using (17) we see that the Lyapunov majorant $h(\rho, \Delta)$ for equation (11), such that $\|\Phi(Z, \delta P)\|_{\mathrm{F}} \le h(\rho, \Delta)$, is [2, 1]

$$
h(\rho, \Delta) = a_0(\Delta) + a_1(\Delta)\rho + \frac{b_0(\Delta) + b_1(\Delta)\rho + b_2(\Delta)\rho^2}{1 - \alpha_3(\Delta) - \beta_3(\Delta)\rho}.
$$

Thus the fundamental equation $h(\rho, \Delta) = \rho$ for determining the non-local bound $\rho = \rho(\Delta)$ for δ_X is quadratic:

(20)
$$
d_2(\Delta)\rho^2 - d_1(\Delta)\rho + d_0(\Delta) = 0,
$$

where

(21)
$$
\begin{aligned}
d_0(\Delta) &:= b_0(\Delta) + a_0(\Delta)(1 - \alpha_3(\Delta)), \\
d_1(\Delta) &:= a_0(\Delta)\beta_3(\Delta) + (1 - \alpha_3(\Delta))(1 - a_1(\Delta)) - b_1(\Delta), \\
d_2(\Delta) &:= b_2(\Delta) + \beta_3(\Delta)(1 - a_1(\Delta)).
\end{aligned}
$$

Suppose that $\Delta \in \Omega$, where

(22)
$$
\Omega := \left\{\Delta \succeq 0 : 2\sqrt{d_0(\Delta)d_2(\Delta)} \le d_1(\Delta)\right\} \subset \mathbb{R}_+^r.
$$

Then equation (20) has non-negative roots $\rho_1 \le \rho_2$,

(23)
$$
\rho_1 = f(\Delta) := \frac{2d_0(\Delta)}{d_1(\Delta) + \sqrt{d_1^2(\Delta) - 4d_0(\Delta)d_2(\Delta)}}.
$$

Hence the operator $\Phi(\cdot, \delta P)$ maps the closed convex ball

$$
\mathcal{B}(\Delta) := \{Z \in \mathbb{R}^{n\times n} \ : \ \|Z\|_{\mathrm{F}} \le f(\Delta)\} \subset \mathbb{R}^{n\times n}
$$

into itself. According to the Schauder fixed point principle there exists a solution $\delta X \in \mathcal{B}(\Delta)$ of equation (11), for which

(24)
$$
\delta_X = \|\delta X\|_{\mathrm{F}} \le f(\Delta), \ \ \Delta \in \Omega.
$$

If $\Delta \in \Omega_1$, where

$$
\Omega_1 := \left\{\Delta \succeq 0 : 2\sqrt{d_0(\Delta)d_2(\Delta)} < d_1(\Delta)\right\} \subset \Omega,
$$

then $\rho_1 < \rho_2$ and the operator $\Phi(\cdot, \Delta)$ is a contraction on $\mathcal{B}(\Delta)$. Hence the solution

δX, for which the estimate (24) holds true, is unique. This means that the perturbed equation has an isolated solution $Y = X + \delta X$, where the elements of δX are analytical functions of the elements of δP.

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ЧУВСТВИТЕЛНОСТ НА ОБЩИ ДИСКРЕТНИ АЛГЕБРИЧНИ УРАВНЕНИЯ НА РИКАТИ

М. М. Константинов, П. Хр. Петков, В. А. Ангелова

Получени са локални и нелокални пертурбационни граници за общи дискретни алгебрични уравнения на Рикати с използване на техниката, предложена в [1]. Разгледаните уравнения обобщават симетричните дискретни матрични уравнения на Рикати в теорията на оптималното управление и филтрацията на линейните дискретни стационарни динамични системи [3].